1. Friday, August 24, 2012

1.1. **Matrix Representations of (Finite) Groups.** Historically, Representation Theory began with matrix representations of groups, i.e. representing a group by an invertible matrix.

Definition 1.1. $GL_n(k)$ = the group of invertible $n \times n$ matrices over k; k can be a field or a commutative ring. A *matrix representation* of G over k is a homomorphism $\rho : G \to GL_n(k)$.

Remark 1.2. Matrix representation is useful for calculating in G if ρ is injective (or faithful). If not, you can only calculate up to equivalence via the kernel.

Example 1.3.

$$\rho : S_n \to GL_n(k).$$

$$\rho(\pi) = A : A_{ij} = \begin{cases} 1 & \text{if } i = \pi(j) \\ 0 & \text{otherwise.} \end{cases}$$

Let $e_1, \dots, e_n \in k^n$; then, $\rho(\pi)(e_j) = e_{\pi(j)}$, so this is a group homomorphism.

In the specific case $n = 3$, if $\pi = (1\,2\,3)$, then

$$\rho(\pi) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

This is the *standard matrix representation* of S_n.

Example 1.4. If G acts on X, and $|X| = n$, then one can map $G \to \mathfrak{S}(X) \cong S_n \to GL_n(k)$. This leads us directly to the *permutation representation* of G.

Example 1.5. G acting on G on the left, so $g \bullet h = gh$; then, $G \to GL_n(k)$, where $n = |G|$. This is called the "regular representation."

Example 1.6. $\rho : G \to GL_n(k)$, sending $\rho(g) = I_n$. This is the *trivial representation* of G.

Example 1.7. Suppose that there is a $\omega \in k$, such that $\omega^n = 1$, for example $e^{2\pi i/n} \in \mathbb{C}$. Then, we have a non-trivial, thus non-permutation representation:

$$\rho : \mathbb{Z}/n\mathbb{Z} \to GL_1(k) = k^\times, \qquad \rho(m) = \omega^m.$$

1.2. **Modern Representation Theory.**

Definition 1.8. A representation of G (over k) is a homomorphism

$$G \to GL(V) = \{k\text{-linear invertible maps } V \to V\}.$$

V is a vector space over k, or a free module over commutative ring k.

Suppose $\dim V < \infty$. Let $(v_1, \dots, v_n)$ be an ordered basis $V \cong k^n$, which means $GL(V) \cong GL_n(k)$. This means that $\rho : G \to GL(V) \cong GL_n(k)$ gives a matrix representation. Conversely, $GL_n(k) \cong GL(k^n)$, so matrix representations give representations on k^n. The other way to define a representation instead of k-linear invertible maps, is action of G on V by linear maps. "gv" $= \rho(g)(v)$.

We also say that V is a G-module, (V, ρ).

Definition 1.9. A G-module homomorphism $\phi : V \to W$ (over k) is a k-linear map s.t.

$$\phi(gv) = g\phi(v), \qquad \forall g \in G, v \in V.$$

Definition 1.10. A submodule (or invariant subspace) is a subspace $W \subseteq V$ such that

$$gw \in W, \forall g \in G, w \in W.$$

(i.e. $W \hookrightarrow V$ is a homomorphism, for some G-action on W.)

Definition 1.11. An isomorphism is an invertible homomorphism (an inverse linear map will also be a homomorphism).

Definition 1.12. Quotient module: given $W \subseteq V$, make G act on V/W by $g(v+W) = g(v) + W$. This is well defined, since:

$$v + W = v' + W \Rightarrow v - v' \in W \Rightarrow g(v - v') \in W \Rightarrow gv + W = gv' + W.$$

Definition 1.13. V is irreducible if $V \neq 0$ and its only submodules are 0 and V.

Question 1.14. When do two matrix representations ρ, ρ' belong to isomorphic G-modules $V \cong W$?

Let V have basis $v_1, \dots, v_n$, and W have basis $w_1, \dots, w_n$. This is the same question as when ρ, ρ' come from two bases of V.

Answer: They are isomorphic iff $\exists S \in GL_n(k)$ s.t. $\rho'(g) = S\rho(g)S^{-1}$, for any $g \in G$.

Example 1.15. Consider again the standard representation, but instead of a homomorphism into $GL_n(k)$, think about it as an action of S_n on $k^n = V$, where $k = \mathbb{C}$.

The subspace $U = \mathbb{C} \cdot \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$ is invariant and the action of S_n on U is trivial. Furthermore, $\dim U = 1$.

Another invariant subspace is $W = \{z : \sum z_i = 0\}$; this has dimension $n-1$. $U \cap W = \varnothing$, so by dimension, $V = U \oplus W$. This implies that $W \cong V/U$ and $U \cong V/W$.

Take S_3, so that $\dim W = 2$. Consider the image of the map $W \hookrightarrow V \to V/U$. The basis $e_1, e_2, e_3 \mapsto \bar{e}_1, \bar{e}_2, \bar{e}_3$. However, these are linearly dependent, so we take only two of them, say $\bar{e}_1, \bar{e}_2 \Rightarrow \bar{e}_3 = -(\bar{e}_1 + \bar{e}_2)$.

$$\rho((12)) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \rho((12))^2 = I_2$$

$$\rho((23)) = \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix} \qquad \rho((23))^2 = I_2$$

$$\rho((123)) = \rho((12))\rho((23)) = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$$

2. Monday, August 27, 2012

Example 2.1. Recall: We consider the standard representation as an action of S_n on $V = k^n$.

The subspace $U = k \cdot \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$ is a 1-dimensional submodule with trivial representation.

$W = \{z : \sum z_i = 0\}$ is an $(n-1)$-dimensional submodule

If char $k = 0$, then $V = U \oplus W$, $W \cong V/U$ and $U \cong V/W$. What if char $k = p | n$? Then $U \subseteq W, V \neq U \oplus W$. In this case $W \subseteq V$ has no $\oplus$ complement U'.

Suppose it does. Every $e_i - e_j \in W$, so V/W is trivial. So a U' would have to be $k \cdot v$ where $\pi \cdot v = v$ for all $\pi \in S_n \Rightarrow v \in U$. Contradiction.

More generally, problems arise when the characteristic divides the order of the group.

Proposition 2.2. *Back to $k = \mathbb{C}$. W is irreducible. Equivalently, every nonzero $v \in W$ generates W; $S_n \cdot v$ spans W.*

Proof. Let $v = \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix}$, such that $\sum z_i = 0$, not all 0, so not all equal. W.l.o.g. take $z_1 \neq z_2$.

Consider $v - \pi v, \pi = (12)$. This is $(z_1 - z_2)\begin{pmatrix} 1 \\ -1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

This means that $e_1 - e_2 \in \mathrm{span}(S_n \cdot v)$; by symmetry, $e_i - e_j \in \mathrm{span}(S_n \cdot v)$ for all i, j. Therefore, $\mathrm{span}(S_n \cdot v) = W$. $\square$

Remark 2.3. Suppose char $p|n$, and

$$0 \subset U \subset W \subset V$$

Then, $U = U/0$ is irreducible, V/W is irreducible, and W/U is irreducible. (perhaps not for $p = 2$).

Definition 2.4. Let k be a commutative ring. The group algebra kG = free k-module with basis G, with multiplication defined by

$$\Big(\sum_{g\in G} a_g g\Big)\Big(\sum_{g\in G} b_g g\Big) = \sum_{g,h\in G} a_g b_h gh.$$

This inherits associativity from the group, and the group identity becomes 1 for kG. (One could define the algebra over a noncommutative ring, but we are interested in the k-algebra structure that commutativity gives it.)

Regard this as a k-algebra by $k \to kG$, sending $a \mapsto a \cdot 1$.

Remark 2.5. The set of G modules over k (i.e. k-module V, plus k-linear action of G on V) can be canonically identified with kG-modules. Given G-module V (over k) make kG act by

$$\Big(\sum a_g g\Big)v = \sum a_g g(v),$$

and it is easy to see that this is a group action.

Conversely, if V is a kG-module, then $G \to kG$ mapping $g \mapsto g$ is a homomorphism $G \to (kG)^\times$ of groups giving a G action on V.

One can also define an algebra where the group acts on the ground ring. The resulting multiplication looks like:

$$\Big(\sum_{g\in G} a_g g\Big)\Big(\sum_{g\in G} b_g g\Big) = \sum_{g,h\in G} a_g g(b_h) \cdot gh.$$

2.1. **Modules over Associative Algebras.** Our convention will be to talk about left modules.

Generalities on A-modules where A is an associative algebra with unit, over some commutative ring k. (If $k = \mathbb{Z}$, we mean non-commutative rings with unit)

Important Concepts:

- Submodule
- quotient V/W
- Irreducible means $V \neq 0$ and only submodules are 0 and V.
- Direct sums
- Indecomposable = not a proper $\oplus$. Irreducible implies indecomposable, but not the converse.
- Module homomorphisms.
- Submodule $W \subset V$, then $0 \to W \to V \to V/W \to 0$ is an exact sequence.
- Given $0 \to K \to V \to Q \to 0$, we say that V is an extension of Q by K.
- An A-module V is Aritinian if every chain of distinct submodules is finite. (Implication: Artinian $\Rightarrow$ Noetherian)

Proposition 2.6. *Every submodule, quotient and extension of Artinian modules is Artinian.*

3. Wednesday, August 30, 2012

Definition 3.1. Recall: an A-module V is Artinian if it has no infinite chain of submodules. (ACC + DCC)

Remark 3.2. If V is Artinian, a maximal chain looks like:

$$0 = V_0 \subset V_1 \subset \cdots \subset V_m = V,$$

where each V_i/V_{i-1} is simple. Such a chain is called a *composition series*. The V_i/V_{i-1} are called composition factors.

Proposition 3.3. *Every submodule, quotient and extension of Artinian modules is Artinian.*

Proof. Submodules and quotients are obvious. For extension, we are given $W \subseteq V$, W Aritinian, V/W Artinian. We want to show that V is Artinian.

Given

$$V' \subseteq V'' \subseteq V,$$

we can send it to a chain in W, by

$$V' \cap W \subseteq V'' \cap W \subseteq W$$

as well as a chain V/W by

$$(V' + W)/W \subseteq (V'' + W)/W \subseteq V/W.$$

The composition factors in each of these modules:

$$(V'' \cap W)/(V' \cap W) \subseteq V''/V', \quad V''/V' \to (V'' + W)/(V' + W).$$

In the corresponding exact sequence:

$$0 \to \frac{V'' \cap W}{V' \cap W} \to \frac{V''}{V'} \to \frac{V'' + W}{V' + W} \to 0.$$

Alternatively, consider the following:

$$(V' + W) \cap V'' = V' + (W \cap V'') \qquad \text{(modular law).}$$

This gives us a W' between V' and V''.

$$V' \cap W = V'' \cap W \text{ and } V' + W = V'' + W \Rightarrow V' = V''.$$

Given (V_i) in V,

$$V_i \mapsto (V_i + W, V_i \cap W),$$

an injective map into the product of finite chains. Therefore, the pre-image chain is also finite.

You can also think about this in the purely formal realm of modular lattices. □

Theorem 3.4 (Jordan-Hölder Theorem)**.** *If (V_i) and (V_i') are two composition series for V, then they have the same length, say n and there is a permutation π of $\{1, 2, \ldots, n\}$ such that $V_i'/V_{i-1}' \cong V_{\pi(i)}/V_{\pi(i)-1}$.*

Proof. The proof follows by induction on the length of composition series. Consider two chains (V_i) and (V_i').

Suppose that $V_1 = V_1'$. Then, $V/V_1 \cong V/V_1'$ have decomposition series of the same length by induction.

