AMCS 602 Fall 2017
Homework Set I, Due Sept. 13, 2017

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Reading: Read the Lecture I - III of Trefethen and Bau: *Numerical Linear Algebra*, which is published by SIAM. Page numbers below refer to this book. The solutions of the following problems should be carefully written up and handed in. A PDF version using LaTeX will be recommended.

1. Show that an $m \times n$ matrix is rank 1 if and only if $A = uv^\star$, where u is an m-vector and v is an n-vector. Then solve the problem 2.6 on page 16.

2. Let $A = (a_{ij})$ be $m \times m$ matrix. The trace of A is defined as

$$\mathrm{tr}(A) = \sum_{i=1}^{m} a_{ii}.$$

If both A and B are matrices, then show that $\mathrm{tr}(AB) = \mathrm{tr}(BA)$. But in general $\mathrm{tr}(AB) \neq \mathrm{tr}(A)\mathrm{tr}(B)$, give a counter-example.

Suppose that a differential matrix field $A(t)$ solves the following differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t} A(t) = [X(t), A(t)],$$

for a matrix valued function $X(t)$. Recall that the commutator $[X, A] := XA - AX$. Then what can we say about $\mathrm{tr}(A(t))$?

In general, if a matrix field $A(t)$ is differentiable with respect to t, then what is the formula for $\frac{\mathrm{d}}{\mathrm{d}t} \det A(t)$?

3. Fix a sequence of complex numbers $\{x_1, \cdots, x_m\}$. We define the Vandermonde matrix as

$$V(x_1, \cdots, x_m) = \begin{bmatrix} 1 & x_1 & \cdots & x_1^{m-1} \\ 1 & x_2 & \cdots & x_2^{m-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_m & \cdots & x_m^{m-1} \end{bmatrix}.$$

Compute that $\det V(x_1, \cdots, x_m)$. From this formula it will be clear that $V(x_1, \cdots, x_m)$ is invertible if those points are distinct.

4. Fix an integer m and define the $p-$norm $(1 \leq p < \infty)$ of $x \in \mathbb{C}^m$ as

$$\|x\|_p = \left(\sum_{i=1}^{m} |x_i|^p \right)^{1/p},$$

where $x^\top = (x_1, \cdots, x_m)$. And

$$\|x\|_\infty := \sup_{1 \le i \le m} |x_i|.$$

Show that for any $1 \le p, q \le \infty$, there exist constants c, C such that

$$c\|x\|_q \le \|x\|_p \le C\|x\|_q.$$

5. Find the induced matrix norm $\|\cdot\|_{(2,2)}$ for the matrix

$$\begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}.$$

6. Page 24, problem 3.6.