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Topological Categories and Algebraic Categories
拓扑范畴 与 代数范畴

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Abstract

While the study of traditional category theory is focused on certain abstract categories and the functors between them, the study of concrete categories is aimed to find the properties of various structures on categories. A crucial viewpoint is that every faithful functor can be seen as a structure over the base category, so the study of structures changes into the study of faithful functors.
Two of the important kinds of structures over categories are topological structures and algebraic structures, but such a saying is only an intuitive one. In my thesis I try to work out what is the essence of topological structures and algebraic structures. They both help greatly for us understanding the base categories, and some nice properties are reflected or preserved by these two kinds of functors.

摘要

传统的范畴学的研究集中在某些抽象范畴，和它们之间的函子，但是对具体范畴的研究却在于探寻范畴之上的结构。一个关键的观点是，任何一个忠实函子都可以被认为是其上域范畴上的一种结构，这样的话对范畴上结构的研究就变成了对忠实函子的研究。
范畴上两种最重要的结构是拓扑结构和代数结构，但是这仅仅是一种直观的说法。这篇论文里我将指出拓扑结构和代数结构两个概念的精华所在。这两种函子都能帮助我们更好地理解它们的上域范畴，而且它们能反映或保持许多优良的性质。

Prerequisite

Throughout the thesis, once the reader is confronted with an undefined term or category, please refer to [ACC]. The theory for sets and classes here we use is from [ACC]. All the familiar concepts are as usually used. For every category, its objects constitute a class, and hom(a, b) is a set for arbitrary objects a, b.
We here use bold letters for categories: **A**, **B**, **C**, … ; and same type of letters for well-known categories: **Set**, **Grp**, **Top**, … . We use capital letters for functors: F, T, U, … . We use lowercase italics for objects: a, b, c, … ; and same type of letters for morphisms: f, g, h, … .

Definition 1.1 Let **X** be a category. A **concrete category** over **X** is a pair (**A**, U), where **A** is a category, and U: **A** →**X** is a faithful functor. U is called the **forgetful functor** or **underlying functor** of the concrete category. **X** is called the **base category** for (**A**, U). A concrete category over **Set** is called a **construct**.

Remark 1.2 If (**A**, U) is concrete over **X**, since faithful functors are injective on hom-sets, we usually assume that $\mathrm{hom}_{\mathbf{A}}(a, b)$ is a subset of $\mathrm{hom}_{\mathbf{X}}(\mathrm{U}a, \mathrm{U}b)$ for each pair (a, b) of **A**-objects. Even though different hom-sets need to be disjoint, such convention never causes problems. Thus, we may express sentences like "for **A**-objects a, b, and **X**-morphism f : Ua→Ub, there exists a (necessarily unique) **A**-morphism a→b with U(a→b) is f " by stating "f : Ua→Ub is an **A**-morphism (from a to b)".

Remark 1.3 Sometimes the underlying functor is obvious so that we will simply regard **A**, instead of (**A**, U), as a concrete category over **X**. In these cases the underlying object of an **A**-object a will sometimes be denoted by $|a|$; *i.e.*, "| |" will serve as a standard notation for underlying functors.

Definition 1.4 Let (**A**, U) be concrete over **X**. The **fibre** of an **X**-object x is the preordered class consisting of all **A**-objects a with Ua = x , ordered by: $a \leq b$ iff id_x: Ua→Ub is an **A**-morphism.

Remark 1.5 Fibres need not be sets, though in most familiar concrete categories they are sets. Intuitively, the fibre of an **X**-object x can be viewed as the class of "structures" on x, where $a \leq b$ in the fibre means the structure a is "thinner" than b. The underlying functor from **Top** to **Set** provides a straightforward example.

Definition 1.6 A **source** in category **X** is a pair $(a, (f_i)_{i \in I})$ consisting of an **X**-object a and a family of **X**-morphisms $f_i : a \to a_i$ with domain a, indexed by some class I.

Remark 1.7 The index class might be a non-empty set or an empty set. It can also be a proper class, *i.e.*, a class that cannot be indexed by any set. We usually denote a source by notation such as $(f_i : a \to a_i)_I$. The dual notion of source is **sink**, which is often written like $(g_i : a_i \to a)_I$.

Remark 1.8 If $S = (a \xrightarrow{f_i} a_i)_{i \in I}$ is a source in **A**, and G : **A**→**B** is a functor, then GS represents the source $(\mathrm{G}a \xrightarrow{\mathrm{G}f_i} \mathrm{G}a_i)_{i \in I}$ in **B**.

Definition 1.9 A **diagram** in category **A** is a functor D : **J**→**A**. Here category **J** is usually small or even finite, although this is not a must.

Definition 1.10 A **limiting cone** of diagram D : **J**→**A** is a source $S = (a \xrightarrow{f_j} \mathrm{D}j)_{j \in \mathrm{Obj}(\mathbf{J})}$ in **A** which satisfies all the following:

(1) For any **J**-morphism $m : j_1 \to j_2$, there is equation $f_{j2} = \mathrm{D}m \circ f_{j1}$; shortly speaking, S commutes with diagram D; and

(2) For any source $T = (b \xrightarrow{h_j} \mathrm{D}j)_{j \in \mathrm{Obj}(\mathbf{J})}$ in **A** which commutes with diagram D, there is a unique A-morphism $k : a \to b$ such that $h_j \circ k = f_j$ for all **J**-objectes j. Here k is usually called the connecting arrow of S and T.

Note In [CWM] Mac Lane use the term "cone" for both sources and sinks, while we adopt the terminology used in [ACC]. We sometimes use the terms "limit" or "limiting source" instead of "limiting cone".

Definition 1.11 A **Colimiting cone** of diagram D : **J**→**A** is a limiting cone of diagram D^{op} : $\mathbf{J}^{\mathrm{op}} \to \mathbf{A}^{\mathrm{op}}$, which is a sink in **A**. "Limiting cone" and "colimiting cone" are dual concepts.

Preface

The idea of definitions of topological categories and algebraic categories arise from a basic viewpoint. That is, to regard an underlying functor from **X** to **A** as a structure over category **A**, which cannot be identified through **A** or **X** alone. Many structures can be decomposed into more basic ones, which often can be classified as “topological” or “algebraic”. The nature of a structure is reflected not so much in properties of its abstract category, but rather in properties of its underlying functor.

Possibly the only topological category over **Set** in everybody’s mind is **Top**. **hTop** (topological spaces with arrows the homotopy classes of continuous maps), **pTop** (pointed topological spaces), **Haus**, or **HComp** are not topological over **Set**, theoretically. However, there are numerous familiar categories that are algebraic over **Set**, such as **Grp**, **Ab**, **Rng**, **R-Mod**, **Mon** (monoids), **SGrp** (semigroups), and all other algebraic systems <*Ω*, *E*>**-Alg** [CWM, page 124]. Underlying functors of all the above algebraic categories have left adjoints, which is equivalent to say “every algebraic system has a free object for an arbitrary set”, a non-trivial proposition [CWM, page 124]. Moreover, such underlying functors not only preserve limits, but create limits [CWM, page 112]. Therefore, we want to know the essence of similarity of these categories. Universal algebra is not a satisfying answer for two reasons. Firstly, the structure of algebraic system is not categorical; secondly, there are other categories not at all “algebraic” intuitively but have the same properties as algebraic categories, such as **HComp**. Recall that **HComp** has a left adjoint, *i.e.*, Stone-Cech compactification of discrete spaces, and its underlying functor to **Set** creates limits; also see [CWM, Chapter VI, Section 9].

Although we may attempt to define “algebraic category” via monads [ACC, §20], such approach is also far from satisfactory because it still rely on the “structure” of the base category and because it is rather complicated. We cannot determine whether a functor is algebraic by means of monads, just as we cannot conclude two given topological spaces are not homeomorphic from failure to find homeomorphisms.

An intuitive observation is “topology is soft, algebra is hard”. “Algebra is hard” probably because, once an algebraic structure is decided on a set, it will not allow any small-range change, since all the elements impose strong bondage upon each other, such as the binary operation in a group. “Topology is soft” because every element is removable in a topological space without being noticed, and with a given topological space we can modify the topology into an arbitrarily coarser or thinner one.

The ugly but practical definitions thus come.

Topological Categories

The evident model for topological categories over **Set** is **Top**, topological spaces with continuous maps.

Our first observations focus on the fibres. On a given set *a*, there are many topologies. These topologies are not mutually irrelevant, but they constitute a complete lattice, where "be thinner than" works as the "≤" relation. That is, every fibre of **Top** over **Set** is a complete lattice. Here a complete lattice is a partially ordered class with a least upper bound and a greatest lower bound for every subclass. Concretely speaking: let Ω be the class composed of all topologies on *a* and let Σ be an arbitrary subclass of Ω, then there is a topology T_1 in Ω with the following property: (1) T_1 is thinner than all topologies in Σ; and (2) among all such topologies in Ω that are thinner than all topologies in Σ, T_1 is the thinnest one. Also, there exists a topology T_2 in Ω with following property: (1) T_2 is coarser than all topologies in Σ; and (2) among all such topologies in Ω that are coarser than all topologies in Σ, T_1 is the coarsest one.

Definition 2.1 A concrete category is called **fibre-complete** iff its fibres are complete lattices. Here complete lattices are allowed to be large.

Proposition 2.2 **Top** over **Set** is fibre-complete.

Our second observation goes beyond topologies on a single set, and considers topologies given by functions. Given set *a*, topological space *b*, and set function $f : a \to b$, the initial topology on *a* is defined to be the coarsest topology that makes *f* continuous. The concept of final topology is dual to initial topology; given $h : b \to a$, the final topology on *a* is defined to be the thinnest topology that makes *f* continuous. When more than one functions are considered, such as an *I*-indexed class of functions $f_i : a \to b_i$ with *a* a set and b_i's topological spaces, then initial topology on *a* can still be found, *i.e.* the coarsest topology that makes all f_i's continuous. The existence of initial and final topologies serves as another important feature of topological categories which leads to nice properties. Within our expectation, the notions of initiality and finality are categorical.