So, we take $V_1 \neq V_1'$. Consider the submodule $V_1 + V_1'$. There exists a composition series (W_i) for $V/(V_1 + V_1')$ by the proposition. So the chain $(0, V_1, W_i)$ and the chain $(0, V_1, W_i)$ have the same length. By induction, the chain (V_i) and (V_i') will therefore have the same length.

Again this could be formulated in a lattice-theoretic way. □

Proposition 3.5. *The following are equivalent (for an A-module V):*

(1) *Every $U \subseteq V$ has a $\oplus$ complement: $V = U \oplus W$.*
(2) *V is a direct sum of simple submodules.*

In this case, V is called completely reducible, or semisimple.

Lemma 3.6. *Property (1) in the proposition is preserved by quotients.*

Proof.

$$\hat{U} \subseteq V/W \leftrightarrow W \subseteq U \subseteq V, \qquad V = U \oplus U' \Rightarrow V/W = U/W + \bar{U}',$$

where $\bar{U}'$ is the image of $U' \subset V \to V/W$.

□

Lemma 3.7. *Property (1) implies that V contains a simple submodule or $V = 0$.*

Proof. Pick a nonzero $v \in V$. Let $M \subseteq V$ be maximal such that $v \notin M$. Let $\bar{v}$ = the image of v in V/M.

Then, $\bar{v} \neq 0$ belongs to every nonzero submodule of V/M i.e. $A\bar{v}$ is the smallest nonzero submodule of V/M, hence simple. So V/M has a simple submodule. Because $V = M \oplus N$, then $N \cong V/M$, so N has a simple submodule and so does V. □

Proof of Proposition 3.5. (1) $\Rightarrow$ (2). Let U_i be a maximal collection of simple submodules, such that $\sum U_i = \oplus U_i$, i.e. if $u_{i_k} \in U_{i_k}$ where $i_1, \ldots, i_m$ are distinct, then $\sum u_{i_k} = 0$ implies that all $u_{i_k} = 0$.

If $V = \sum U_i$, we are done. If $V = U \oplus W$, then $W \cong V/U$. By the lemma, W has a simple submodule U'. But $\sum U_i + U' = \bigoplus U_i \oplus U'$, contradicting maximality.

□

4. Friday, August 31, 2012

Proof of Proposition 3.5 Continued. (2) $\Rightarrow$ (1). Let $V = \bigoplus_I U_i$. Given U, let $J \subseteq I$ be a maximal subset such that $W := \bigoplus_{i \in J} U_i$ has $W \cap U = 0$.

Claim: $W + U = V$, i.e. the quotient of $U \to V/W$ is surjective. $V/W \cong \bigoplus\limits_{i \notin J} U_i$. Then some U_i ($i \notin J$) is not contained in

$$U' = im(U \to V/W \cong \bigoplus_{i \notin J} U_i).$$

So $U' \cap U_i = 0$, with U_i simple. Therefore, $(U + W) \cap (U_i + W) = W \Rightarrow U \cap (W + U_i) = 0$. This contradicts maximality of J; therefore, $U + W$ must be the whole module.

□

Recall that, under these conditions, V semi simple or completely reducible.

Corollary 4.1. *Any submodule, quotient, or direct sum of semi simple modules is semisimple.*

Next question: When are all A-modules semisimple?
More about simple modules:

Lemma 4.2 (Schur's Lemma)**.** *If V, W simple modules, then any homomorphism $\varphi : V \to W$ is either 0 or an isomorphism.*

Proof. Consider the possible submodules to play the role of kernel and image. □

Corollary 4.3. *If $V \ncong W$, then $Hom_A(V, W) = 0$. $End_A(V) = Hom(V, V)$ is a division ring.*

We can also make statements about semisimple modules.

If $V = \bigoplus V_i$ and $W = \bigoplus_j W_j$, then

$$\mathrm{Hom}(V,W) = \prod_i \mathrm{Hom}(V_i, W).$$

Assuming both index sets are finite, then

$$\mathrm{Hom}(V,W) = \bigoplus_i \bigoplus_j \mathrm{Hom}(V_i, W_j).$$

Finally, if we index the modules as $V = \oplus_\lambda V_\lambda^{m_\lambda}$, and $W = \oplus_\lambda V_\lambda^{n_\lambda}$, such that $\lambda \neq \mu \Rightarrow V_\lambda \neq V_\mu$, with the index set finite. Then,

$$\mathrm{Hom}(V,W) = \bigoplus_\lambda \mathrm{Hom}(V_\lambda^{m_\lambda}, V_\lambda^{n_\lambda}) = M_{m_\lambda \times m_\lambda}(End_A(V_\lambda)).$$

$$\text{and } End_A(V) = \bigoplus_\lambda M_{n_\lambda \times n_\lambda}(D_\lambda),$$

where D_λ is the division ring $End(V_\lambda)$.

4.1. **Jacobson Radical of A.** Let V be a simple A-module. Any $0 \neq v \in V$ generates V as $Av = V$. Let $\mathfrak{m} = \mathrm{ann}_A(v)$. Then $\mathfrak{m}$ is a maximal left ideal.

$$A/\mathfrak{m} \xrightarrow{\sim} V, \qquad a \mapsto a \cdot v.$$

Therefore, $A/\mathfrak{m} \cong A/\mathfrak{m}' \cong V$, when $\mathfrak{m}, \mathfrak{m}'$ are annihilators of different $v, v' \in V$.

Example 4.4. $A = M_n(k)$. Then, $V = k^n$. $\mathfrak{m} = \{$ matrices killing a vector $v\}$.

We can group the maximal ideals as those that kill elements in the same simple module.

$$\bigcap_{\mathfrak{m} \cdot A/\mathfrak{m} \cong V} \mathfrak{m} = \mathrm{ann}(V) = \ker(A \to \mathrm{End}_k(V))$$

is a two-sided ideal.

Definition 4.5. The (left) Jacobson radical $J(A)$ is defined as

$$J(A) = \bigcap_{\mathfrak{m} \text{ maximal left ideal}} \mathfrak{m} = \bigcap_{V \text{ of each iso. type}} \mathrm{ann}(V).$$

By theorem, the right and left Jacobson radicals are equal.

Proposition 4.6.

$$A \to \prod_V End_{\,k}(V) \to End_{\,k}\left(\bigoplus_V V\right).$$

By Schur's Lemma, the second arrow is isomorphism. $J(A)$ is the kernel of the first map. A has a faithful semisimple module if and only if $J(A) = 0$. $A/J(A)$ has a faithful semisimple module, since $J(A/J(A)) = 0$.

Theorem 4.7. *The following are equivalent (for k-algebra A):*

(1) A is a semisimple left A-module.
(2) A is isomorphic to a finite product $\prod_{i=1}^m M_{n_i}(D_i)$ of matrix algebras over division rings.
(3) A is Artinian (as a left A-module) and $J(A) = 0$.
(4) Every left A-module is semisimple.

Proof. (4) $\Rightarrow$ (1) trivial. (1) $\Rightarrow$ (4) almost trivial: Every A-module is a quotient of a free module $\bigoplus A$. Quotients of semisimple modules are semisimple. (2) $\Rightarrow$ (3) not too bad. $\square$

5. Wednesday, September 5, 2012

Proof of Theorem 4.7 Continued. (1) ⇒ (2). Note that any A with unit has $A^{op} \to \text{End}_A(A)$, with $a \mapsto \varphi(x) = xa$. This is actually an isomorphism, with inverse $\varphi \mapsto \varphi(1)$. Suppose A is semisimple, i.e. $A = \bigoplus_{i=1}^k V_i^{n_i}$, with V_i simple and $V_i \neq V_j$ for $i \neq j$. (1) ⇒ A is a finite direct sum of simples because A is finitely generated.

$$A^{op} \cong \text{End}_A\left(\bigoplus_{i=1}^k V_i^{n_i}\right) = \prod_{i=1}^k \text{End}\,(V_i^{n_i}) = \prod_{i=1}^k M_{n_i}(D_i^{op}),\ \text{where } D_i^{op} = \text{End}\,V_i.$$

(2) ⇒ (3). $M_n(D_i)$ (and also A) has a simple module $D_i^n = V_i$. $e_1 = (1,0,\dots,0)$ generates. For $0 \neq v \in D_i^n$ there is a matrix X such that $X \cdot v = e_1$. This implies that A acting on $\bigoplus D_i^{n_i}$ is faithful, so $J(A) = 0$.

Note that $A \cong_{A-mod} \bigoplus_i V_i^{n_i} \Rightarrow A$ is Artinian.

(3) ⇒ (1). General fact: For a module V, and U_i a family of submodules, such that $\bigcap_{i=1}^k U_i = 0$ and $U_j + \bigcap_{i=1}^{j-1} U_i = V$, for all $j = 1,\dots,k$, and given $x \in V/U_j$ there is a $v \in V$ such that $v \mapsto x$ and $v \in \ker \phi_{j-1}$. . Then the map $\varphi : V \to \bigoplus_{i=1}^k V/U_i$ is an isomorphism.

Proof: It is injective, because $\ker\varphi = \cap U_i = 0$.

By the second hypothesis, if ϕ_{j-1} is surjective, so is $\phi_j : V \to \bigoplus_{i=1}^j V/U_j$. By induction, φ is surjective.

Given (3), construct maximal left ideals $\mathfrak{m}_1, \mathfrak{m}_2, \dots, \mathfrak{m}_k$ as follows. If $\mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_{j-1} = 0$ for all j, then we are done (Since $J(A) = 0 \Rightarrow A$ is semisimple). If not, then...

□

Corollary 5.1. *Assume A is semi simple. Then,*

(1) A has finitely many simple modules V_i up to isomorphism.
(2) V_i is $D_i^{n_i}$ as an A-module via $M_{n_i}(D_i)$.
(3) A has central, orthogonal idempotents e_i such that e_i acts on V_i as 1 and on $V_j, j \neq i$ as 0.
(4) In $V = \bigoplus_{i=1}^k (\bigoplus_J V_i)$ (where J is a possibly infinite indexing set), the summands $\bigoplus_J (V_i) =$ im e_1 are unique (isotopic components).
(5) A-mod is categorically equivalent to $\prod_{i=1}^k D_i$-mod.

5.1. Back to kG-modules (G modules over k).

Remark 5.2. $\text{Hom}_k(V,W)$, is the group (k-module) of k-module homomorphisms from V to W. Given $W \xrightarrow{\varphi} W'$, we get a map

$$\text{Hom}(V,W) \xrightarrow{\text{Hom}(V,\varphi)} \text{Hom}(V,W'), \qquad \alpha \mapsto \phi \circ \alpha.$$

This respects commutative diagrams, so $\text{Hom}(V,-)$ is a functor from k-mod → Ab (or to k-mod, if k is commutative).

Similarly, $\text{Hom}(-,V)$ is a functor from $(k\text{-mod})^{op} \to (k\text{-mod})$.

If G acts on V and H acts on W, then $G^{op} \times H$ acts on $\text{Hom}_k(V,W)$. We check when $H = G$. Then,

$$G \to G \times G \to G^{op} \times G \text{ acts on } \text{Hom}_k(V,W), \qquad g \cdot \varphi = g \circ \varphi \circ g^{-1}.$$

So $\text{Hom}_k(V,W)$ is a G-module.

$$\text{Hom}_k(V,W)^G = \{\varphi : g\varphi = \varphi,\ \forall g \in G\} = \text{Hom}_{kG}(V,W).$$

where $g\varphi = \varphi$ means $g \circ \varphi = \varphi \circ g$.

Theorem 5.3 (Maschke's Theorem)**.** *Let k be a field, and G a finite group, such that char $k \not| G$. Then kG is semi simple.*

Proof. Given $U \subseteq V$ kG-modules, we want $W \subseteq V$ such that $V = U \oplus W$.

$$0 \to U \xrightarrow{i} V \xrightarrow{p} V/U \to 0.$$

We want to split this – i.e. we want $h : V/U \to V$ such that $ph = 1_{V/U}$.

