AMCS 602 Fall 2017
Homework Set VII, Due Nov. 29, 2017

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Reading: Read the Lecture 32-40 of Trefethen and Bau: *Numerical Linear Algebra*, which is published by SIAM. Read Chapter 6, in particular section 6.1 to section 6.6 of Demmel: *Applied numerical linear algebra*, which is also published by SIAM.

Page numbers below refer to the book by Trefethen and Bau. The solutions of the following problems should be carefully written up and handed in.

1. Page 249, Problem 32.2. Do part a), b) and c).

2. Page 255, Problem 33.2.(Hint for part e): you can use Cayley-Hamilton theorem as in `https://en.wikipedia.org/wiki/Cayley%E2%80%93Hamilton_theorem`.)

3. Recall that the matrix T_N we obtained from discretizing the 1D Poisson problem with Dirichlet boundary condition

$$T_N = \begin{bmatrix} 2 & -1 & & 0 \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ 0 & & -1 & 2 \end{bmatrix}_{N \times N} . \tag{1}$$

Prove that the eigenvalues of T_N are $\lambda_j = 2(1-\cos(\frac{\pi j}{N+1}))$ and the corresponding eigenvectors are z_j, where the $k-$th component of z_j is

$$z_j(k) = \sqrt{\frac{2}{N+1}} \sin\left(\frac{jk\pi}{N+1}\right).$$

4. Denote by $\rho(A)$ the spectral radius of the matrix A. Prove that

$$\rho(A) = \inf\{\|A\| \,\Big|\, \|\cdot\| \textit{ can be any operator norm}\}.$$

This problem has two parts. First you need to show that $\rho(A) \leq \|A\|$. Then for any fixed $\epsilon > 0$, construct an operator norm $\|\cdot\|_\star$, such that $\|A\|_\star \leq \rho(A) + \epsilon$.

5. Recall in the lecture of the iterative methods. We define $A = D - \tilde{L} - \tilde{U}$, where D is the diagonal of A, while $-\tilde{L}$ and $-\tilde{U}$ are the strictly lower triangular

part of A and the strictly upper triangular part of A respectively. Also we set $DL = \tilde{L}$ and $DU = \tilde{U}$. In the SOR(ω) method, the iteration scheme reads

$$x^{(m+1)} = (I - \omega L)^{-1} \left((1-\omega)I + \omega U \right) x^{(m)} + \omega (I - \omega L)^{-1} D^{-1} b.$$

Define that

$$R_{SOR(\omega)} = (I - \omega L)^{-1} \left((1-\omega)I + \omega U \right).$$

Prove that $\rho(R_{SOR(\omega)}) \geq |\omega - 1|$. This shows that $0 < \omega < 2$ is a necessary condition for the convergence of the SOR(ω) method. Prove also that if A is symmetric positive definite, then $\rho(R_{SOR(\omega)}) < 1$ for all $0 < \omega < 2$, so SOR(ω) converges for all $0 < \omega < 2$.