Definition 2.3 Let (**A**, U) be concrete over **X**. An **A**-morphism $f : a \to b$ is called **initial** iff: for any **A**-object *c* and **X**-morphism $h : |c| \to |a|$, that $f \circ h : |c| \to |b|$ is an **A**-morphism implies that *h* is an **A**-morphism. An **A**-morphism $g : a \to b$ is called **final** iff: for any **A**-object *c* and **X**-morphism $h : |b| \to |c|$, that $h \circ g : |a| \to |c|$ is an **A**-morphism implies that *h* is an **A**-morphism.

Definition 2.4 Let (**A**, U) be concrete over **X**. A source $S = (a \xrightarrow{f_i} a_i)_{i \in I}$ is called **initial** iff: for any **X**-morphism $h : |b| \to |a|$ with $(f_i \circ h)_I$ all **A**-morphisms, *h* must be an **A**-morphism. **Final sink** is the dual notion for initial source.

It seems that we copy the definitions word for word, from section on initial and final topologies of some topology textbook, if we view **A** as **Top** and **X** as **Set** [GT, definition 4.15]. Not only **Top** but also lots of other familiar categories satisfy these requirements (fibre-completeness, existence of initial and final structures). For instance, **PMet** (pseudo-metric spaces with non-expanding maps [ACC, page 132]) over **Set**; **Rel** (relations with relation-preserving maps) over **Set**; **Prost** (pre-ordered sets with order-preserving maps) over **Set**; **TopGrp** (topological groups with continuous homomorphisms) over **Grp**, and other topological structures over other categories.

Virtually fibre-completeness and existence of initial and final arrows suffice to claim whether a functor is topological, but general textbooks on categories do not use them as definition; instead, Herrlich expressed the definition in a quite succinct though very abstract way [ACC, Definition 21.1]. I now exhibit the definition before explain some of its terms. The definitions of underlined words will follow immediately.

Definition 2.5 A functor G : **A**→**B** is called **topological** iff every <u>G-structured source</u> $(f_i : b \to \mathrm{G}a_i)_I$ has a unique <u>G-initial</u> lift $(f'_i : a \to a_i)_I$. (Here "lift" means $\mathrm{G}f'_i = f_i$ for all i.)

Definition 2.6 Let G : **A**→**B** be a functor. A **G-structured source** is a pair $(a, (f_i, b_i)_I)$, with a an **A**-object and $(f_i, b_i)_I$ a family of pairs indexed by some class I, where for every i, b_i is a **B**-object and $a \xrightarrow{f_i} \mathrm{G}b_i$ is an **A**-morphism. Such G-structured source is usually denoted by $(a \xrightarrow{f_i} \mathrm{G}b_i)_{i \in I}$. If G is an underlying functor, we usually use the term **structured source** instead of G-structured source.
G-structured sink and **structured sink** are their dual notions.
When the index family I is a singleton set, we usually call the G-structured source a **G-structured morphism** (or **G-structured arrow**). Its dual notion is **G-costructured morphism**.

Note The G-structured source $(f_i : a \to \mathrm{G}b_i)_I$ is not a source, since it contains information not only of $\mathrm{G}b_i$'s but also of b_i's; however, their definitions are similar.

Definition 2.7 Let G : **A**→**B** be a functor. A source $S = (f_i : a \to a_i)_I$ is called **G-initial** iff: for each source $(g_i : b \to a_i)_I$ in **A** with the same codomain of S and **B**-morphism $h : \mathrm{G}b \to \mathrm{G}a$ which satisfy $\mathrm{G}f_i \circ h = \mathrm{G}g_i$ for all i, there is a unique **A**-morphism $h' : b \to a$ satisfying $\mathrm{G}h' = h$ and $f_i \circ h' = g_i$ for all i.
G-final sink is the dual notion.

Remark 2.8 If G is an underlying functor, then the definition of G-initial source coincides with initial source.

Remark 2.9 In the definition of topological functors, we do not require faithfulness; the reason is not that absence of such requirement may lead to greater generality, but that faithfulness can be deduced from the present definition.

I now claim that a functor G is topological iff it is faithful, fibre-complete, and has initial and final structures for any base object. The "if" part is much easier than the "only if" part, so we try to prove the “if” part first.

Proposition 2.10 (Characterization of topological functors) G : **A**→**B** is a functor. Then G is topological iff it satisfies all the following:

(1) G is faithful; and

(2) G is fibre-complete as a concrete functor; and

(3) for any **B**–morphism $f : b \to |a|$ (a is an **A**–object), it has a (necessarily unique) initial lift, *i.e.*, an initial **A**-morphism $f' : a_b \to a$ with $\mathrm{G}f' = f$; and

(4) for any **B**–morphism $g : |a| \to b$ (a is an **A**–object), it has a (necessarily unique) final lift, *i.e.*, a final **A**-morphism $g' : a \to a_b$ with $\mathrm{G}g' = g$.

Proof of the “if” part of Proposition 2.10

Idea We need to find initial lift for every G-structured sources $(f_i : b \to |a_i|)_I$. We are directed to find from the fibre of b the greatest one that makes every f_i an **A**-morphism. If $|a| = b$, then $f_i : |a| \to |a_i|$ is an **A**-morphism iff $|a| \leq$ ‘the initial lift of $f_i : b \to |a_i|$’. Therefore, it suffices to find the greatest one that is less than ‘the initial lift of $f_i : b \to |a_i|$’ for all i in I. Fortunately, there is one such **A**-object because the fibre of b is a complete lattice.

Proof For every i, let $f_i : c_i \to a_i$ be the initial lift of $f_i : b \to |a_i|$. Then let c_b be the greatest lower bound of $\{c_i\}_I$. For any i, since $c_b \leq c_i$ in the fibre of b, there exists $h_i : c_b \to c_i$ with $\mathrm{G}h_i = id_b$. Thus the **A**-arrow $h_i \circ f_i : c_b \to a_i$ is a lift of $f_i : b \to |a_i|$. To show that $f_i : c_b \to a_i$ is initial, suppose that $(c \xrightarrow{f_i} a_i)_{i \in I}$ is a lift of $(f_i : b \to |a_i|)_I$, then $c \leq c_i$ for any i, hence $c \leq c_b$. That is, $id_b : c \to c_b$ is an A-morphism, so $(c_b \xrightarrow{f_i} a_i)_{i \in I}$ is an initial lift of . For the fibre is a partially ordered class, initial lift is necessarily unique.

Note Property (4) is not used in the above proof, so the first three properties combined are sufficient to detect topological functors.

The characterization of **Proposition 2.10** shows excellent properties for topological functors, e.g. the existence of initial lifts for sources implies the existence of discrete structures on objects, and hence implies the existence of a left adjoint; and dually, the existence of final lifts for sinks implies the existence of indiscrete structures on objects, and hence implies the existence of a right adjoint. One result of such properties is that, as is well-known, U : **Top**→**Set** preserves limits and colimits.

Although an intuitive characterization helps us understanding how the definition comes, there are two points yet to be explained in the proposition. First, it remains unknown why topological functors are faithful. Second, the characterization is self-dual, since faithfulness is self-dual, fibre-completeness is self-dual, and initiality and finality are dual, but **Definition 2.5** is definitely not.

Proposition 2.11 Topological functors are faithful.

Proof Let G : **A**→**B** be a topological functor. Assume that $r, s : a \to a'$ are a parallel pair of **A**-morphisms, with $\mathrm{G}r = \mathrm{G}s$. Let $I = Mor(\mathbf{A})$ be the class composed of all morphisms in **A**, and let the source S be $(f_i : \mathrm{G}a \to \mathrm{G}a_i)_I$ with $f_i = \mathrm{G}r$ and $a_i = \mathrm{G}a'$ for all i. (Notice that i runs all over morphisms in **A**.) Then S has a G-initial lift, say $(g_i : a_o \to a_i)_I$. Define source $T = (h_i : a \to a_i)_I$ to be:

$$h_i = \begin{cases} r, & \text{if } g_i \text{ and } i \text{ are composable and } g_i \circ i = s \\ s, & \text{otherwise} \end{cases}$$

Obviously $Gh_i = Gg_i \circ id_{Ga}$ for all i. Since $(g_i : a_o \to a_i)_I$ is G-initial, there is an **A**-morphism k: $a \to a_o$ such that $Gk = id_{Ga}$ and $h_i = g_i \circ k$ for all i. Since k is also an **A**-morphism, we know $h_k = g_k \circ k$, so g_k and k are composable. If $g_k \circ k \neq s$, then by definition, $h_k = s$, contradiction. So, the only possibility is $g_k \circ k = s$, then $h_k = g_k \circ k = r$. So $r = s$.

Note This proof may be associated with Gödel's proof of the incompleteness theorem and Cantor's proof of the uncountability of real numbers, for they all use the "diagonal method".

Remark Much of the strength of being topological lies in the fact that the G-structured sources are allowed to be large, as in the preceding proof, and such fact will be used frequently later on [ACC, remark 21.4]. From now on we prefer the notation "U : **A**→**X**" to "G : **A**→**B**", because topological functors are faithful.

Definition 2.12 A concrete category (**A**, U) over **X** is called **topological** iff U is topological.

Proof of "topological functors are fibre-complete"

For any fibre $U^{-1}(x)$ of topological functor U : **A**→**X**, and any subclass C of $U^{-1}(x)$, we need to find a greatest lower bound for C. For the structured-source in **X**, $(x \xrightarrow{id_x} |a_i|)_{i \in I}$ where I is defined to be C and $a_i = i$ for all I, there is a unique initial lift, say $(f_i : a \to a_i)_I$ in **A**. Then by the definition of initiality, a is the greatest lower bound for C. By uniqueness of the initial lift, equivalent elements must be equal, so $U^{-1}(x)$ is not only a preordered class but also a partially-ordered class.