We can find a k-linear map $j : V/U \to V$ such that $pj = 1_{V/U}$, where $j \in \mathrm{Hom}_k(V/U, V)$. We want $h \in \mathrm{Hom}_k(V/U, V)^G$, invariant under G action.

The desired map is $R = \frac{1}{|G|}\sum_g g \in kG$. □

6. Friday, September 7, 2012

Lemma 6.1 (Properties of the Reynolds Operator)**.** *$R^2 = R$, and R acts in any kG-module V as a projection on*

$$V^G = \{v \in V : gv = v, \forall g \in G\}.$$

$R(V) = V^G$ and $R|_{V^G} = 1_{V^G}$.

Proof.

$$h \cdot Rv = \frac{1}{|G|}\sum_{g\in G} hg \cdot v = Rv \Rightarrow Rv \in V^G.$$

$$v \in V^G \Rightarrow Rv = \frac{1}{|G|}|G|v = v.$$

□

Proof of Theorem 5.3 Continued. Define $h = R(j)$, where j was the vector-space map from above.

$$h \in \mathrm{Hom}_k(V/U, V) = \mathrm{Hom}_{kG}(V/U, V).$$

Note that

$$\mathrm{Hom}_k(V/U, V) \longrightarrow \mathrm{End}\,_k(V/U), \qquad j \mapsto pj$$

is G-homomorphism, since p is G-invariant:

$$g(pj) = gpjg^{-1} = pgjg^{-1} = pg(j).$$

So, $ph = pR(j) = R(pj) = R(1_{V/U}) = 1_{V/U} \in \mathrm{End}\,_k(V/U)$. □

Recall that if A is semi simple, in A we have central orthogonal idempotents e_i, act as projections on isotypic components in $V = \bigoplus_i(\bigoplus_I V_i)$. Specifically, $A \cong \prod_i M_n(D_i)$.

What is R? It is the e_i for $V_i = k$ (the trivial representation).

In some sense, the representations over an algebraically closed field will be finer than the representations in non-algebraically closed subfields; the representations will "clump together" when you pass to the subfield.

Lemma 6.2. *If $k = \bar{k}$, the only finite dimensional division algebra over k is k.*

Proof. $k \subseteq D, x \in D$, then $k(x)$ is commutative, hence $k(x) = k$. □

Corollary 6.3. *If $k = \bar{k}$, char $k \not| \ |G|$, then $kG \cong \prod_i M_{n_i}(k)$. The non-isomorphic simple kG-modules are $V_i = M_{n_i}(k)$-module k^{n_i}.*

Corollary 6.4. *For $k = \bar{k}$ and char k ok, if $V_1, \ldots, V_r$ are the simple kG modules up to isomorphism, then $\sum_i \dim(V_i)^2 = |G|$.*

Example 6.5. $G = S_3$. We have $V_1 = \mathbb{C}$ trivial representation. $\mathbb{C}^3 \cong V_1 \oplus W$ gives us V_2 a 2-dimensional representation. So $6 = 1^2 + 2^2 + n^2 \Rightarrow n = 1$, so we have a missing one-dimensional representation. The missing one is the sign rep V_ϵ, which sends each permutation to its sign. This set of representations gives us a map $G \to M_1 \times M_2 \times M_1$, which implies $kG \cong M_1 \times M_2 \times M_1$.

($k = \bar{k}$, char $k \nmid |G|$). Then, $\dim Z(kG)$ = number of simple modules. Alternatively,

$$Z(kG) = \{\sum_g a_g g : a_g \text{ is constant on conjugacy classes}\},$$

since being in the center also means that it is invariant under conjugation.

Suppose G is abelian ($k = \bar{k}$, char $k \nmid |G|$). Then $kG \cong \prod_i M_1(k)$, i.e. all simple modules have dimension 1. They are given by homomorphisms $G \to k^\times$, more specifically $G \to$ roots of 1 in $k \to k^\times$; the set of roots $\cong \mathbb{Q}/\mathbb{Z}$ in char 0, or "enough" of $\mathbb{Q}/\mathbb{Z}$ in char p.

Definition 6.6. The dual group $\hat{G} = \mathrm{Hom}_{Ab}(G, \mathbb{Q}/\mathbb{Z})$. So $|\hat{G}| = |G|$, but there is no canonical isomorphism.

Corollary 6.7. *$kG \cong \bigoplus V_i^{\dim V_i}$, as a kG-module.*

Corollary 6.8. *Maschke's Theorem implies that all $g \in G$ acting on V are simultaneously diagonalizable.*

Definition 6.9. Let A be a ring, N an A-module, and M an A^{op} module. Then, $M \otimes_A N$ is an abelian group generated by symbols $m \otimes n$, where $m \in M$, $n \in N$, with relations $(m + m') \otimes n = m \otimes n + m' \otimes n, m \otimes (n + n') = m \otimes n + m \otimes n', ma \otimes n = m \otimes an$.

The motivating property is that $M \times N \to M \otimes N$ is $\mathbb{Z}$-linear in each variable and "A-associative."

It satisfies the universal property:

$$\begin{tikzcd} M \otimes N \arrow[r, "\phi"] & Q \\ M \times N \arrow[u, "\otimes"] \arrow[ur, "\bullet"'] & \end{tikzcd}$$

where $\bullet$ is a bilinear map with $\bullet(ma, n) = \bullet(m, an)$.

7. Monday, September 10, 2012

More on the tensor product: If $f : M \to M'$ is a map of left A-modules and $g : N \to N'$ is a map of right A-modules.

$$\begin{tikzcd} & M \otimes_A N \arrow[d, "f\otimes g"] \\ M \times N \arrow[ur] \arrow[r] & M' \otimes_A N' \end{tikzcd}$$

where $m \otimes n \mapsto f(m) \otimes g(n)$. This map will satisfy:

$$f(ma) \otimes g(n) = f(m)a \otimes g(n) = f(m) \otimes ag(n) = f(m) \otimes g(an).$$

We can put left and right module structures on the tensor product, by multiplying in the right or left component.

If k is commutative with M and N k-modules. View them as k-bimodules with same left and right action. As a result, k acts on $M \otimes_k N$ from both right and left, and these actions are the same. Therefore, $M \otimes_k N$ has a natural k-module structure.

Remark 7.1. In some ways the tensor product $M \otimes N$ "eats up" the action on M from the right and N from the left, so that its only actions are inherited from the left action on M and the right action on N.

Remark 7.2. Let M, N, P, Q be A-modules. The functor $P \otimes \cdot$ is right-exact. Specifically, given an exact sequence:

$$M \xrightarrow{f} N \xrightarrow{g} Q \to 0,$$

the following sequence is exact:

$$P \otimes_A M \xrightarrow{1\otimes f} P \otimes_A N \xrightarrow{1\otimes g} P \otimes_A Q \to 0.$$

Example 7.3. Consider the sequence

$$0 \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z} \to 0.$$

We apply the functor $\mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \cdot$ to the sequence, and obtain:

$$0 \to \mathbb{Z}/2\mathbb{Z} \xrightarrow{\cdot 2 = 0} \mathbb{Z}/2\mathbb{Z} \xrightarrow{1} \mathbb{Z}/2\mathbb{Z} \to 0.$$

So, the surjections are preserved, but the injection is not; thus "right-exact".

Remark 7.4. In the example, we used the basic fact that $A \otimes_A M \cong M$, by the canonical maps $a \otimes m \mapsto am, m \mapsto 1 \otimes m$.

Corollary 7.5. *Let $M \cong (\bigoplus_I A)$, a free right A-module. $\Rightarrow M \otimes_A N = \bigoplus_I N$.*

Corollary 7.6. *Let k be a field, and M, N vector spaces with bases $\{m_i\}, \{n_j\}$ respectively. Then, $M \otimes_k N$ is a vector space with basis $\{m_i \otimes n_j\}$.*

Proposition 7.7. *Let V, W be kG-modules. Then $V \otimes_k W$ is again a kG-module.*

$g(v \otimes w) = gv \otimes gw$. In other words, we have linear maps $g : V \to V, g : W \to W$; so, we can tensor them to obtain $g \otimes g : V \otimes W \to V \otimes W$.

Let A, B be k-algebras, with k commutative. Let V be an A-module, and W be a B-module. $A \otimes_k B$ is a k-algebra with multiplication

$$(a \otimes b)(a' \otimes b') = (aa' \otimes bb').$$

Additionally, $V \otimes_k W$ is an $A \otimes_k B$-module.

Based on this, V and W as described in the Proposition will have $V \otimes_k W$ be a priori a $kG \otimes_k kG$-module. The tensor product can also be expressed as $k(G \times G)$, since the basis is described by pairs of elements, but the diagonal map $G \to G \times G$ induces a kG structure on the module.

To further explain this, keep V and W A-modules.

$A \otimes_k A$ acts on $V \otimes_k W$, and $A^{op} \otimes_k A$ acts on $\mathrm{Hom}_k(V, W)$.

We have a bunch of maps:

$$kG \xrightarrow{\Delta} kG \otimes_k kG, \qquad g \mapsto g \otimes g.$$

$$kG \xrightarrow{s} kG^{op}, \qquad g \mapsto g^{-1}.$$

$(s \otimes 1) \circ \Delta$ gives the map:

$$kG \to kG^{op} \otimes kG, \qquad g \mapsto g^{-1} \otimes g.$$

Then given $\varphi : V \to W$, $g(\varphi) = g \circ \varphi \circ g^{-1}$.

We need the following to commute:

We also have a counit $\varepsilon : kG \to k = \mathrm{End}_{\,k}(k)$, mapping $g \mapsto 1$, and the unit $u : k \to A$, with $x \mapsto x$.

Then kG is a co-commutative Hopf Algebra. $k \otimes_k V \cong V$. And $V^* = \mathrm{Hom}_k(V, k)$. V finite dimensional, $V^* \otimes_k W \cong \mathrm{Hom}_k(V, W)$.

$kG^{op} \xrightarrow{S} kG$.

$$\begin{tikzcd} kG \arrow[r, "\Delta"] \arrow[d, "\Delta"] & kG \otimes kG \arrow[d, "1\otimes\Delta"] \\ kG \otimes kG \arrow[r, "\Delta\otimes 1"] & kG \otimes kG \otimes kG \end{tikzcd}$$

8. Wednesday, September 12, 2012

8.1. **Characters.** Assume that char $k = 0$, and that $k = \bar{k}$. (Specifically, $k = \mathbb{C}$). Let V be a finite-dimensional kG-module.

Definition 8.1. The Character $\chi_V(g) = tr_V(g)$, i.e. the trace of its action as a matrix on the vector space.

Remark 8.2. Note that $\chi_V(hgh^{-1}) = tr_V(hgh^{-1}) = tr_V(g) = \chi_V(g)$. So the character is constant on conjugacy classes. So, χ_V is a "class function."

Remark 8.3. What is $\chi_{V\oplus W}$? Choosing bases for V and W, a given g has an action on V determined by a matrix M, and on W by a matrix N. So, the action of the direct sum is determined by the matrix

$$\left(\begin{array}{c|c} M & \mathbf{0} \\ \hline \mathbf{0} & N \end{array}\right)$$

Therefore, $tr_{V\oplus W}(g) = tr_V(g) + tr_W(g)$.

Remark 8.4. Let the k-algebra A be a finite dimensional V-module, with $a \in A$. Take a subspace $U \subseteq V$, and the quotient V/U.

The matrix representing the action of a will look like:

$$\left(\begin{array}{c|c} M & * \\ \hline \mathbf{0} & N \end{array}\right)$$

where M describes the action on U and N describes the action on V. Therefore, $tr_V(a) = tr_U(a) + tr_{V/U}(a)$, even though they are not direct summands.

Remark 8.5. What is the character on a tensor product $V \otimes W$? The vector space $V \otimes W$ has basis $\{v_i \otimes w_j\}$. The matrix of g will have $(i,j),(i',j')$ entries, i.e. $M_{ii'}N_{jj'}$. Therefore,

$$\chi_{V\otimes W}(g) = \sum_{i,j} M_{ii}N_{jj} = tr(M)tr(N) = \chi_V(g)\chi_W(g).$$

Remark 8.6. Consider now, $\chi_{V^*}(g)$, the character on the dual space. Let $\{\xi_i\}$ be the dual basis of V^*. Define an inner product $\langle \xi_i, v_j\rangle = \delta_{ij}$, which satisfies:

$$\langle g\xi_i, v_j\rangle = \langle \xi_i, g^{-1}v_j\rangle.$$

(This fact comes from the functorial definition by which G acts compatibly on everything.) The matrix of g on V^* is $(M^{-1})^T$.

In the case of $k = \mathbb{C}$, this implies $\chi_V(g^{-1}) = \overline{\chi_V(g)}$.

Remark 8.7. These results immediately give us:

$$\chi_{\mathrm{Hom}(V,W)}(g) = \chi_{V^*\otimes W}(g) = \chi_V(g^{-1})\chi_W(g).$$

Remark 8.8. Recall the Reynolds operator, defined as:

$$R = \frac{1}{|G|}\sum_{g\in G} g \in kG.$$

Then R acts on V by projection onto V^G. So, $\chi_V(R) = \dim V^G$. We can also explicitly calculate the character from the formula, which leads to:

$$\chi_V(R) = \dim V^G = \frac{1}{|G|}\sum_{g\in G}\chi_V(g).$$

We observe:

$$\dim \operatorname{Hom}_G(V,W) = \dim \operatorname{Hom}_k(V,W)^G = \frac{1}{|G|}\sum_{g\in G}\chi_V(g^{-1})\chi_W(g) = \langle \chi_V, \chi_W\rangle.$$

This leads to the following definition:

Definition 8.9.

$$\langle x, \varphi\rangle = \frac{1}{|G|}\sum_{g\in G}\chi(g^{-1})\varphi(g)$$

a symmetric, bilinear form on (class) functions $G \to k$.

[When $k = \mathbb{C}$, we can define this as

$$(x,\varphi) := \frac{1}{|G|}\sum_{g\in G} x(g)\overline{\varphi(g)},$$

a Hermitian, positive definite form.]

Lemma 8.10. *The number of irreps is the same as the number of conjugacy classes.*

Proof. A semisimple module is the direct sum of modules over division algebras. Each matrix is an irreducible representation.

Any element in the center of kG corresponds to scalar multiples of the identity matrix in each module, so the dimension of the center is also equal to this value. (See Example 6.5) □

Schur's Lemma + the formula above + the fact that the number of irreps = the number of conjugacy classes imply

Theorem 8.11. *The characters $\{\chi_V : V$ irreducible $\}$ form an orthonormal basis of the space of class functions. In particular, if $V = \bigoplus_i V_i^{d_i}$, then*

$$d_i = \langle \chi_V, \chi_{V_i}\rangle \Rightarrow \chi_V = \sum d_i \chi_{V_i} \Rightarrow \langle \chi_V \chi_V\rangle = \sum d_i^2$$

Corollary 8.12.

$$\langle \chi_V, \chi_V\rangle = 1 \Leftrightarrow V \text{ irreducible.}$$

Example 8.13. $V = kX$, permutation representation of G acting on X. Then $\chi_V(g) = |X^g|$, the subset fixed by g. This implies:

$$\dim V^G = \#\text{ orbits } = \frac{1}{|G|}\sum_{g\in G}|x^g|,$$

known as Burnside's formula.

Example 8.14. $V = \mathbb{C}^n$ acted on by S^n. We start with the trivial representation $= \mathbb{C}(1,\dots,1)$. We are left with $V = 1 \oplus W$.

$$\chi_V(\pi) = |[n]^\pi|.$$

$$\langle \chi_V, \chi_V\rangle = \frac{1}{n!}\sum_\pi |[n]^\pi|^2 = \frac{1}{n!}\sum_{\pi\in S_n}|([n]\times[n])^\pi| = \frac{1}{n!}\sum_\pi \chi_{V\otimes V}(g)$$

which is the number of orbits of S_n on $[n]\times[n]$, i.e. 2. Therefore, W is irreducible.

Proposition 8.15. $kX = 1\oplus$ *irreducible if and only if G's action on X is doubly transitive.*

9. Friday, September 14, 2012

Example 9.1. $G = GL_n(\mathbb{F}_q)$. Note that

$$\mathbb{P}^{n-1}(\mathbb{F}_q) = \{(x_1 : \cdots : x_n) | \text{ not all } x_i = 0\}/\mathbb{F}_q^\times$$

G acting on $\mathbb{P}^{n-1}(\mathbb{F}_q)$ is doubly transitive. $\mathbb{C}\cdot\mathbb{P}^{n-1}(\mathbb{F}_q) = 1 \oplus V$, where

$$\dim V = |\mathbb{P}^{n-1}(\mathbb{F}_q)| - 1 = \frac{q^n - 1}{q-1} - 1 = q(1 + \cdots + q^{n-2}).$$

Example 9.2 (Characters of S_4)**.** The number of conjugacy classes is equal to the number of partitions of 4, i.e. 5. The irreducible representations: we have the trivial rep 1, we know that $\mathbb{C}^4$ decomposes as $1 \oplus W$, for W irreducible from the Proposition, and we have ε, the sign representation. We also guess a tensor, $\varepsilon \otimes W$ (since any tensor with 1 would be trivial).

We obtain the character χ_W by finding the values of $\chi_{\mathbb{C}^4}$ and subtracting the trivial representation.

To obtain the character $\chi_?$, we observe that we have a representation kG, with

$$\chi_{kG}(g) = \begin{cases} |G| & g = 1 \\ 0 & g \neq 1 \end{cases}$$

If we decompose $kG \cong \oplus V_i^{d_i}$ for V_i irreducible, we will have $24 = 1+1+9+9+x^2$, for the remaining representation of dimension x. So the dimension is 2.

$$\langle \chi_{kG}, \chi_V \rangle = \langle \chi_{kG}, \chi_V \rangle = \frac{1}{|G|} \sum_{g\in G} \chi_{kG}(g^{-1})\chi_V(g) = \chi_V(1).$$

So, we take $(24,0,0,0,0)-(1,1,1,1,1)-(1,-1,1,1,-1)-(9,3,-3,0,-3)-(9,-3,-3,0,3) = (2,0,2,-1,0)$.

	(1^4) [1] 1	$(2,1^2)$ [6] (12)	$(2,2)$ [3] (12)(34)	$(3,1)$ [8] (123)	(4) [6] (1234)
χ_1	1	1	1	1	1
χ_W	3	1	-1	0	-1
$\chi_?$	2	0	2	-1	0
$\chi_{\varepsilon\otimes W}$	3	-1	-1	0	1
χ_ε	1	-1	1	1	-1

The last mysterious irrep corresponds to the quotient of S_4 by its normal subgroup.

Given simple kG-modules $V_1, V_2, \ldots, V_m$, such that $kG \cong \bigoplus \mathrm{End}_k(V_i)$, we have a projection operation $e_i \in Z(kG)$ such that e_i acts on V_i as the identity, and on any other V_j as 0.

How do we find this e_i? We know that $e_i \in \sum a_g g$ where a_g is constant on conjugacy classes.

$$\chi_{V_j}(e_i) = \delta_{ij}\chi_{V_i}(1) = \sum a_g \chi_{V_j}(g) = \sum_g \alpha(g)\chi_{V_j^*}(g^{-1}) = |G|\langle \alpha, \chi_{V_j^*}\rangle.$$

where we define $\alpha(g) = a_g$. This last expression implies that $\alpha = \chi_{V_i}\frac{\chi_{V_i}(1)}{|G|}$.

So $e_i = \frac{\chi_{V_i}(1)}{|G|}\sum \chi_{V_i^*}(g)g$.

Remark 9.3. Let $H \subseteq G$, the G-module V has a corresponding H-module $\mathrm{Res}_H^G(V)$, i.e. $kH \subseteq kG$. Starting with k-algebras $B \subseteq A$, an A-module V maps to a B-module $\mathrm{Res}_B^A(V)$.

Additionally, if we start with a B-module W, the tensor product $A \otimes_B W$ is a left A-module $\mathrm{Ind}_B^A(W)$. In fact, these functors are adjoint.

Theorem 9.4. *Let A, B be groups and V, W be modules such that A acts on V, B acts on W. Then,*

$$Hom_B(W, Res_B^A V) \cong Hom_A(Ind_B^A W, V).$$

Specifically, Ind_B^A and Res_B^A are adjoint.

10. Wednesday, September 19. 2012

[Was absent for Monday, September 17 for Rosh Hashanah]

Example 10.1. Let S_n act on $V = \mathbb{C}^n$ be the permutation representation. Let $S_k \times S_{n-k} \subset S_n$. Let $\varepsilon \boxtimes 1$ be the 1-dimensional representation of $S_k \times S_{n-k}$ in which $\rho(w, z) = \text{sign}(w)$.

Proposition 10.2.

$$Ind_{S_k \times S_{n-k}}^{S_n} \varepsilon \boxtimes 1 \cong \wedge^k V.$$

Proof. To give

$$\varphi : \text{Ind}_{S_k \times S_{n-k}}^{S_n} \varepsilon \otimes 1 \underset{S_n}{\to} \wedge^k V$$

is equivalent to giving

$$\psi : \varepsilon \otimes 1 \underset{S_k \times S_{n-k}}{\to} \text{Res} \wedge^k V$$

Let v be a generator of $\varepsilon \otimes 1$. Simply send $v \mapsto v_1 \wedge \cdots \wedge v_k$, where $\{v_i\}$ is a basis of V, and extend. Therefore, φ is surjective, and it is bijective since the dimension is $\binom{n}{k}$ on both sides.

$$\langle \chi_{\wedge^k V}, \chi_{\wedge^k V} \rangle = \langle \varepsilon \boxtimes 1, \text{Res} \wedge^k V \rangle_{S_k \times S_{n-k}}.$$

$$= \dim \text{Hom}_{S_k \times S_{n-k}}(\varepsilon \otimes 1, \wedge^k V) = \langle u \in \wedge^k V : u \ S_{n-k}\text{-invariant }, w(u) = \text{sign}(w)u \text{ for } w \in S_k \rangle$$

The operator

$$A = \frac{1}{k!} \sum_{w \in S_k} \text{sign}(w) w \in \mathbb{C} S_k$$

acts in any S_k-module as projection on the ε component.

$$I = \{i_1 < \cdots < i_k\} \subseteq \{1, \ldots, n\} = [n].$$

The wedge products $v_I = v_{i_1} \wedge \cdots v_{i_k}$ are a basis of $\wedge^k V$.

We know:

$Av_I = 0$ if $|I \cap [k]| < k - 1$.

$Av_{[k]} = v_{[k]}$.

For $j > k$, we have $A(v_{[k] \setminus i_0} \cap v_j) = \pm \sum_{i=1}^k (-1)^i v_{[k] \setminus i} \wedge v_j$.

So this gives us

$$\text{Hom}_{S_k \times S_{n-k}}(\varepsilon \otimes 1, \wedge^k V) = \text{ span}(v_{[k]}, \sum_{j=k+1}^{n} \sum_{i=1}^{k} v_{[k] \setminus i} \wedge v_j).$$

So the inner product $\langle \chi_{\wedge^k V}, \chi_{\wedge^k V} \rangle = 2$, and $0 < k < n$. $\Rightarrow \wedge^k V$ is direct sum of 2 distinct irreducibles.