Proposition 2.11 If (**A**, U) is topological over **X**, then ($\mathbf{A}^{op}$, U^{op}) is topological over $\mathbf{X}^{op}$.

Idea The claim is equivalent to "For a faithful functor U : **A**→**X** for which every structured source $(f_i : x \to |a_i|)_I$ in **X** has a unique initial lift $(f'_i : a \to a_i)_I$, then every structured sink $(f_i : |a_i| \to x)_I$ in **X** has a unique final lift $(f'_i : a_i \to a)_I$ ". If **X** is the category **1** (the category with only one object with its identity the only morphism), then the proposition will be like "For a partially-ordered class in which every subclass has a greatest lower bound, then every subclass has a least upper bound", or shortly, a meet-complete partially-ordered class must be join-complete. Here "meet" stands for "great lower bound", and "join" means "least upper bound". Recall that the proof of this theorem relies on the fact, that for a subclass C, the meet of subclass $C' =$ {elements greater than all elements of C} is the join of C. Since "category" is a generalization of pre-ordered class, and "topological category" is a generalization of complete lattice, the following proof is merely a generalization of the preceding proof.

Proof We need to find a final lift for every structured sink $(f_i : |a_i| \to x)_I$ in **X**. For such purpose, construct structured source $S = (x \xrightarrow{g_j} c_j)_{j \in J}$ consisting of all possible $g_j : x \to |c_j|$ such that the **X**-morphism $g_j \circ f_i : |a_i| \to |c_j|$ is an **A**-morphism for all i. The U-structured source S has an initial lift, say $(\gamma_j : c \to c_j)_J$. By initiality each $f_i : |a_i| \to x$ in **X** can be lifted into $\varphi_i : a_i \to c$ in **A**. The resulting sink $(\varphi_i : a_i \to c)_I$ is the final lift structured sink $(f_i : |a_i| \to x)_I$, which can be easily verified from the definition of finality. The uniqueness is straightforward from fibre-completeness.

Remark The above Topological Duality Theorem implies a duality principle for topological categories. However, $\mathbf{A}^{op}$ is not concrete over **X** but $\mathbf{X}^{op}$, so this does not imply a duality principle over a fixed category. For example, there is no similarity between **Set** and its dual, $\mathbf{Set}^{op}$.

Proof of the "only if" part of Proposition 2.10

Properties (1) (2) (3) has been proved already, and (4) is deduced from (3) and proposition **2.11**.

Note Since in **2.11** (1), (2), and (3) combined are sufficient to detect topological functors, we now know that (1), (2), and (4) combined are also sufficient.

Up to present we have constructed the equivalence of the succinct definition and the intuitive one. From now on we can use either definition freely. The following step will be looking for pleasant properties of topological functors.

Proposition 2.12 Topological functor has a left adjoint and a right adjoint. Its left adjoint and right adjoints are both full embeddings of categories.

Idea As for U : **Top**→**Set**, its left adjoint is the discrete functor which gives a set its discrete topology, while its right adjoint is the indiscrete functor which gives a set its indiscrete topology. Therefore, we seek to find the least and the greatest elements on fibres so as to construct adjoints. The formal definition of "discrete" is: if (**A**, U) is concrete over **X**, then the **A**-object *a* is called **discrete** iff every possible **X**-morphism $|a|\to|b|$ is an **A**-morphism. Dual definition works for "**indiscrete**".

Proof By duality principle, we only prove the existence of left adjoints.

Let (**A**, U) be topological over **X**. we need to find for every **X**-object *x* a universal arrow from *x* to U, see [CWM, page 55]. Let a_x be the least one in the fibre $U^{-1}(x)$, or equivalently, let $(h_i : a_x\to a_i)_I$ be a initial lift of the U-structured source $(f_i : x\to|a_i|)_I$ consisting of all possible U-structured morphisms $f_i : x\to|a_i|$. We claim $id_x : x\to|a_x|$ is universal from *x* to U. The claim is easily justified from initiality of $(h_i : a_x\to a_i)_I$ and faithfulness of U. From the existence of universal arrow to U from every **X**-object, U has a left adjoint [CWM, page 83, Theorem 2 (ii)].

If we denote the left adjoint by F, then $id_x : x\to \mathrm{UF}x$ is the universal arrow, so $\mathrm{U}\circ\mathrm{F} = \mathrm{id}_{\mathbf{X}}$, and F is injective on objects.

We then have $\mathrm{Hom}_{\mathbf{A}}(\mathrm{F}x_1, \mathrm{F}x_2) \cong \mathrm{Hom}_{\mathbf{X}}(x_1, \mathrm{UF}x_2) = \mathrm{Hom}_{\mathbf{X}}(x_1, x_2)$. Since the universal arrows are identities, the equation above coincides with the hom-set function defined by F, so F is bijective on hom-sets. Therefore, F is an full embedding of categories.

Corollary 2.13 Topological functors preserve limits and colimits.

Proof Since topological functors have left and right adjoints, they preserve limits and colimits. [CWM, page 118]

If U : **A**→**X** is topological, the preceding corollary does not suffice to help us find limits in **A**. Given a diagram D : **J**→**A** with limiting cone *S*, we only know that U*S* is the limiting cone of U∘D, but given U*S* we do not know how to determine what *S* is.

Recall that when learning product spaces in topology, infinite products cause problems in that we should not define open sets naturally as "arbitrary union of open boxes in the sense that they have open projection on every coordinates", but like "arbitrary union of boxes which have open projection on finite coordinates and full-space projection on other dimensions". Both definitions use the Cartesian product as the underlying set. However, it turns out that the latter definition is categorical product in **Top**. Previous knowledge does not tell us why it is the case, so we now point out that the topology on the product space should be the initial topology for the structured source of projections.

Proposition 2.14 Functor U : **A**→**X** is topological, then limiting cones in **A** are U-initial sources.
Dual: Colimiting cones in topological categories are final sinks.
Proof Suppose **A**-source $S = (f_i : a \to a_i)_I$ is a limiting cone for diagram D : **J**→**A**, where I = Obj(**J**), a_i = Di for all i. To show that S is initial, we need to show that for any **X**-morphism h : $|b| \to |a|$ with $(f_i \circ h)_I$ all **A**-morphisms, h must be an **A**-morphism. This is obvious if we just find the unique connecting **A**-arrow $k : b \to a$ such that the **A**-arrow $(f_i \circ h) = f_i \circ k$ for all i, and Uk = h by faithfulness.

Proposition 2.15 Functor U : **A**→**X** is topological. Functor D : **J**→**A** is a diagram in **A**. Diagram U∘D has a limiting cone $S = (g_i : x \to |a_i|)_I$ in **X**. Here I = Obj(**J**), a_i = UDi for all i. Then U has a limiting cone which is a lift of S, and it is the only lift of S among all the limiting cones of diagram U. Shortly speaking, U lifts limits uniquely.
Dual: Topological functors lift colimits uniquely.
Proof A lift of S which is also a limiting cone must be the initial lift of S, so the uniqueness is obvious. Let $T = (f_i : a \to a_i)_I$ be the initial lift of S; to show that T is a limiting cone of U, we need to find connecting morphism $b \to a$ for every source $R = (\varphi_i : b \to a_i)_I$ which commutes with arrows in diagram D. If we check UT and UR in **X**, we know that there is no more than one choice, namely, the connecting morphism for UT and UR in **X**, say $h : |b| \to |a|$. By initiality, h is an A-morphism, so the A-arrow h is the unique connecting morphism for T and R. Therefore, U lifts limits uniquely.

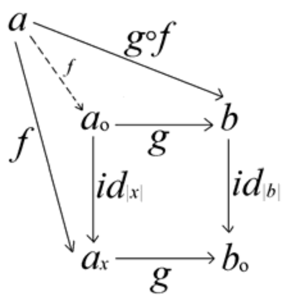
Corollary 2.16 U : **A**→**X** is topological, then **A** is (co)complete iff **X** is (co)complete.

The lifting of limits is not an amazing property for forgetful functor, for many familiar forgetful functors lift limits uniquely, like **Grp**, **Ab**, **Vec**… over **Set**, and such "algebraic" functors [CWM, page 112]. But the algebraic functors does not preserve colimits, for example, the free products of groups have quite complicated underlying sets, and the direct sum of vector spaces is not the disjoint union of spaces. A left adjoint implies preservation of limits, and dually. While a left adjoint of underlying functor is often called a "free" functor, right adjoints (so-called "co-free functors" or "indiscrete functors") for underlying functors are not as ubiquitous. Therefore owning a right adjoint is the special property of topological functors.

Proposition 2.17 A concrete category (**A**, U) over **X** is topological iff all the followings hold:
(1) U lifts limits uniquely; and
(2) (**A**, U) has indiscrete structures, *i.e.*, every **X**-object has an indiscrete lift.
Proof The "only if" part is proved in **2.12** and **2.15**. To prove the "if" part, we need to show, by **2.10**, (i) U is fibre-complete and (ii) every costructured arrow $f : |a| \to x$ in **X** has an final lift.
(i) For an arbitrary subclass C of fibre $U^{-1}(x)$, we will find its unique greatest lower bound. If C

is empty, then the indiscrete lift of x is the greatest in the fibre $U^{-1}(x)$, hence also the greatest lower bound of C. If C is non-empty, define category **D** to be the subcategory of fibre $U^{-1}(x)$ (as preordered class) consisting of all such objects that are greater than at least one object in C. **D** is not empty because the indiscrete lift of x is in **D**. Let F denote the embedding of categories **D**→**A**. Now the image of U∘F is rather trivial, namely, it has the same image x for objects and the images of morphisms are all id_x. The diagram is connected and non-empty, so U∘F has the limit source $L = (id_x : x \to x)_{\mathrm{Obj}(\mathbf{D})}$. Since U lifts limits uniquely, L has a unique lift which is a limit of F. The lift of limit is the unique greatest lower bound of C.