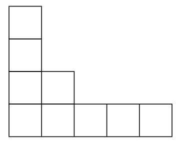
$$V = 1 \oplus W \longrightarrow \wedge^k V = \wedge^k W \oplus \wedge^{k-1} W.$$

□

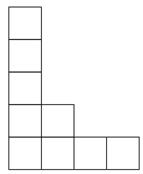
10.1. **Character Theory of S_n.**

Definition 10.3. A partition of n is a sequence $(\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l > 0)$ such that $\sum \lambda_i = n$.
l = the length of the partition.

Example 10.4. $(5,2,1,1)$ is a partition of 9 of length 4. The Young diagram for this partition is:



We obtain its transpose by reflecting across the line $y = x$. So, the transpose is:



Reading off the new partition, we obtain $\lambda^* = (4,2,1,1,1)$. Explicitly, λ_i^* is the number of parts $\geq i$ in λ.

We induce a partial order on partitions of n, saying $\lambda \leq \mu$ if and only if$\lambda_1 + \cdots + \lambda_k \leq \mu_1 + \cdots + \mu_k$, for all k. For example, $(5,2,1,1) \geq (4,3,1,1)$.

The conjugacy classes of S_n are in bijection with the partitions of n.
Given $|\lambda| = n$ define the representations of S_n. Let $S_\lambda = S_{\lambda_1} \times \cdots \times S_{\lambda_l} \subseteq S_n$.
$\Sigma_\lambda = \mathrm{Ind}_{S_\lambda^*}^{S_n} 1$
$W_\lambda = \mathrm{Ind}_{S_\lambda}^{S_n} \varepsilon$.

$$\dim(\Sigma_\lambda) = \dim(W_\lambda^*) = \frac{n!}{\lambda_1! \cdots \lambda_l!} = \binom{n}{\lambda_1, \ldots, \lambda_l}$$
$$\Sigma \mathbb{C} X_\lambda = \mathbb{C} T_\lambda$$
where $X_\lambda = \{$ words in letters $1^{\lambda_1}, 2^{\lambda_2}, \ldots\}$.
$\mathrm{Stab}(1 \cdots 1 2 \cdots 2 \cdots l \cdots l) = S_\lambda$.

Example 10.5.

$$w = 12213311.$$
$$\updownarrow$$
$$\begin{array}{|c|c|cc}\hline 5 & 6 & \multicolumn{2}{c}{} \\ \cline{1-2} 2 & 3 & \multicolumn{2}{c}{} \\ \hline 1 & 4 & \multicolumn{1}{c|}{7} & \multicolumn{1}{c|}{8} \\ \hline \end{array}$$

S_8 acts by permuting the numbers. T_λ = "row tabloids" = fillings $Y(\lambda) \to [8]$ modulo row permutation.

W_λ = action of S_n on "column tabloids" = $\mathbb{C}\{$ fillings $Y(\lambda) \to [n]\}$ modulo signed column permutation.

Under this equivalence, e.g.,

$$\begin{array}{|c|cc}\cline{1-1} 6 & \multicolumn{2}{c}{} \\ \hline 2 & \multicolumn{1}{c|}{5} & \multicolumn{1}{c|}{4} \\ \hline 1 & \multicolumn{1}{c|}{3} & \multicolumn{1}{c|}{7} \\ \hline \end{array} \quad = \quad \begin{array}{|c|cc}\cline{1-1} 1 & \multicolumn{2}{c}{} \\ \hline 2 & \multicolumn{1}{c|}{3} & \multicolumn{1}{c|}{7} \\ \hline 6 & \multicolumn{1}{c|}{5} & \multicolumn{1}{c|}{4} \\ \hline \end{array}.$$

$$\langle \chi_{\Sigma_\lambda}, \chi_{W_\mu} \rangle = \dim \mathrm{Hom}_{S_n}(\Sigma_\lambda, W_\mu).$$

This is 0 if $\lambda \nleq \mu$ and 1 if $\lambda = \mu$. This means that they have exactly one common simple component, V_λ. The V_λ's are distinct.

11. Friday, September 21, 2012

We defined the following representations:

Definition 11.1.

$$\Sigma_\lambda = \mathrm{Ind}_{S_\lambda}^{S_n} 1.$$

This is the permutation representation on row tabloids, in which you permute the entries of each row.

$$W_\lambda = \mathrm{Ind}_{S_{\lambda^*}}^{S_n} \varepsilon.$$

with action on $\mathbb{C}X$, where $X = \{\text{column tabloids}\}$.

For convenience, let us designate the blocks acted on by $S_{\lambda_1}, \dots, S_{\lambda_l}$ as

$$B_1 = \{1, \dots, \lambda_1\}, B_2 = \{\lambda_1 + 1, \dots, \lambda_1 + \lambda_2\}, B_3 = \cdots.$$

Theorem 11.2. *The dimension* $\dim Hom(\Sigma_\lambda, W_\mu)$ *(equivalently,* $\langle \chi_{\Sigma_\lambda}, \chi_{W_\mu} \rangle$*) has:*

$$\dim Hom(\Sigma_\lambda, W_\mu) = \begin{cases} 0 & \lambda \nleq \mu \\ 1 & \lambda = \mu \end{cases}.$$

Proof.

$$\dim \mathrm{Hom}(\Sigma_\lambda, W_\mu) = \dim W_\mu^{S_\lambda}.$$

R_{S_λ} kills a column tabloid T if any B_k has two members in one column of T.

$$R_{S_\lambda}(T) \neq 0 \Rightarrow \lambda_1 + \cdots + \lambda_k \leq \sum_i \min(\mu_i^*, k) = \mu_1 + \cdots + \mu_k.$$

If $\lambda = \mu$ then equality holds above, so $R_{S_\lambda}(T) \neq 0 \Rightarrow$ up to sign, T is of the form represented in Figure 1. Then $R_{S_\lambda}(T)$ doesn't depend (up to sign) on T. Therefore, it is 1.

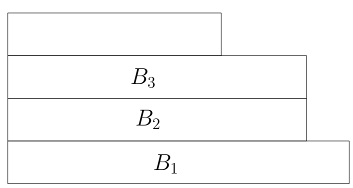


Figure 1. Tableaux with blocks in each row.

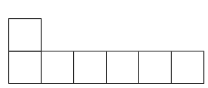
□

Recalling $\Sigma_\lambda = \mathrm{Ind}_{S_\lambda}^{S_n} 1$, and $W_\lambda = \mathrm{Ind}_{S_{\lambda^*}}^{S_n} \epsilon$.

Let $\Sigma_\lambda = \bigoplus V_i$ and $W_\lambda = \bigoplus W_j$, where the V_i's and W_i's are irreducible.

Then Σ_λ and W_λ have a unique $V_i = W_j$ in common (with only one in each). This is due to the theorem about the dimension above. Call it V_λ. If V_λ occurs in W_μ then $\lambda \leq \mu$ i.e. $W_\mu \cong V_\mu \oplus \bigoplus_{\lambda < \mu} V_\lambda^{K_{\lambda\mu}}$. However, we know that V_λ is in S_λ, so $\mu \leq \lambda \Rightarrow \mu = \lambda$. Hence the V_λ's are distinct.

Example 11.3. $W_{(2,1,\dots,1)} = \varepsilon \otimes V_{(n-1,1)}$.



Example 11.4 ($\lambda = (2,2)$)**.** W_λ has basis $\begin{array}{|c|c|}\hline 1&3\\\hline 2&4\\\hline\end{array}, \begin{array}{|c|c|}\hline 1&2\\\hline 3&4\\\hline\end{array}$. The irrep $V_\lambda \subseteq W_\lambda$. Claim:

$$b = \begin{array}{|c|c|}\hline 1&2\\\hline 3&4\\\hline\end{array} + \begin{array}{|c|c|}\hline 1&2\\\hline 4&3\\\hline\end{array} + \begin{array}{|c|c|}\hline 2&1\\\hline 3&4\\\hline\end{array} + \begin{array}{|c|c|}\hline 2&1\\\hline 4&3\\\hline\end{array}.$$

generates $V_\lambda \subseteq W_\lambda$.

$$c = (13)b = -\begin{array}{|c|c|}\hline 1&2\\\hline 3&4\\\hline\end{array} - \begin{array}{|c|c|}\hline 3&1\\\hline 4&2\\\hline\end{array} - \begin{array}{|c|c|}\hline 1&3\\\hline 2&4\\\hline\end{array} - \begin{array}{|c|c|}\hline 2&1\\\hline 4&3\\\hline\end{array}.$$

$(12)b = b. \qquad (12)c = -(b+c). \qquad (23)b = (12)c \qquad (23)c = c \qquad (34)c = -(b+c).$

Having explored the group actions, we can write the matrices of the representation:

$$\rho((12)) = \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}, \qquad \rho((23)) = \begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix}, \qquad \rho((34)) = \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}$$

We also saw this as a representation of S_4 by mapping to S_3.

12. Monday, September 24, 2012

We know all of the irreps of S_n, but we don't know anything about them.

Definition 12.1. The Frobenius ring is a graded ring $R = \bigoplus_n X(S_n)$, where $X(S_n) = \{f : S_n \to \mathbb{C}$ such that f is constant on conjugacy classes$\}$. $R_0 = \mathbb{C}$. Multiplication from $X(S_k) \times X(S_l) \to X(S_{k+l})$ is given by

$$x \cdot \varphi = \mathrm{Ind}_{S_k \times S_l}^{S_{k+l}} x \boxtimes \varphi,$$

where $x \boxtimes \varphi$ is the function that acts as $x \boxtimes \varphi(w,z) = x(w)\varphi(z)$.

This multiplication is associative.

Additionally, the multiplication is commutative.

$$\begin{tikzcd} S_k \times S_l \arrow[r] \arrow[d] & S_{k+l} \\ S_l \times S_k \arrow[ur] & \end{tikzcd}$$

The down arrow is isomorphism, by conjugating by the obvious permutation $\pi \in S_n$; so the ring R is commutative.

What should we take as the basis for the graded piece $X(S_n)$?

An obvious strategy would be to have a basis element for each conjugacy class, taking the value 1 on that class and 0 on all others. However, it is better to normalize.

Definition 12.2. Let λ be a partition of n. We define the basis element $\pi_\lambda \in R_n = X(S_n)$ be $z_\lambda 1_{C_\lambda}$ where

$$z_\lambda = \frac{n!}{|C_\lambda|} = |Z(g)|, \text{ for some } g \in C_\lambda = \prod_i i^{r_i} r_i!.$$

Remark 12.3. There are two ways to reach this formula. You can describe the centralizer of a permutation of this type, which will be anything that swaps blocks, or rotates a cycle.

Alternatively count the size of a conjugacy class directly by first choosing blocks, then dividing by the different ways to order blocks of equal size, then multiplying the different ways to order the elements within each block. You arrive at $|C_\lambda| = \prod \lambda_i \prod r_i!$. Dividing $n!$ by this quantity gets the desired result.

Remark 12.4. $G, C \subseteq G$ conjugacy class. $z_C = \frac{|G|}{|C|} = |Z(g)|, g \in C$.

$$\langle x, z_c 1_c \rangle = \frac{1}{|G|} \sum_{g \in G} z_C 1_C(g) x(g^{-1}) = x(g^{-1}), g \in C.$$

Remark 12.5. Let $H \subseteq G$ subgroup, with $C \subseteq H$ conjugacy class, and $\hat{C} \supseteq C$ a conjugacy class in G.

$$\langle x, \mathrm{Ind}_H^G z_C 1_C \rangle_G = \langle \mathrm{Res}_H^G x, z_C 1_C \rangle_H = x(h^{-1}), h \in C$$

Therefore, $\mathrm{Ind}_H^G z_C 1_C = z_{\hat{C}} 1_{\hat{C}}$.

Remark 12.6. When you multiply conjugacy classes on different groups, you get the right function on the conjugacy class of the bigger group containing the product. Specifically, take $C \subseteq G$, and $C' \subseteq H$, then $C \times C' \subseteq G \times H$.

$$1_C \boxtimes 1_{C'} = 1_{C \times C'} \Rightarrow Z_{C \times C'} = \frac{|G \times H|}{|C \times C'|} = z_C z_{C'} \Rightarrow z_C 1_C \boxtimes z_{C'} 1_{C'} = z_{C \times C'} 1_{C \times C'}.$$

Therefore, $\pi_\lambda \pi_\mu = \pi_{\lambda \sqcup \mu}$. So if we define $\pi_k = \pi_{(k)}$, then for any partition λ, $\pi_\lambda = \pi_{\lambda_1} \cdots \pi_{\lambda_l}$. This brings us to the following conclusion:

Theorem 12.7. *The Frobenius ring $R \cong \mathbb{C}[\pi_1, \pi_2, \ldots]$ as a graded ring, where $deg(\pi_i) = i$.*

The inner product $\langle , \rangle_{S_n}$ gives an inner product on R_n, which extends to a homogeneous $\langle , \rangle$ on R. Specifically,

$$\langle \pi_\lambda, \pi_\mu \rangle = \delta_{\lambda\mu} z_\lambda.$$

This gives us the following convenient fact for a character χ,

$$\langle \chi, \pi_\lambda \rangle = \chi(g).$$

Proposition 12.8.

$$1_{S_n} = \sum_{|\lambda|=n} \frac{1}{z_\lambda} \pi_\lambda.$$

$$\varepsilon_{S_n} = \sum_{|\lambda|=n} (-1)^{n-l(\lambda)} \frac{1}{z_\lambda} \pi_\lambda.$$

$$\Sigma_\lambda = 1_{S_{\lambda_1}} \cdots 1_{S_{\lambda_l}}.$$

$$W_\lambda = \varepsilon_{S_{\lambda_1^*}} \cdots 1_{S_{\lambda_k^*}}.$$

Proof. The first fact is a tautology.

Taking a permutation $w \in C_\lambda$, $\mathrm{sign}(w) = (-1)^{\text{even parts of } \lambda}$. The number of even parts of $\lambda \equiv l(\lambda) - \#$ odd parts $\equiv l(\lambda) - n \mod 2 \equiv n - l(\lambda) \mod 2$. ☐

Remark 12.9. The irreducible characters χ_λ of S_n (i.e. the character of the representation V_λ), where $n = |\lambda|$ satisfy the following;

$$\langle \chi_\lambda, \chi_\mu \rangle = \delta_{\lambda\mu}.$$

$$\Sigma_\mu = \chi_\mu + \sum_{\lambda > \mu} K_{\lambda\mu} \chi_\lambda.$$

$$W_\mu = \chi_\mu + \sum_{\lambda < \mu} M_{\lambda\mu} \chi_\lambda.$$

where $K_{\lambda\mu}$ and $M_{\lambda\mu}$ are positive integers, and "<" is the typical partial order on partitions.

Note that any two of those properties determine $\{\chi_\lambda\}$.

Remark 12.10. Define $H_k = 1_{S_k}$, and $E_k = \varepsilon_{S_k}$. Then $R = \mathbb{C}[H_1, H_2, \ldots] = \mathbb{C}[E_1, E_2, \ldots]$. Even more than that: $\mathbb{Z} \cdot \{\chi_\lambda\} = \mathbb{Z}[H_1, H_2, \ldots] = \mathbb{Z}[E_1, E_2, \ldots]$. (Homogeneous and elementary polynomials are integer bases for the symmetric functions.)

13. Friday, September 28, 2012

[Class on Wednesday missed for Yom Kippur]

Last class we defined:

$$\Lambda := \text{ Symmetric functions in } x_1, x_2, \dots$$

$$m_\lambda = x_1^{\lambda_1} \cdots x_l^{\lambda^l} + \cdots.$$

$$e_k = \sum_{i_1 < \cdots < i_k} x_{i_1} \cdots x_{i_k} = m_{(1^k)}.$$

$$e_\lambda = e_{\lambda_1} \cdots e_{\lambda_l}.$$

$$p_k = \sum x_i^k = m_{(k)}.$$

$$p_\lambda = p_{\lambda_1} \cdots p_{\lambda_l}.$$

We proved that $\{m_\lambda\}, \{e_\lambda\}, \{p_\lambda\}$ are bases of Λ.

We now define:

$$h_k = \sum_{|\lambda| = k} m_\lambda.$$

$$h_\lambda = h_{\lambda_1} \cdots h_{\lambda_l}.$$

$$H(z; \underline{x}) = H(z) := \sum_{n=0} z^n h_n = \prod_i \frac{1}{1 - z x_i},$$

since $\prod_i \frac{1}{1-zx_i} = \sum_n z^n x_i^n$, so the degree n part will result from all partitions of n into the contributing variables.

$$E(z; \underline{x}) := \sum_{n=0}^{\infty} z^n e_n = \prod_i (1 + z x_i).$$

Observe:

$$E(z) = \frac{1}{H(-z)} \Leftrightarrow E(z) H(-z) = 1.$$

This gives us an identity in the coefficients:

$$\sum_{k+l=n} (-1)^l e_k h_l = \delta_{n0}.$$

(we define $e_0 = h_0 = 1$ by convention.)

This in turn gives us a recursive formula for each! So that,

$$h_k \in \mathbb{C}[e_1, \dots, e_k], \qquad e_k \in \mathbb{C}[h_1, \dots, h_k].$$

i.e. $\mathbb{C}[e_1, \dots, e_k] = \mathbb{C}[h_1, \dots, h_k]$ for all k.

This implies that $\{h_\lambda\}$ is a basis.

Remark 13.1. Additionally, the formula for the h_k's in terms of the e_k's is the same was the formula for the e_k's in terms of the h_k's! This leads us to define the involution $\omega : \Lambda \to \Lambda$, such that

$$\omega(e_k) = h_k \Rightarrow \omega(h_k) = e_k \Rightarrow \omega^2 = id.$$

Now we consider a generating function for the power sums:

$$\sum_{k=1}^{\infty} p_k \frac{z^k}{k} = \sum_{k,i} \frac{x_i^k z^k}{k} = \sum_i \ln \frac{1}{1 - z x_i} = \ln(\prod_i \frac{1}{1 - z x_i}) = \ln H(z) = -\ln E(-z).$$

This gives us the formula:

$$H(z) = \exp \sum_{k=1}^{\infty} p_k \frac{z^k}{k}.$$

Mapping the log above via the involution ω, we obtain

$$\ln E(z) = \sum_{k \geq 1} p_k \frac{(-z)^k}{k} = \sum_{k \geq 1} (-1)^{k-1} p_k \frac{z^k}{k}.$$

This gives us the identity:

$$\omega(p_k) = (-1)^{k-1} p_k \Rightarrow \omega(p_\lambda) = (-1)^{|\lambda| - l(\lambda)} p_\lambda.$$

This sign coefficient is the sign of ω as an element of $C_\lambda \subset S_n$.

$$\exp \sum_{k=1}^{\infty} p_k \frac{z^k}{k} = \prod_{k=1}^{\infty} \exp(p_k \frac{z^k}{k}) = \prod_{k=1}^{\infty} \left(\sum_{r=0}^{\infty} p_k^r \frac{z^{kr}}{k^r r!} \right).$$

Observe that:

$$h_n = \sum_{\lambda = (1^{r_1}, 2^{r_2}, \dots)} p_1^{r_1} p_2^{r_2} \cdots / 1^{r_1} 2^{r_2} \cdots r_1! r_2! \cdots = \sum_{|\lambda| = n} \frac{p_\lambda}{z_\lambda},$$

where z_λ is the size of the conjugacy class of λ.
By the involution this means that

$$e_n = \sum_{|\lambda| = n} \frac{(-1)^{n - l(\lambda)} p_\lambda}{z_\lambda}.$$

We now have a map, $F : R \xrightarrow{\sim} \Lambda$, sending

$$\pi_\lambda \mapsto p_\lambda$$

$$1_{S_k} = H_k \mapsto h_k.$$

$$\chi_{\Sigma_\lambda} = H_\lambda \mapsto h_\lambda.$$

$$\varepsilon_{S_k} = E_k \mapsto e_k.$$

$$\chi_{W_\lambda^*} = E_\lambda \mapsto e_\lambda.$$

$$- \otimes \varepsilon \mapsto \omega.$$

$$\langle \, , \, \rangle \mapsto \text{ Hall inner product }.$$

Definition 13.2. The Hall Inner Product is defined taking the following identity in R:

$$\langle \pi_\lambda, \pi_\mu \rangle = \delta_{\lambda\mu} z_\lambda,$$

and mapping to the corresponding identity in Λ:

$$\langle p_\lambda, p_\mu \rangle := \delta_{\lambda\mu} z_\lambda,$$

and simply extend linearly.

13.1. **Cauchy Identity.** Let V be a finite dimensional vector space, and $\langle\ ,\ \rangle : V \otimes V \to k$ a non-degenerate bilinear form. Then:

$$\langle\ ,\ \rangle \to V \xrightarrow{\sim} V^*, v \mapsto \langle -, v \rangle.$$

So,

$$V \otimes V^* \cong \operatorname{End}_k(V).$$

which contains the identity. In this map, we have:

$$v \otimes \lambda \mapsto \varphi(x) = \lambda(x)v.$$

Let $v_1, \ldots, v_n$ be a basis of V, and $\xi_1, \ldots, \xi_n$ be the dual basis in V^*. Then,

$$\sum_i v_i \otimes \xi_i \mapsto 1 \in \operatorname{End}(V).$$

This means that $\varphi(x) = \sum v_i \xi_i(x) = x$.

So, as a general fact, if $\{v_i\}$ and $\{w_i\}$ are bases of V dual w.r.t. $\langle\ ,\ \rangle$ i.e. $\langle v_i, w_j \rangle = \delta_{ij}$, this gives an identity element $\sum_i v_i \otimes w_i \in V \otimes V \cong V \otimes V^*$.

So using the inner product from earlier, we obtain a special element

$$\sum_{|\lambda|=n} \frac{p_\lambda \otimes p_\lambda}{z_\lambda} \in \Lambda_n \otimes \Lambda_n.$$

We obtain a map:

$$\Lambda \otimes_{\mathbb{C}} \Lambda \to \Lambda(\underline{x}, \underline{y}),$$

the symmetric functions in the two sets of variables, separately symmetric in each. In particular:

$$f \otimes g \mapsto f(\underline{x}) g(\underline{y}).$$

An obvious basis for this ring is $\{m_\lambda(\underline{x}) m_\mu(\underline{y})\}$, but we want a better one.