(ii) For every costructured arrow $f : |a| \to x$ in **X**, we view f as a diagram in **X**. The limit of f is obviously $|a|$. By the lift of limits, $f : a \to x_o$ (x_o is the indiscrete lift of x) has a limit a_x with $|a_x| = x$. We guess that $f : a \to a_x$ in **A** is the final lift of $f : |a| \to x$ in **X**. For an **X**-arrow $g : |a_x| \to |b|$ which satisfies $g \circ f : |a| \to |b|$ is an **A**-arrow, we need to show g is an **A**-arrow. Let b_o be the indiscrete lift of $|b|$, then the pullback of $|a_x| \xrightarrow{g} |b_o| \xleftarrow{id_{|b|}} |b|$ is $|a_x| \xleftarrow{id_{|x|}} x \xrightarrow{g} |b|$. Since U lifts limits uniquely, there is an a_o in $U^{-1}(x)$ such that the pullback of $a_x \xrightarrow{g} b_o \xleftarrow{id_{|b|}} b$ is $a_x \xleftarrow{id_{|x|}} a_o \xrightarrow{g} b$. As in the diagram below, the outer quadrangle commutes, and the only possible connecting arrow $a \to a_o$ is f, so $a_o \le a_x$. But since $a \xrightarrow{f} a_x$ is initial, $a_x \le a_o$. Therefore, $a_x = a_o$, and $g : a_x \to b$ is an **A**-arrow.



Remark Unique lifting of limits is a widespread property shared by many reasonable forgetful functors not only in topology, but also in algebra. Hence the above theorem shows that the existence of indiscrete structures is the crucial condition that makes (**A**,U) topological. [ACC, remark 21.19]

When studying monomorphisms and epimorphisms, we take injective maps and surjective maps as their models. However, non-surjective epimorphisms may exist in concrete categories over **Set**, such as the inclusion **Z**→**Q** in **Rng**. However, the notions of injection and monomorphism coincide in most categories familiar to us, while in topological concrete categories over **Set**, it is also true that epimorphisms and surjections coincide. The essence of such properties lies in the faithfulness and preservation of limits and colimits.

Proposition 2.18 Faithful functors reflect monomorphisms, *i.e.*, if F : **A**→**B** is a functor and $a \xrightarrow{f} b$ is an **A**-morphism, Ff is monic implies that f is monic.

Proof Let $h_1, h_2 : c \to a$ be a parallel pair with $f \circ h_1 = f \circ h_2$. $\mathrm{F}f \circ \mathrm{F}h_1 = \mathrm{F}f \circ \mathrm{F}h_2$, so $\mathrm{F}h_1 = \mathrm{F}h_2$. Then $h_1 = h_2$ by faithfulness.

Remark Since faithfulness is self-dual, faithful functors also reflect epimorphisms.

Proposition 2.19 Functors that preserves pullback (limit of diagram $\{\cdot \to \cdot \leftarrow \cdot\}$) squares preserves monomorphisms.

Proof Suppose $\mathrm{F} : \mathbf{A} \to \mathbf{B}$ is a functor that preserves pullback squares, and $a \xrightarrow{f} b$ is an **A**-morphism. As easily seen, f is monic iff the left square below is a pullback square.

$$\begin{array}{ccc} a & \xrightarrow{id_a} & a \\ {\scriptstyle id_a}\downarrow & & \downarrow{\scriptstyle f} \\ a & \xrightarrow[f]{} & b \end{array} \qquad\qquad \begin{array}{ccc} \mathrm{F}a & \xrightarrow{id_{\mathrm{F}a}} & \mathrm{F}a \\ {\scriptstyle id_{\mathrm{F}a}}\downarrow & & \downarrow{\scriptstyle \mathrm{F}f} \\ \mathrm{F}a & \xrightarrow[\mathrm{F}f]{} & \mathrm{F}b \end{array}$$

Then, the right square above is a pullback square, so $\mathrm{F}f$ is monic.

Remark Since pullback is a kind of limit, functors preserving limits must preserve pullback, hence preserves monomorphisms. The proposition follows:

Proposition 2.20 $\mathrm{U} : \mathbf{A} \to \mathbf{X}$ is a topological functor, then for every **A**-morphism f,

(1) f is monic iff $\mathrm{F}f$ is monic; and dually

(2) f is epi iff $\mathrm{F}f$ is epi.

Notes

Just as category is a generalization of monoid, and groupoid is a generalization of group, I think topological concrete category is a generalization of complete lattice. This is one of the reasons why topological structures are strongly connected with orders.

As in the characterization of topological functors, fibre-completeness is a very strong property which rules out lots of concrete categories seemingly "topological". **Met** (metric spaces with continuous maps) is not topological because its fibre is not a partially ordered set, *i.e.*, there are different structures on one set that are equivalent. If we require the morphisms to be contractions, there will not be equivalent but different metric structures on one set. But then another question emerges, that is, given a set there is no "free metric space" over it, since no number is greater than all numbers. **Haus** is not topological since it has no indiscrete structures, unless for a singleton or empty set. **HComp** is not topological for the same reason. **hTop** is not even a concrete category over **Set** in any natural way. However, **Rel**, **Prost**, **Unif**, and so on, are topological constructs [ACC, page 475], which have no "additional" requirements, such as Hausdorff, compactness, or connectedness, other than basic ones.

On the other hand, the underlying functors of "algebraic" categories do not satisfy the definition of topological functor, because no "algebraic" category is fibre-complete, namely, the identity map on a set cannot be a homomorphism between different "algebraic" structures. However, all identity functors are trivially topological.

There are also topological functors with codomains not **Set**, such as U : **TopGrp→Grp**. We know from our deduction that the limits and colimits in **TopGrp** can be constructed just as in **Grp** with initial (for limits) or final (for colimits) topologies.

Roughly speaking, the essence of **topologicality** lies in existence of indiscrete (or "co-free") objects, *i.e.*, existence of a right adjoint.

My work so far is my ideas of topological concrete functors. The ideas for definitions are mainly from [ACC, Chapter VI], while the observations, proofs and remarks are by myself.

Algebraic Categories

The evident models for algebraic categories over **Set** are **Grp**, **Ab**, **R-Mod**, **Vec**, **Rng**, **Mon**, **SGrp**, and other equational classes of algebras [CWM, page 124]. For detailed definition, refer to [ACUA, Definition 11.7]

From now on denote every equational class of algebras as $<\Omega, E>$-**Alg**, where Ω is the set of operators and E is the set of identities, and call its objects $<\Omega, E>$-algebras. As mentioned in [CWM, page 124], every equational class of algebras of a given type $\tau = (\Omega, E)$ has a left adjoint, so its underlying functor preserves limits. Moreover, the underlying functor of $<\Omega, E>$-**Alg** creates limits.

Definition 3.1 Let F : **A**→**B** be a functor. F is said to **create limits** iff :

For every diagram D : **J**→**A** and every limiting source $L = (f_i : b \to b_i)_I$ of F∘D in **B** with I = Obj(**J**) and b_i = F∘D(i) for all i in I, (1) there exists a unique source $S = (h_i : a \to a_i)_I$ in **A** with a_i = D(i) for all i, such that each $h_i = \mathrm{F}f_i$; and (2) this S is a limiting cone of D.

Proposition 3.2 The underlying functor to **Set** of $<\Omega, E>$-**Alg** creates limits.

Proof [CWM, page 111, Theorem 2; page 112, Theorem 3]

Proposition 3.3 If F : **A**→**B** creates limits, then F lifts limits uniquely.

Proof Trivial from the definitions.

Definition 3.4 Functor F : **A**→**B** is said to **create isomorphisms** iff, for any F-structured arrow $h : b \to \mathrm{F}a$ which is an isomorphism in **B**,

(1) there is a unique **A**-morphism $f : a_b \to a$ such that Ff = h , and

(2) f is an **A**-isomorphism.

Note "Create isomorphisms" is not self-dually defined, but it really is a self-dual property just as it sounds, and the proof is easy but not trivial.

Proposition 3.5 If F : **A**→**B** creates isomorphisms, then $\mathrm{F}^{op} : \mathbf{A}^{op} \to \mathbf{B}^{op}$ creates isomorphisms.

Proof Suppose $h : \mathrm{F}a \to b$ is an isomorphism in **B**, we are asked to find a unique **A**-morphism $a \xrightarrow{f} a_o$ such that Ff = h and then to show that f is an **A**-isomorphism.

First find the inverse $h^{-1} : b \to \mathrm{F}a$ of h. Since h^{-1} is an **B**-isomorphism, it has a unique lift $k : a_b \to a$ such that $\mathrm{F}k = h^{-1}$. Then we may find the inverse $f : a \to a_b$ of the **A**-isomorphism k. Obviously $\mathrm{F}f = (\mathrm{F}k)^{-1} = (h^{-1})^{-1} = h$ and f is an **A**-isomorphism.

Assume there is another g satisfying Fg = h, then: the isomorphism and F-structured arrow $b \xrightarrow{id_b} \mathrm{F}a_b$ has a unique lift, namely, $a_b \xrightarrow{id_{a_b}} a_b$, but $a_b \xrightarrow{f} a \xrightarrow{g} a_b$ is also a lift of $b \xrightarrow{id_b} \mathrm{F}a_b$, so $g \circ f = id = k \circ f$. We immediately know $g = k$, for the isomorphism f is cancellable. Therefore the lift f of h is unique.

Proposition 3.6 If F : **A**→**B** creates limits, then F creates isomorphisms.
Proof Just regard the isomorphism in **B** as a limiting cone.

Definition 3.7 (**A**, U) is concrete over **X**. U is called **fibre-discrete** iff its fibres are ordered by identities, *i.e.*, all its fibres are disconnected partially ordered classes.