We observe that:

$$p_k(x)p_k(y) = \sum x_i^k \sum y_j^k = \sum_{i,j} (x_i y_j)^k := p_k(XY) = p_k(x_1y_1, x_1y_2, x_2y_1, \ldots).$$

Then, for the power sum tensor from earlier:

$$\sum_{|\lambda|=n} \frac{p_\lambda(x)p_\lambda(y)}{z_\lambda} = \sum_{|\lambda|=n} \frac{p_\lambda(XY)}{z_\lambda} = h_n(XY).$$

Then,

$$\sum_\lambda t^{|\lambda|} \frac{p_\lambda(x)p_\lambda(y)}{z_\lambda} = \sum_n t^n h_n(XY) = H(t; XY) = \prod \frac{1}{1 - t x_i y_j}.$$

This implies that:

$$\sum_\lambda \frac{p_\lambda(x)p_\lambda(y)}{z_\lambda} = \prod_{i,j} \frac{1}{1 - x_i y_j}.$$

This leads us to the Cauchy identity:

Theorem 13.3 (Cauchy Identity). *For p_λ,*

$$\sum_{|\lambda|=n} \frac{p_\lambda(x)p_\lambda(y)}{z_\lambda} = \prod_{i,j} \frac{1}{1 - x_i y_j}.$$

Proposition 13.4. *Homogeneous bases $\{u_\lambda\}, \{v_\lambda\}$ of Λ satisfy*

$$\langle u_\lambda, v_\mu \rangle = \delta_{\lambda\mu} \Leftrightarrow \sum_\lambda u_\lambda(x) v_\lambda(y) = \prod_{i,j} \frac{1}{1 - x_i y_j}.$$

14. Friday, October 5, 2012

[Missed class on Monday and Wednesday for Sukkot]

We have the following bases of Λ:

$$m_\lambda, p_\lambda, e_\lambda, h_\lambda, s_\lambda.$$

We know:

$$h_\mu = \sum_{\lambda \geq \mu} K_{\lambda\mu} s_\lambda, \qquad K_{\lambda\mu} = |SSYT(\lambda, \mu)|.$$

$$e_\mu = \sum_{\lambda \geq \mu} s_{\lambda^*}$$

$$e_{\mu^*} = \sum_{\lambda^* \leq \mu^*} K_{\lambda^*\mu^*} s_\lambda.$$

Corollary 14.1.

$$F : R \to \Lambda, \qquad F(x_\lambda) = s_\lambda.$$

Corollary 14.2.

$$\langle s_\lambda, s_\mu \rangle = \delta_{\lambda\mu}.$$

Corollary 14.3.

$$\chi_\lambda(w_\mu) = \langle s_\lambda, p_\mu \rangle.$$

Corollary 14.4.

$$p_\mu = \sum_{|\lambda|=n} \chi_\lambda(w_\mu) s_\lambda.$$

$$\mathrm{Ind}_{S_\lambda}^{S_n} 1 = \chi_\lambda + \sum_{\mu < \lambda} (?) \chi_\mu.$$

$$\mathrm{Ind}_{S_{\lambda^*}}^{S_n} \varepsilon = \chi_\lambda + \sum_{\mu > \lambda} (?) \chi_\mu.$$

This implies that the transition matrix is both upper and lower triangular, which means that it must be unique.

All but $\{p_\lambda\}$ are $\mathbb{Z}$-bases of $\Lambda_\mathbb{Z}$.

Corollary 14.5.

$$\chi_\lambda(w_\mu) \in \mathbb{Z}.$$

This is actually an unusual property for finite groups.

Corollary 14.6.

$$\dim(V_\mu)^{S_\lambda} = \langle \chi_\lambda, Ind_{S_m}^{S_n} 1 \rangle = K_{\lambda\mu}.$$

Corollary 14.7.

$$\sum_\lambda s_\lambda(x) s_\lambda(y) = \prod_{i,j} \frac{1}{1 - x_i y_j}.$$

Corollary 14.8.

$$\omega(s_\lambda) = s_{\lambda^*}.$$

Corollary 14.9.

$$K_{\lambda\mu} = \langle s_\lambda, h_\mu \rangle.$$

$$\Rightarrow s_\lambda = \sum_\mu K_{\lambda\mu} m_\mu = \sum_{T \in SSYT(\lambda)} x^T.$$

where $x^T = \prod_{a \in \lambda} x_{T(a)}$.

In particular, $\dim V_\lambda = K_{\lambda,(1^n)} = |SYT(\lambda)|$.

Example 14.10. Consider $V_{\ydiagram{2,3}}$. We list all of the Standard Young Tableaux:

$$\begin{ytableau} 4 & 5 \\ 1 & 2 & 3 \end{ytableau}, \begin{ytableau} 2 & 4 \\ 1 & 3 & 5 \end{ytableau}, \begin{ytableau} 2 & 5 \\ 1 & 3 & 4 \end{ytableau}, \begin{ytableau} 3 & 5 \\ 1 & 2 & 4 \end{ytableau}, \begin{ytableau} 3 & 4 \\ 1 & 2 & 5 \end{ytableau},$$

So $\dim V_{\ydiagram{2,3}} = 5$.

14.1. **New Basis (?)** Recall: V_λ sits inside $\mathrm{Ind}_{S_{\lambda^*}}^{S_n} \varepsilon = \mathbb{C}\cdot\{\text{column antisymmetric tabloids}\}$.

$C(T_0)$ = column tabloid given by T_0.
$R(T_0)$ = row tab loud given by T_0.
Note that s_λ is the stabilizer of $R(T_0)$.
$V_\lambda \subseteq \mathbb{C}\cdot\{\text{column antisymmetric tabloids}\}$, has an element $v_0 = \sum_{w\in s_\lambda} w\cdot C(T_0)$.

Example 14.11. $\lambda = (2,2)$.

$$v_0 = \begin{array}{c|c} 3 & 4 \\ 1 & 2 \end{array} + \begin{array}{c|c} 3 & 4 \\ 2 & 1 \end{array} + \begin{array}{c|c} 4 & 3 \\ 1 & 2 \end{array} + \begin{array}{c|c} 4 & 3 \\ 2 & 1 \end{array} \neq 0.$$

Proposition 14.12. $v_0 \neq 0$.

Proof. Column tabloids $wC(T_0)$ are distinct (allowing for the signs), hence linearly independent. $\square$

Let $S = w(T_0)$, map $: \lambda \to \{1,\dots,n\}$, i.e. S is a filling with no constraints. Then,

$$w(v_0) = \text{“}C(wT_0)\text{”} = \sum_{R(S')=R(S)} C(S')$$

$$\begin{tikzcd} \mathrm{Ind}_{S_\lambda}^{S_n} 1 \arrow[rr] \arrow[dr] & & \mathrm{Ind}_{S_{\lambda^*}}^{S_n} \varepsilon \\ & V_\lambda \arrow[ur] & \end{tikzcd}$$

Indeed,

$$\sum_{R(S')=R(S)} C(S') = \text{“}V_{R(S)}\text{”}.$$

These span V_λ.

Theorem 14.13. *The elements $V_T = w(v_0)$ where $T = w(T_0)$ for $T \in SYT(\lambda)$ are linearly independent, hence a basis of V_λ.*

Lemma 14.14. *There exists a total ordering on $\{S : \lambda \to \{1,\dots,n\}\}$, such that in each row orbit, the one with increasing rows is least, and in each column orbit the one with increasing columns is greatest.*

In other words, we read the tableaux from northwest to southeast.

Example 14.15.

$$\begin{ytableau} 4 & 1 \\ 2 & 3 & 5 \end{ytableau} \quad \to \quad 42135.$$

$$\begin{ytableau} 4 & 3 \\ 2 & 1 & 5 \end{ytableau} \quad \to \quad 42315.$$

Consider a term $\pm C_S$ in V_T always has $S \supseteq T$. Order $SYT(\lambda)$ using $\prec$. The matrix giving coefficient of C_S in V_T for $S, T \in SYT$ is upper triangular.

15. Wednesday, October 10

[Class on Monday missed for Shemini Atzeret.]

15.1. **Affine Algebraic Varieties over $\mathbb{C}$.** Let $X \subseteq \mathbb{C}^n$ be the zero locus of some polynomials $f \in \mathbb{C}[x_1, \dots, x_n]$.

We define the ideal $\mathcal{I}(X) = \{f(\underline{x}) \in \mathbb{C}[\underline{x}] : f|_X = 0\}$.

Define $\mathbb{C}[\underline{x}]/\mathcal{I}(X) = \mathcal{O}(X)$ as the ring of regular functions on X = the subring of $X^{\mathbb{C}}$ generated by the x_i's.

Example 15.1. GL_n is the vanishing locus of $\det M * t - 1$, where t is a dummy variable. Specifically, $\mathcal{O}(GL_n) = \mathbb{C}[x_{11}, \dots, x_{nn}, t]/(t \cdot \det(\underline{x}) - 1)$.

Example 15.2. SL_n does not need the dummy variable. $\mathcal{O}(SL_n) = \mathbb{C}[x_{11}, \dots, x_{nn}]/(\det(\underline{x}) - 1)$.

Let $f : X \to Y$ be a morphism if $f^{\#} : Y^{\mathbb{C}} \to X^{\mathbb{C}}$, sending $\alpha \mapsto \alpha \circ f$ sends $\mathcal{O}(Y)$ into $\mathcal{O}(X)$.

This means that if these varieties are embedded into affine space, the map of spaces is a polynomial map.

Theorem 15.3.

$$(\textit{Morphisms } X \to Y) \longleftrightarrow (\mathbb{C}\textit{-algebra homomorphisms } \mathcal{O}(Y) \to \mathcal{O}(X)).$$

We have a commutative diagram:

$$\begin{tikzcd}
\mathcal{O}(Y) \arrow[r] & \mathcal{O}(X) \\
\mathbb{C}[y_1, \dots, y_m] \arrow[u] \arrow[r] & \mathbb{C}[x_1, \dots, x_m] \arrow[u]
\end{tikzcd}$$

An algebraic group G is a group and a variety such that $\mu : G \times G \to G$ and $(\cdot)^{-1} : G \to G$, $(1 : pt \to G)$ are morphisms.

Remark 15.4. Any projective algebraic group is abelian. Any arbitrary algebraic group must factor through the affine algebraic group. Therefore, from the perspective of representation theory, affine algebraic groups are the only ones of interest.

An algebraic (rational/regular) representation is a morphism of algebraic groups $G \to GL_n$ or $G \to GL(V)$, where $V \cong \mathbb{C}^n$

Remark 15.5. A morphism of algebraic groups is a morphism of varieties and a group homomorphism (one does not imply the other).

Remark 15.6. Let $\rho : G \to GL(V)$ be an algebraic representation, with $W \subseteq V$ a G-invariant subspace. Pick a basis of V starting with a basis of W. Then,

$$\rho(g) = \left(\begin{array}{c|c} \alpha(g) & \gamma(g) \\ \hline 0 & \beta(g) \end{array}\right),$$

where $\alpha : G \to GL(W), \beta : G \to GL(V/W)$. This shows that $W, V/W$ are algebraic.

Remark 15.7. Let V, W be algebraic representations of G. $\alpha : G \to GL(V), \beta : G \to GL(W)$. Then $V \oplus W$ has the representation

$$\rho(g) = \left(\begin{array}{c|c} \alpha(g) & 0 \\ \hline 0 & \beta(g) \end{array}\right).$$

Therefore, $V \oplus W$ is algebraic.

Remark 15.8. Note that for $V \otimes W$, we can take

$$\rho(g)_{(i,i'),(j,j')} = \alpha_{ij}(g)\beta_{i'j'}(g).$$

Therefore, $V \otimes W$ is algebraic.

All the familiar constructions associated to representations, e.g. $V^*, \mathrm{Hom}(V,W), S^nV, \wedge^n V \subset V^{\otimes n}$, are algebraic.

We observed that the group multiplication $\mu : G \times G \to G$ corresponds to a morphism of varieties

$$\mathcal{O}(G) \to \mathcal{O}(G \times G) = \mathcal{O}(G) \otimes \mathcal{O}(G).$$

(Note that $\mathcal{O}(X \times Y) = \mathcal{O}(X) \otimes_{\mathbb{C}} \mathcal{O}(Y)$.)