Proposition 3.8 The underlying functor to **Set** of <*Ω*, *E*>-**Alg** is fibre-discrete.
Proof Out of creation of isomorphisms.
Let the underlying functor be U : <*Ω*, *E*>-**Alg**→**Set**. If $a \le b$ in fibre $U^{-1}(x)$, then there is a morphism $a \xrightarrow{f} b$ with Uf = id_x. Obviously f is a lift of the isomorphism $x \xrightarrow{id_x} Ub$. Since the morphism id_b is also a lift of id_x, $f = id_x$ by uniqueness.

Proposition 3.9 The underlying functor to **Set** of <*Ω*, *E*>-**Alg** has a left adjoint; or equivalently, there is a free <*Ω*, *E*>-algebra for any set.
Proof [CWM, page 124-125]

Although operators and identities are typical for "algebraic" categories over **Set**, we cannot define algebraic category through this way, because such method would rule out other categories behaving just as algebraic ones, like **HComp**, and because such method is not categorical. We want to find a definition which does not rely on additional structures on category, but does lead to nice properties as mentioned above.

One problem is, all familiar "algebraic" categories over **Set** have colimits, or equivalently, they have coproducts and coequalizers.
As for coequalizers, we are tempted to guess that in <*Ω*, *E*>-**Alg**, the coequalizer of f and $g : A \to B$ is the quotient of b that identifies $f(a)$ and $g(a)$ for all a in A but only identifies those have to be identified by such requirement, the smallest congruence relationship containing all $f(a) \sim g(a)$. It is plausible that <*Ω*, *E*>-**Alg** has coequalizers constructed in this way. Another construction of the coequalizer of f and $g : a \to b$ is like the following. Let $(b \xrightarrow{h_i} c_i)_I$ be the source consisting of all arrows $b \xrightarrow{h_i} c_i$ that satisfy $h_i \circ f = h_i \circ g$. We cannot represent the whole source by a single <*Ω*, *E*>-**Alg** morphism, but if we allow large <*Ω*, *E*>-algebras (that is, classes with <*Ω*, *E*> structures on them), we can form the product of $(c_i)_I$. Thus the source $(b \xrightarrow{h_i} c_i)_I$ can be expressed by a single large <*Ω*, *E*>-algebra homomorphism $b \xrightarrow{h} \prod_{i \in I} c_i$. Factorize h into a composite of an epimorphism and a monomorphism in the canonical method, *i.e.*, $b \xrightarrow{p} c \xrightarrow{e} \prod_{i \in I} c_i$ where p is the projection to the image and e is the embedding to the codomain. As a quotient of b, c can be indexed by a set, so we can regard c as a real <*Ω*, *E*>-algebra. Then I claim $p : b \to c$ is a coequalizer of f and g, and its proof is trivial.

For coproducts: In **Grp**, there are free products; in **Rng**, there are tensor products, etc.; but the underlying set of the coproduct of a and b seems irrelevant to the underlying sets of a and of b. My observation is, the coproduct of a family M indexed by a set I in <Ω, E>-**Alg** can be formed by two successive operations: first, find a generating set s_i for every <Ω, E>-algebra m in M, and let m_o be the free object over set $\coprod_{i\in I} s_i$; second, find a quotient <Ω, E>-algebra of m_o in which all elements that should collapse do collapse. Just take **Grp** for example, we know that free product of groups are formed very much like construction of free groups. The construction of free product of $(G_i)_I$ is the set of finite sequence of words $(a_1 , \dots , a_n)$ where each a_i belongs to some G_i and any two adjacent words are not in the same G_i. Such requirement for adjacent words is forming the quotient of free group over set $\coprod_{i\in I} G_i$. The step of forming quotient is the same as constructing coequalizers in the last paragraph, that is, get the image-embedding factorization of $m_o \to \prod_{i\in I} m_i$ and the embedding of image represents the very coproduct of $M = (m_i)_I$.

From the above observations, the existence of colimits should be a property of algebraic functors in the absence of right adjoints. The existence of colimits, as in the construction procedure, requires factorizations of homomorphism from a <Ω, E>-algebra m_o to a large <Ω, E>-algebra $\prod_{i\in I} m_i$. We can represent such a homomorphism by a source $(m_o \to m_i)_I$, so the factorization of sources are required.

As in the introduction of topological functors, we first give the formal definition of essentially algebraic functors [ACC, Definition 23.1], and then explain the terms within it. The definition of algebraic functors is a bit more complicated and seems unnatural, so I do not introduce it much.

Definition 3.10 A functor is called **essentially algebraic** provided that it creates isomorphisms and is (Generating, Mono-Source)-factorizable.

Definition 3.11 G : **A**→**B** is a functor. Then an G-structured arrow $f : b \to \mathrm{G}a$ in **B** is called **generating** iff: for any pair of **A**-arrows $h_1, h_2 : a \to c$, $\mathrm{G}h_1 \circ f = \mathrm{G}h_2 \circ f$ implies $h_1 = h_2$.
Remark If G has a left adjoint F with φ an natural isomorphism $\mathrm{Hom}_{\mathbf{A}}(\mathrm{F}-,-) \cong \mathrm{Hom}_{\mathbf{B}}(-,\mathrm{G}-)$, then $f : b \to \mathrm{G}a$ is generating iff $\varphi f : \mathrm{F}b \to a$ is epi. The proof is trivial.

Definition 3.12 A source $(h_i : a \to a_i)_I$ in category **A** is called a **mono-source** iff: for any parallel pair of **A**-arrows $f_1, f_2 : c \to a$ satisfying $h_i \circ f_1 = h_i \circ f_2$ for all i, there must be $f_1 = f_2$.
Remark If $(a_i)_I$ has a product, then $(h_i : a \to a_i)_I$ is a mono-source iff its product map $a \to \prod_{i\in I} a_i$ is a monomorphism. This grabs the essence of mono-sources, even when the product does not exist. In $b \xrightarrow{p} c \xrightarrow{e} \prod_{i\in I} c_i$ mentioned when constructing coequalizers, the embedding e is theoretically illegal, but it can be considered as a mono-source legally.

Definition 3.13 A functor F : **A**→**B** is called **(Generating, Mono-Source)-factorizable** iff: for any F-structured source $R = (h_i : b \to \mathrm{F}a_i)_I$, there exists a generating F-structured arrow $b \xrightarrow{f} \mathrm{F}a$ in **B** and a mono-source $S = (a \xrightarrow{g_i} a_i)_{i \in I}$ in **A** such that $\mathrm{F}g_i \circ f = h_i$ for all i. Such a factorization into f and S is often called a **(Generating, Mono-Source)-factorization** of the F-structured source R.

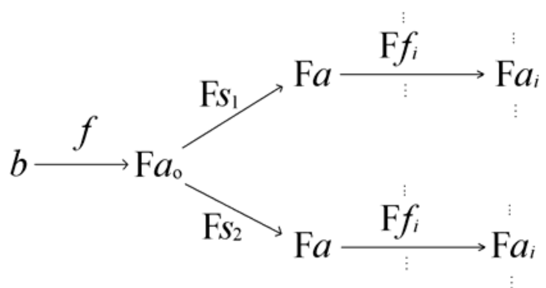
Definition 3.14 A category **A** is called **(Epi, Mono-Source)-factorizable** iff:

for any source $R = (h_i : b \to a_i)_I$ in **A**, there exists a epimorphism $b \xrightarrow{f} a$ and a mono-source $S = (a \xrightarrow{g_i} a_i)_{i \in I}$ such that $g_i \circ f = h_i$ for all i. Such a factorization into f and S is often called an **(Epi, Mono-Source)-factorization** of the source R.

Notice that in the definition faithfulness is again not pre-assumed, and the reason is again that the requirements in the definition imply faithfulness. An essentially algebraic functor sends only isomorphisms and nothing else to isomorphisms, which might suggest faithfulness. Notice also that in the definition, existence of a left adjoint is not included, which is also because it is implied hence not expressed.

Lemma 3.15 Essentially algebraic functors preserve mono-source.

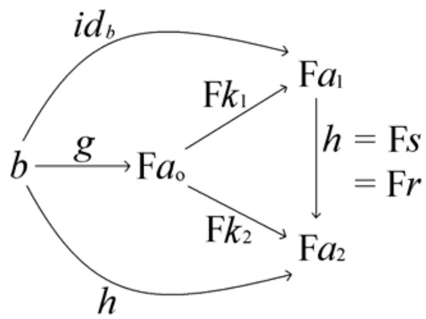
Proof Suppose $\mathrm{F} : \mathbf{A} \to \mathbf{B}$ is essentially algebraic. For any mono-source $S = (a \xrightarrow{f_i} a_i)_{i \in I}$ in **A**, to prove $\mathrm{F}S = (\mathrm{F}a \xrightarrow{\mathrm{F}f_i} \mathrm{F}a_i)_I$ is a mono-soure in **B**, we need to show $r_1 = r_2$ for every parallel pair $r_1, r_2 : b \to \mathrm{F}a$ of **B**-arrows satisfying $\mathrm{F}f_i \circ r_1 = \mathrm{F}f_i \circ r_2$ for all i. Define a source $(b \xrightarrow{r_j} \mathrm{F}a)_{j=1,2}$, and let its (Generating, Mono-Source)-factorization be: a generating F-structured arrow $f : b \to \mathrm{F}a_o$ in **B** and a mono-source $(a_o \xrightarrow{s_j} a)_{j=1,2}$ in **A**, with $\mathrm{F}s_j \circ f = r_j$ ($j = 1, 2$). Since $\mathrm{F}f_i \circ \mathrm{F}s_1 \circ f = \mathrm{F}f_i \circ \mathrm{F}s_2 \circ f$ for all i, $f_i \circ s_1 = f_i \circ s_2$ for all i because f is generating. For $S = (a \xrightarrow{f_i} a_i)_I$ is a mono-source, $s_1 = s_2$. Therefore $r_1 = \mathrm{F}s_1 \circ f = \mathrm{F}s_2 \circ f = r_2$.



Proposition 3.16 Essentially algebraic functors are faithful.

Proof Suppose functor $\mathrm{F} : \mathbf{A} \to \mathbf{B}$ is essentially algebraic. Let r, s be a parallel pair of **A**-arrows $a_1 \to a_2$ satisfying $\mathrm{F}r = \mathrm{F}s = h$ in **B**. Let $b = \mathrm{F}a_1$. We define an F-structured source

$S = (b \xrightarrow{f_j} \mathrm{F}a_j)_{j=1,2}$ where $f_1 = id_b$ and $f_2 = h$. By definition, S has a (Generating, Mono-Source)-factorization, say $b \xrightarrow{g} \mathrm{F}a_0$ is generating and $M = (a_0 \xrightarrow{k_j} a_j)_{j=1,2}$ is a mono-source with $\mathrm{F}k_j \circ g = f_j$ (j = 1, 2). Since $\mathrm{F}s \circ \mathrm{F}k_1 \circ g = \mathrm{F}r \circ \mathrm{F}k_1 \circ g = \mathrm{F}k_2 \circ g$, we know that $s \circ k_1 = r \circ k_1 = k_2$ from that $g : b \to \mathrm{F}a_0$ is generating. We now can deduce that $\mathrm{F}k_1$ is a monomorphism: if $\mathrm{F}k_1 \circ m_1 = \mathrm{F}k_1 \circ m_2$ then $\mathrm{F}k_2 \circ m_1 = \mathrm{F}s \circ \mathrm{F}k_1 \circ m_1 = \mathrm{F}s \circ \mathrm{F}k_1 \circ m_2 = \mathrm{F}k_2 \circ m_2$, so $m_1 = m_2$ from that $\mathrm{F}M = (\mathrm{F}a \xrightarrow{\mathrm{F}k_j} \mathrm{F}a_j)_{j=1,2}$ is a mono-source. Being monic and right-invertible by g, $\mathrm{F}k_1$ is an isomorphism with inverse g. F creates isomorphisms, so k_1 is an isomorphism. Therefore $s \circ k_1 = r \circ k_1$ implies $s = r$.



Definition 3.17 Concrete functor (**A**, U) over **X** is called **essentially algebraic** iff U is essentially algebraic.

Proposition 3.18 Essentially algebraic functors have left adjoints.

Idea In every <*Ω*, *E*>-**Alg**, "free object" over a set b is generated by set b, and among all objects generated by set b, the free object has fewest relations, which means the embedding from b to every object generated by b can be factored through the embedding from b to its free object.

Proof Suppose functor U : **A**→**B** is essentially algebraic. We need to find for every **B**-object b a universal arrow from b to U. [CWM, page 55] [ACC, Definition 18.1]

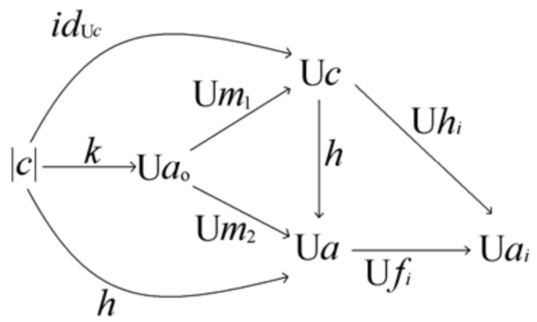
Define F-structured source $S = (b \xrightarrow{f_i} \mathrm{U}a_i)_{i \in I}$ consisting of all possible U-structured arrows $b \xrightarrow{f_i} \mathrm{U}a_i$. Then S has a (Generating, Mono-Source)-factorization, say $b \xrightarrow{f} \mathrm{U}a$ is generating and $(a \xrightarrow{h_i} a_i)_{i \in I}$ is a mono-source with $b \xrightarrow{f} \mathrm{U}a \xrightarrow{\mathrm{U}h_i} \mathrm{U}a_i$ is equal to $b \xrightarrow{f_i} \mathrm{U}a_i$ for all i. I now claim that $b \xrightarrow{f} \mathrm{U}a$ is universal from b to U. For any possible F-structured arrow $b \xrightarrow{f_i} \mathrm{U}a_i$, there is a $a \xrightarrow{h_i} a_i$ such that $f_i = \mathrm{U}h_i \circ f$, and the uniqueness comes from that $b \xrightarrow{f} \mathrm{U}a$ is generating.

Remark Here we did not use the property of mono-source or the creation of isomorphisms, so we know that each F-structured source can be factored through a generating structured arrow is sufficient to give the functor F a left-adjoint.

Corollary 3.19 Essentially algebraic functors preserve limits.

Corollary 3.20 Essentially algebraic functors create limits.
Its proof is after the following lemma.

Lemma 3.21 Let functor U : **A**→**X** be an essentially algebraic functor. Then every mono-source in **A** is initial.



Proof Suppose source $S = (a \xrightarrow{f_i} a_i)_{i \in I}$ is a mono-source in **A**. See the diagram above.

For any **X**-arrow $\mathrm{U}c \xrightarrow{h} \mathrm{U}a$ with Uf_i∘h **A**-arrows for all i, we need to show that h is an **A**-arrow. Let Uf_i∘h = Uh_i. First, find a (Generating, Mono-Source)-factorization for the U-structured source $(|c| \xrightarrow{k_j} \mathrm{U}a_j)_{j=1,2}$ where (k_1, a_1) = ($id_{|c|}$, c) and (k_2, a_2) = (h, a), say generating U-structured arrow $|c| \xrightarrow{k} \mathrm{U}a_o$ in **X** and mono-source $(a_o \xrightarrow{m_j} a_j)_{j=1,2}$ in **A**. For all i, Uh_i∘Um_1∘k = Uf_i∘h∘Um_1∘k = Uf_i∘h = Uf_i∘Um_2∘k. Since k is generating, h_i∘m_1 = f_i∘m_2 in **A**, and Uf_i∘Um_2 = U(h_i∘m_1) = Uf_i∘h∘Um_1 . For $\mathrm{U}S = (\mathrm{U}a \xrightarrow{\mathrm{U}f_i} \mathrm{U}a_i)_{i \in I}$ is a mono-source (recall that essentially algebraic functors preserve mono-source), Um_2 = h∘Um_1 . That $(a_o \xrightarrow{m_j} a_j)_{j=1,2}$ is a mono-source implies $(\mathrm{U}a_o \xrightarrow{\mathrm{U}m_j} \mathrm{U}a_j)_{j=1,2}$ is a mono-source, and hence that Um_1 is monic. Being monic and right-invertible by k, Um_1 is an isomorphism. So m_1 is an isomorphism, and h = Um_2∘ U$(m_1)^{-1}$. So h is an **A**-arrow.

Proof of Corollary 3.20
Suppose functor U : **A**→**X** is essentially algebraic. For any diagram D : **J**→**A**, if U∘D has a limiting source $S = (x \xrightarrow{f_j} \mathrm{UD}j)_{j \in \mathrm{Obj}(j)}$ in **X**, we need to find the unique source $T = (a_x \xrightarrow{h_j} \mathrm{D}j)_{j \in \mathrm{Obj}(j)}$ such that UT = S, and also show that T is a limiting cone in **A**.
First step: We get a (Generating, Mono-Source)-factorization for the U-structured source $S = (x \xrightarrow{f_j} \mathrm{UD}j)_{j \in \mathrm{Obj}(j)}$, say the generating structured arrow $x \xrightarrow{f} \mathrm{U}a$ and the

mono-source $R = (a \xrightarrow{k_j} \mathrm{D}j)_{j \in \mathrm{Obj}(j)}$ in **A** which satisfy $\mathrm{U}k_j \circ f = f_j$ for all j.

Second step: We show that $R = (a \xrightarrow{k_j} \mathrm{D}j)_{j \in \mathrm{Obj}(j)}$ is a limiting cone in **A**. For arbitrary source $(c \xrightarrow{m_j} \mathrm{D}j)_{j \in \mathrm{Obj}(j)}$ in **A** which is a cone over diagram D : **J**→**A** (*i.e.*, the cone commutes with all arrows within the image of diagram D), $(\mathrm{U}c \xrightarrow{\mathrm{U}m_j} \mathrm{UD}j)_{j \in \mathrm{Obj}(j)}$ is a cone over diagram UD : **J**→**X**. Therefore there is a connecting arrow $\mathrm{U}c \xrightarrow{g} x$ such that $f_j \circ g = \mathrm{U}m_j$ for all j. Since R is a mono-source, UR is a mono-source, and hence UR is U-initial. So $f \circ g$ is an **A**-morphism which serves as the connecting arrow from $(c \xrightarrow{m_j} \mathrm{D}j)_{j \in \mathrm{Obj}(j)}$ to R. The uniqueness of such connecting arrow follows immediately because R is a mono-source.

Third step: We can now show that U creates limits. That R is a limiting cone in A and that U preserves limits imply UR is a limiting cone. Therefore $x \xrightarrow{f} \mathrm{U}a$ is an isomorphism as the connecting arrow of two limiting cones. Since U creates isomorphisms, there is an A-isomorphism $a_x \xrightarrow{h} a$ with Uh = f. Then limiting cone $T = (a_x \xrightarrow{h \circ k_j} \mathrm{D}j)_{j \in \mathrm{Obj}(j)}$ is the required lifting of S. Such a lifting is inevitably a limiting cone, and hence equivalent to h, so it is unique since U creates isomorphisms.

Proposition 3.22 The underlying functor U : <*Ω*, *E*>-**Alg**→**Set** is essentially algebraic.

Proof Functor U obviously creates isomorphisms.

For any U-structured source $S = (x \xrightarrow{f_i} \mathrm{U}a_i)_{i \in I}$ in **Set**, we may construct $\prod_{i \in I} a_i$, a large <*Ω*, *E*>-algebra (Cartesian product with natural algebraic structure) and large product $x \xrightarrow{f} \prod a_i$ of $(f_i)_I$. Factor f into a composite of a generating map (in the common algebraic sense) and an embedding, say $x \xrightarrow{g} a \xrightarrow{e} \prod a_i$. Being generated by x, hence a surjective image of the free <*Ω*, *E*>-algebra over x, a can be indexed by a set, so a can be chosen as a set. The embedding e can be represented by an **A**-source $(a \xrightarrow{e_i} a_i)_{i \in I}$ which is a mono-source. Meanwhile g is generating (categorically) from the definition. So the structured source S is (Generating, Mono-source)-factorizable.

Proposition 3.23 Essentially algebraic functors are fibre-discrete.

Proof By the creation of isomorphisms.

From work above, we know that "Essentially algebraic" is not a self-dual property: an essentially algebraic functor preserves monomorphisms, but it does not preserves epimorphisms, for example **Z**→**Q** is epi in **Rng** and but not in **Set**. Note that monomorphisms are mono-sources with the index set a singleton set.

The study so far tells us that the essence of being "algebraic" is the existence of free objects, fibre-discreteness, and the ability to factor a morphism $a \xrightarrow{f} b$ into $a \xrightarrow{e} c \xrightarrow{m} b$ where e is epi and m is monic. In fact, $h : x \to \mathrm{U}a$ is generating iff $\varphi(h) : \mathrm{F}x \to a$ is epi, here F is the left adjoint of U, "free functor" and φ is the natural equivalence $\mathrm{Hom}_{\mathbf{X}}(-,\mathrm{U}-) \to \mathrm{Hom}_{\mathbf{A}}(\mathrm{F}-,-)$.

By now we have not considered the lifting of colimits in essentially algebraic concrete categories. We do not expect essentially algebraic functors to create colimits, to lift colimits, or even to preserve colimits, because there are examples that they do not. The underlying functors of **Grp**, **Ab**, **R-Mod**, **Vec**, … do not preserve coproducts.
However, they have colimits for all diagram D : $\mathbf{J} \to \langle \Omega, E \rangle$-**Alg** with **J** small. Here "**J** is small" means that the class $Mor(\mathbf{J})$ is a set.
Nevertheless, by computing the coproducts and coequalizers in $\langle \Omega, E \rangle$-**Alg**, the colimits in **Set** generate (in the algebraic sense) the colimits in $\langle \Omega, E \rangle$-**Alg**, which gives us the idea of constructing colimits in $\langle \Omega, E \rangle$-**Alg**.

Proposition 3.24 Functor U : **A**→**X** is essentially algebraic, then for any functor D : **J**→**A**, the diagram D has a colimiting cone in **A** if U∘D : **J**→**X** has a colimiting cone in **X**.
Note: Here **J** is not required to be small.

Proof Suppose U∘D : **J**→**X** has a colimiting cone $(\mathrm{UD}j \xrightarrow{f_j} x)_{j \in \mathrm{Obj}(J)}$, then define a source $S = (x \xrightarrow{h_i} \mathrm{U}a_i)_{i \in I}$ consisting of all possible structured arrows $h_i : x \to \mathrm{U}a_i$ satisfying $h_i \circ f_j : |\mathrm{D}j| \to |a_i|$ are **A**-arrows for all j. Then S has a (Generating, Mono-Source)-factorization, say the generating structured arrow $x \xrightarrow{h} \mathrm{U}a$ in **X** and the mono-source $(a \xrightarrow{k_i} a_i)_{i \in I}$ in **A** with $\mathrm{U}k_i \circ h = h_i$ for all i. By the lemma above, every mono-source is initial, so every composition $|\mathrm{D}j| \xrightarrow{f_j} x \xrightarrow{h} |a|$ is an **A**-arrow. We claim that $(\mathrm{D}j \xrightarrow{h \circ f_j} a)_{j \in \mathrm{Obj}(J)}$ is a colimiting cone of diagram D : **J**→**A**.
For an arbitrary sink $R = (\mathrm{D}j \xrightarrow{m_j} b)_{j \in \mathrm{Obj}(J)}$ from diagram D to an **A**-object b, we may factor $\mathrm{U}R = (\mathrm{UD}j \xrightarrow{\mathrm{U}m_j} \mathrm{U}b)_{j \in \mathrm{Obj}(J)}$ through x, say by $(\mathrm{UD}j \xrightarrow{f_j} x)_{j \in \mathrm{Obj}(J)}$ and $x \xrightarrow{m} \mathrm{U}b$. By definition of $S = (x \xrightarrow{h_i} \mathrm{U}a_i)_{i \in I}$, (m, b) is one of the (h_i, a_i), so m can be factored through $x \xrightarrow{h} \mathrm{U}a$. Thus R can be factored through a. The uniqueness of the connecting arrow comes from that $x \xrightarrow{h} \mathrm{U}a$ is generating.
Remark From the proof above, we see that the colimits in "algebraic" categories are constructed as the object generated by the colimits of the underlying diagram, satisfying the some relationships, as I said before.

Corollary 3.25 Functor U : **A**→**X** is essentially algebraic, then

(1) **A** is complete iff **X** is complete,
(2) if **X** is co-complete, then **A** is co-complete.

Corollary 3.26 Every $<\Omega, E>$-**Alg** is complete and co-complete.

As is known to all, monads provide a method to define "algebraic systems", so it is useful to find out whether monadic functors are essentially algebraic. It is not true that every monadic functor is essentially algebraic, for the following reason: It is easy to construct a category **C** which has no epimorphisms or monomorphisms except identities, then in **C** we cannot factor a non-identity arrow into first an epimorphism then a monomorphism, so the identity functor of **C** is not essentially algebraic, while the identity functor is of course monadic. Therefore we have to tune down our expectation.
[CWM, page 137-143] [ACC, Section 20]

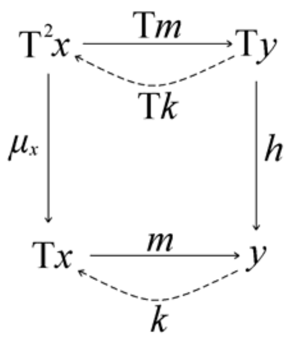
Remark The terms involving monads and monadic functors are from [CWM, Chapter VI].

Proposition 3.27 Monadic functors over **Set** are essentially algebraic.
Proof By definition, a monadic functor U : **A**→**X** can be factored into first an isomorphism $\mathbf{A}\to\mathbf{X}^{\mathrm{T}}$, and then an underlying functor $\mathrm{G}^{\mathrm{T}} : \mathbf{X}^{\mathrm{T}}\to\mathbf{X}$, for some monad $\mathrm{T} = <\mathrm{T}, \eta, \mu>$.
[CWM, page 140, definition; page 142, theorem 1]
We can identify the categories **A** and $\mathbf{X}^{\mathrm{T}}$. Therefore it suffices to prove $\mathrm{U} : \mathbf{Set}^{\mathrm{T}}\to\mathbf{Set}$ is essentially algebraic for any monad T.

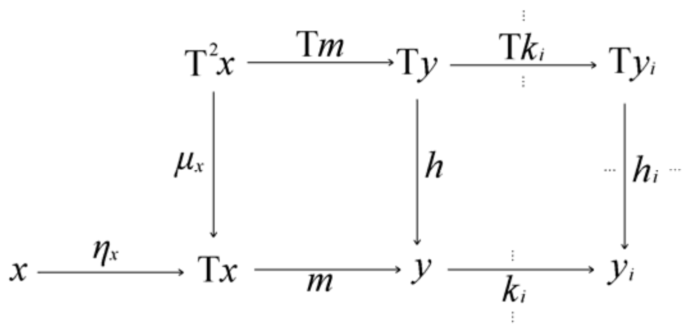
$$\begin{array}{ccc} \mathrm{T}x & \xrightarrow{\mathrm{T}f} & \mathrm{T}y \\ k\downarrow & & \downarrow h \\ x & \xrightarrow[f]{} & y \end{array}$$

The creation of isomorphisms seems obvious, as in the diagram above: given a U-structured arrow $f : x\to\mathrm{U}<y, h>$, or equivalently, a bijective set-function $f : x\to y$ with T-algebra $h : \mathrm{T}y\to y$, there is a unique set-function $k : \mathrm{T}x\to x$ such that $h\circ\mathrm{T}f = f\circ k$, namely, $h\circ\mathrm{T}f\circ f^{-1}$.

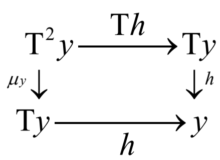
Suppose $S = (x \xrightarrow{f_i} \mathrm{U} < y_i, h_i >)_{i\in I}$ is an arbitrary U-structured source in **Set**. We are asked to find an (Generating, Mono-Source)-factorization for S. We can first factor each f_i into $x \xrightarrow{\eta_x} \mathrm{U} < \mathrm{T}x, \mu_x > \xrightarrow{\mathrm{U}m_i} \mathrm{U} < y_i, h_i >$, since η_x is a universal arrow from x to U. Here m_i stands for both morphisms in **Set** and morphisms in $\mathbf{Set}^{\mathrm{T}}$. Because in **Set** things are quite simple, we can find the unique (Epi, Mono-Source)-factorization of source $(\mathrm{T}x \xrightarrow{m_i} y_i)_{i\in I}$, say surjection $\mathrm{T}x \xrightarrow{m} y$ and mono-source $(y \xrightarrow{k_i} y_i)_{i\in I}$. To define a T-algebra structure $h : \mathrm{T}y\to y$ over y, we will copy the process in defining group structures in quotient groups: Given group G and its normal subgroup N, how to define a group structure on the set of cosets G/N.



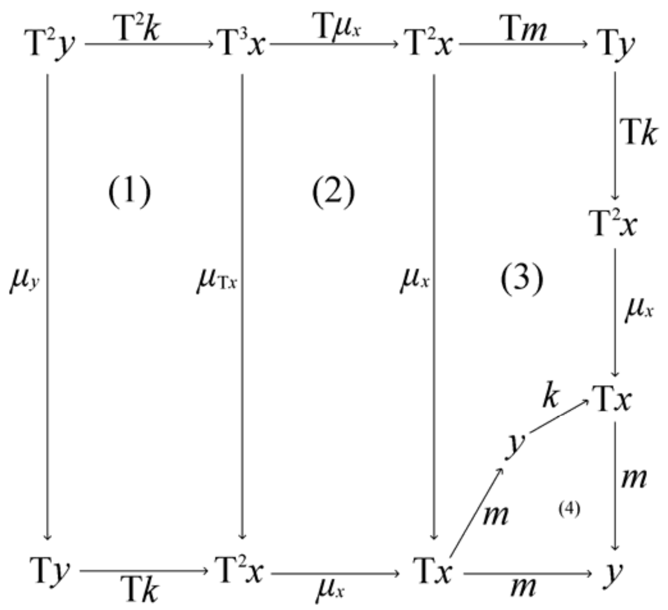
In **Set** every epimorphism is a retraction, *i.e.*, right invertible, unless the domain is empty. If Tx is empty, then x is empty since there is a set function $x \xrightarrow{\eta_x} \mathrm{T}x$, so the source $S = (x \xrightarrow{f_i} \mathrm{U} < y_i, h_i >)_{i \in I}$ can be factorized into first the generating structured arrow $x \to \mathrm{U} < \varnothing, id_\varnothing >$ and then the mono-source $(< \varnothing, id_\varnothing > \longrightarrow < y_i, h_i >)_{i \in I}$. If T$x$ is non-empty, let k be a right inverse of the surjection m. Define h : Ty→y to be the composition $k \circ \eta_x \circ \mathrm{T}k$, as in the diagram above. Then the left square below automatically commutes. Also, the right square in the diagram below commutes for each i, because Tm is right invertible by Tk, hence an epimorphism.



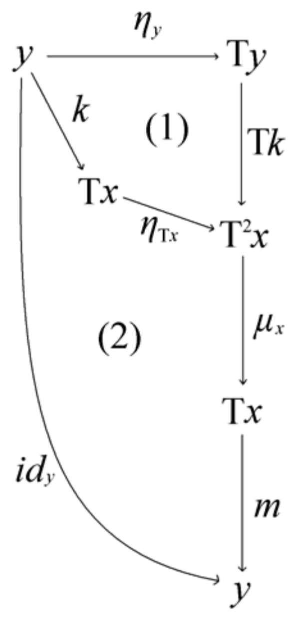
Our final task is to prove h : Ty→y thus defined is really a T-algebra, *i.e.*, it makes the two diagrams [CWM, page 140, diagram 1] commute. First we shall prove the commutativity of the following diagram:



See the diagram below. The commutativity of rectangle (1) and of polygon (3) rely on that μ is a natural transformation, the commutativity of rectangle (2) comes from the axiom of monad [CWM, page 137, diagram 2], and the commutativity of quadrangle (4) comes from $k \circ m = id_y$. The big outer square is equal to the smaller square above.



The commutativity of the other diagram of axiom for T-algebra [CWM, page 140, diagram 1] is proved in the following diagram. The commutativity of quadrangle (1) follows the naturality of η, and the commutativity of polygon (2) comes from that $\mu_x \circ \eta_{Tx} = id_{Tx}$ and $k \circ m = id_y$.



We claim that the structured arrow $x \xrightarrow{\eta_x} Tx \xrightarrow{m} U < y, h >$ together with the source in **Set**^{T} $(< y, h > \xrightarrow{k_i} < y_i, h_i >)_{i \in I}$ provide a (Generating, Mono-Source)-factorization of S, namely, $(x \xrightarrow{f_i} U < y_i, h_i >)_{i \in I}$.

Faithful functors reflect mono-source just as they reflect monomorphisms. The source

$(<y,h> \xrightarrow{k_i} <y_i,h_i>)_{i \in I}$ in **Set**^{T} is a mono-source because its underlying source $(y \xrightarrow{k_i} y_i)_{i \in I}$ in **Set** is a mono-source, and because U is faithful.

It remains to check whether the U-structured arrow $x \xrightarrow{\eta_x} \mathrm{T}x \xrightarrow{m} \mathrm{U}<y,h>$ is generating. Define the functor F : **Set**→**Set**^{T} sending every set a to $<\mathrm{T}a, \eta_a>$ and sending every arrow to itself. By [CWM, page 140, theorem 1], F is the left adjoint of U. Since $\mathrm{F}x \xrightarrow{id_{\mathrm{F}x}} <\mathrm{T}x, \mu_x>$ is an epimorphism, the U-structured arrow $x \xrightarrow{\eta_x} \mathrm{U}<\mathrm{T}x, \mu_x>$ is generating [see Definition 3.11 of this paper, and its following remark]. Faithful functors reflect epimorphisms, so that m is an epi in **Set** implies that the **Set**^{T}-arrow $<\mathrm{T}x, \mu_x> \xrightarrow{m} <y,h>$ is an epimorphism. A simple result is, the fact that the U-structured arrow $x \xrightarrow{\eta_x} \mathrm{UF}x$ is generating combined with that $\mathrm{F}x \xrightarrow{m} <y,h>$ is epi in **Set**^{T} is sufficient to imply that the U-structured arrow $x \xrightarrow{\eta_x} \mathrm{UF}x \xrightarrow{m} \mathrm{U}<y,h>$ is generating.

Remark We see that the proof requires nice properties of **Set**, namely, all epimorphisms are right-invertible except those with empty domain (Zorn's Lemma), and we cannot define such a T-algebra structure on a quotient set without a right-inverse. Recall that in defining quotient group we define the multiplication of cosets by choosing representatives from them.

Corollary 3.28 The underlying functor of **HComp** to **Set** is essentially algebraic.
Proof Underlying functor **HComp**→**Set** is monadic by [CWM, page 157, theorem 1].

The concept of "essentially algebraic" sounds weaker than "algebraic", because with certain additional assumptions, in algebraic categories one can find enough projectives [ACC, proposition 23.28], which is a must for doing homological algebra. I am not going to introduce algebraic functors any more than stating the definition.

Definition 3.29 In a category **C**, an epimorphism e is called an **extremal epimorphism** iff: for any factorization $e = m \circ h$ with m a monomorphism, m must be an isomorphism. [ACC, definition 7.74]
Dual notion: extremal monomorphisms.

Definition 3.30 A functor is called **algebraic** iff it is essentially algebraic and preserves extremal epimorphisms.

Definition 3.31 A concrete category (**A**, U) over **X** is called **algebraic** iff U is algebraic.

Proposition 3.32 Monadic functors over **Set** are algebraic.
Proof We already know that U : **Set**^{T}→**Set** is essentially algebraic, so it remains to show that U

preserves extremal epimorphisms, which is available in [ACC, example 23.20 (1)].

Finally, as a conclusion of my thesis, I will study what kinds of functors are both algebraic and topological.

Proposition 3.33 If functor U : **A**→**X** is both essentially algebraic and topological, then U is an isomorphism between categories **X** and **A**.

Proof Suppose U : **A**→**X** is both essentially algebraic and topological.

(1) For any **X**-object x, the fibre $U^{-1}(x)$ is a complete lattice and a preordered class ordered by equality, hence a singleton set. Therefore U is bijective on objects.

(2) The faithfulness guarantees that U is injective on hom-sets.

(3) The existence of an initial lift of U-structured arrow shows that U is surjective on hom-sets.

All in all, U is an isomorphism.

Remark Though algebraic structures and topological structures are compatible in various situations, like **TopGrp**, **n-Mfd**, and the category of Lie groups, there is no possibilities that a structure is both topological and algebraic.

Also notice that we should not determine whether a concrete category is algebraic or topological by its appearance. **Rel** is topological over **Set**, since a relation on set a is given by a subset of the Cartesian product a^2. Although **Lat** (lattices and lattice homomorphisms) is a subcategory of **Rel**, it is algebraic over **Set**, because a lattice structure on a set a is given by two certain functions $a^2 \to a$ satisfying some properties. **HComp** is algebraic over **Set**, even if it seems topological.

Notes

While the definition of topological functors is well-established, the definition of algebraic functors is controversial. See [CTop1], [CTop2].

I first grabbed the concept of "algebra" in a seminar presentation given by Liang Hao (Zhejiang University), and finds out that the different "algebraic systems" are all defined by operators and identity relations. In learning category theory, the underlying functors of algebraic categories show amazing similarity [CWM, page 111, page 124]. The method through universal algebra provides us with sufficient models of "algebraic categories" [CT, page 236].

In spring 2007 I studies monads by myself and realized various algebraic structures can be defined by means of free objects, e.g., a group structure on set a can be given by a set function to a from the free group generated by a. Under suitable requirements (commutativity of diagrams), these structure suffice to define categories like **Grp**, **Ab**, **R-Mod**, **Vec**, **Mon**, etc.

[ACC, Section 23] and [CT, Section 32] provide yet another way for studying the similarity of algebraic categories, namely, through certain factorization properties of categories. Such method depicts the essence of "algebraic categories", that is, fibre-discreteness and the existence of free-objects and (Epi, Mono-Source)-factorizations. [ACC, 23.8]

Among the three method, I think the last one is the most general one which focus completely on the intrinsic structures inside categories, while the "monad" way works quite nicely in concrete categories over **Set** but it depends whether it goes well on other categories. Universal algebra rules out lots of "algebraically-behaved" categories like **HComp**, and only contain those with given algebraic structures, but even though such a way is not at all categorical, it provides the rough ideas and motivations for algebraic categories and functors, so it should be frequently inspected to study properties of categories.

The ideas are mainly from [ACC, Chapter VI], while the observations, proofs and remarks are by myself.

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