Every part of the group structure induces a "co-" structure on $\mathcal{O}(G)$.

$$\begin{aligned} \mu : G \times G \to G \qquad &\leftrightarrow \qquad \Delta : \mathcal{O}(G) \to \mathcal{O}(G) \otimes \mathcal{O}(G). \\ (\cdot)^{-1} : G \to G \qquad &\leftrightarrow \qquad S : \mathcal{O}(G) \to \mathcal{O}(G). \\ 1 : pt \to G \qquad &\leftrightarrow \qquad ev_1 : \mathcal{O}(G) \to \mathbb{C}. \end{aligned}$$

These maps make $\mathcal{O}(G)$ a Hopf algebra.

[For G finite, $\mathcal{O}(G) = G^{\mathbb{C}} = (kG)^*$.]

16. Friday, October 12, 2012

Consider the additive and multiplicative groups G_a and G_m. We want a representation that is also a homomorphism of algebraic groups.

$$G \to GL(V). \qquad G \times V \to V.$$

This induces a ring homomorphism:

$$S(V^*) = \mathcal{O}(V) \to \mathcal{O}(V) \otimes \mathcal{O}(G) = S(V^*) \otimes \mathcal{O}(G).$$

$S(V^*)$ is the symmetric algebra generated by V^*. Restricting to V^* we claim that the image will be contained in $V^* \otimes \mathcal{O}(G)$ as below:

$$V^* \to V^* \otimes \mathcal{O}(G).$$

We have the following diagrams that $\mathcal{O}(G)$ satisfies:

$$\begin{tikzcd} \overset{G\times G\times G}{(g,h,v)} \arrow[r, "1_G \times \rho"] \arrow[d, "\mu_G \times 1_V"'] & \overset{G\times V}{(g,hv)} \arrow[d, "\rho"'] \\ \overset{G\times V}{(gh,v)} \arrow[r, "\rho"'] & \overset{V}{ghv} \end{tikzcd}$$

$$\begin{tikzcd} V \arrow[d] \arrow[dr, "1_V"] & \\ G \times V \arrow[r, "\rho"'] & V \end{tikzcd}$$

$$\begin{tikzcd} V^* \arrow[r, "\sigma"] \arrow[d, "\sigma"'] & V^* \otimes \mathcal{O}(G) \arrow[d, "1_{V^*} \otimes \Delta^{op}"'] \\ V^* \otimes \mathcal{O}(G) \arrow[r, "\sigma \otimes 1_{\mathcal{O}(G)}"'] & V^* \otimes \mathcal{O}(G) \otimes \mathcal{O}(G) \end{tikzcd}$$

where $\mathcal{O}(G) \xrightarrow{\Delta} \mathcal{O}(G) \otimes \mathcal{O}(G)$.

$$\begin{tikzcd} V^* \arrow[d, "\sigma"'] \arrow[dr, "1_V"] & \\ V^* \otimes \mathcal{O}(G) \arrow[r, "1_{V^*} \otimes e_{V_1}"'] & V \end{tikzcd}$$

Definition 16.1. Right coaction of coalgebra $\mathcal{O}(G)$, $\Delta : \mathcal{O}(G) \to \mathcal{O}(G) \otimes_{\mathbb{C}} \mathcal{O}(G)$ with counit $\mathcal{O}(G) \to \varepsilon$ on V, means

$$\sigma : V \to V \otimes \mathcal{O}(G)$$

such that the diagrams commute. If V is a left comodule, then V^* is a right comodule. Starting with the map:

$$G \times V \to V \qquad \rightsquigarrow \qquad V^* \times G \to V^* \qquad \rightsquigarrow \qquad V \to V \otimes \mathcal{O}(G) \to V.$$

$\mathcal{O}(G)^*$ is an algebra. We have a comultiplication:

$$\Delta : \mathcal{O}(G) \to \mathcal{O}(G) \otimes \mathcal{O}(G).$$

which induces:

$$\Delta^* : \mathcal{O}(G)^* \leftarrow \mathcal{O}(G)^* \otimes \mathcal{O}(G)^*$$

with unit $ev_1 : \mathcal{O}(G) \to \mathbb{C}$.

The map

$$g \mapsto ev_g : \mathcal{O}(G) \to \mathbb{C}$$

is a homomorphism from $G \to (\mathcal{O}(G)^*)^*$.

If G is finite, then $\mathcal{O}(G)^* = kG$. For infinite groups G, this space is huge. In some sense, $\mathcal{O}(G)^*$ is the double dual of $\mathbb{C}G$, but it is not the whole double dual.

Example 16.2. Let $G_m = (\mathbb{C}^\times, \cdot) = GL_1(\mathbb{C})$.

$$\mathcal{O}(G_m) = \mathbb{C}[t, t^{-1}].$$

$$G_m \times G_m \to G_m, \qquad s, t \mapsto st.$$

$$\mathbb{C}[t, t^{-1}] \to \mathbb{C}[s, t, s^{-1}, t^{-1}] = \mathbb{C}[t, t^{-1}] \otimes \mathbb{C}[t, t^{-1}] = \mathbb{C}[s, s^{-1}] \otimes \mathbb{C}[t, t^{-1}], \qquad t \mapsto st = t \otimes t.$$

The comultiplication is $\Delta(t) = t \otimes t, \Delta(t^{-1}) = t^{-1} \otimes t^{-1}$.

If G_m acts on V, then we have a right comodule structure:

$$\sigma : V \to V \otimes \mathbb{C}[t, t^{-1}].$$

$$a \in G_m = \mathbb{C}^\times, \qquad a \cdot v = \langle \sigma(v), ev_a \rangle = \langle \sum v_i \otimes f_i(t), ev_a \rangle = \sum v_i f_i(a).$$

$\langle a_m, f \rangle$ = the coefficient of t^m in $f(t)$. Specifically, $a_m \in \mathcal{O}(G_m)^*$ and $a_m \mapsto E_m \in \operatorname{End} V$.

$\langle a_m \cdot a_n, f \rangle$ = the coeficient of $s^m t^n$ in $f(st)$.

If $f = \sum a_k t^k$, then $f(st) = \sum a_k s^k t^k$. So, $a_m \cdot a_n = 0$ if $m \neq n$, and $a_n \cdot a_n = a_n$ for all $n \in \mathbb{Z}$.

This implies that $V = \bigoplus_{n \in \mathbb{Z}} V_n$, where a_n = projection on V_n. Explicitly:

$$ev_t = \sum_{n \in \mathbb{Z}} a_n t^n.$$

For any finite f, this is a finite sum. The G-action on each V_n is $t \mapsto t^n$.

In this way, G_m is like a finite group; i.e. Maschke's Theorem holds – the algebraic representations are completely reducible.

17. Monday, October 15, 2012

17.1. **Review of G_m.** Recall that $G_m = (\mathbb{C}^\times, \cdot), \mathcal{O}(G) = \mathbb{C}[t, t^{-1}]$.

A representation $\rho : V \to V \otimes \mathcal{O}(G)$ acts as

$$gv = \rho(v)(g) = \langle \rho(v), \varepsilon_g \rangle.$$

where $\varepsilon_g = ev_g \in \mathcal{O}(G)^*$. Then, we can write:

$$\rho(v) = \sum v_i \otimes f_i, \qquad gv = \sum f_i(g) v_i.$$

$\mathcal{O}(G)^*$ is an algebra containing G in its group of units as $g \mapsto \varepsilon_g$.

In the case of G_m we distinguished special elements of $\mathcal{O}(G)^*$. Let $\langle e_n, f(t) \rangle$ = coefficient of t^n in $f(t)$. $e_n \in \mathcal{O}(G_m)^*$, for all $n \in \mathbb{Z}$.

$$e_k e_n = \delta_{kn} e_n.$$

$$V = \bigoplus_{n \in \mathbb{Z}} V_n.$$

$$e_n = \text{ projection onto } V_n.$$

G_m acts on V_n by $t(v) = t^n v$, i.e. it has one irrep, which is one-dimensional with matrix (t^n) for each $n \in \mathbb{Z}$. Every comodule is a direct sum of these.

17.2. **Additive group G_a.** The group $G_a = (\mathbb{C}, +)$. The ring of functions is $\mathcal{O}(G_a) = \mathbb{C}[x]$. Define $a_m \in \mathcal{O}(G_a)^*$ for each $m \in \mathbb{N}$, by $\langle a_m, f(x) \rangle$ = the coefficient of x^m in $f(x)$.

The multiplication is

$$G_a \times G_a \to G_a, \qquad (x, y) \mapsto x + y.$$

The comultiplication is

$$\Delta : \mathcal{O}(G) \to \mathcal{O}(G) \otimes \mathcal{O}(G), \qquad x \mapsto x \otimes 1 + 1 \otimes x.$$

(In other words, x is a primitive element of the Hopf algebra.)

It acts on powers of x by

$$\Delta(x^m) = (x \otimes 1 + 1 \otimes x)^m = \sum \binom{m}{k} x^k \otimes x^{m-k}.$$

(This means that x^m is a grouplike element.)

We define the product $\cdot$ by:

$$\langle a_k \cdot a_l, f \rangle := \langle a_k \otimes a_l, \Delta f \rangle$$

the coefficient of $x^k y^l$ in $f(x+y)$.

$$a_k a_l = \binom{k+l}{k} a_{k+l}.$$

This implies that

$$a_k = \frac{a_1^k}{k!}.$$

The whole action of G_a on V is determined by the operator $a_1 : V \to V$.

If $v_1, \dots, v_n$ are a basis of V, then

$$\rho(v_i) = \sum_j v_j \otimes f_{ij}(x).$$

So, there is an n such that a_n kills v. Therefore, a_1 is a nilpotent operator.

(If V is a comodule but $\dim V = \infty$, a_1 is locally nilpotent, i.e. for all $v \in V$, there is an n such that $a_1^n v = 0$.)

We can describe

$$x \cdot v = \langle \rho(v), ev_x \rangle.$$

In other words, "$ev_x = \sum_{m\geq 0} a_m x^m$". This appears to be an infinite sum, but by nilpotency, depending on x, this will be a polynomial. So,

$$ev_x = \sum_{m\geq 0} \frac{a_1^m}{m!} x^m = e^{a_1 x}.$$

This means that we can describe G_a-modules as finite dimensional vector spaces V with nilpotent endomorphism $a \in \text{End}(V)$.

There exist subspaces $0 \subset V_1 \subset \cdots \subset V_d = V$ such that $aV_k \subseteq V_{k-1}$ i.e. a acts as 0 on V_k/V_{k-1}. G_a acts trivially on each factor.

So for contrast: G_m: there is an irrep. V_m for each $m \in \mathbb{Z}$. All comodules are completely reducible. (This is a reductive group.)

G_a: There is a unique irrep – the trivial representation. All nontrivial reps are not completely reducible. Example:

$$x \mapsto \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}.$$

which is *unipotent* (i.e. if you subtract I, the result is nilpotent).

Theorem 17.1. *Every affine algebraic group G over $\mathbb{C}$ has a unique maximal unipotent normal subgroup U, and G/U is reductive.*

Example 17.2. Let $G = B = \{\text{upper-triangular matrices } A \in GL_2\}$.

$$\begin{pmatrix} t & a \\ 0 & u \end{pmatrix} \qquad t, u \neq 0. \qquad \Rightarrow \mathcal{O}(G) = \mathbb{C}[t^{\pm 1}, u^{\pm 1}, a].$$

The normal subgroup is

$$U = \left\{ \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \right\} \cong G_a.$$

The following exact sequence of groups

$$0 \to U \hookrightarrow B \to G_m^2 \to 0$$

implies that $B/U = G_m^2$, which is reductive.

18. Wednesday, October 17, 2012

18.1. **Classical Reductive Groups $/\mathbb{C}$.** This will be a less formal review.

(1) GL_n.
(2) SL_n. (group of matrices of determinant 1)
(3) $SO_n = \{X \in SL_n : XX^T = I\} \subset O_n$ (group of matrices that preserve some bilinear form.) We take only the det 1 piece, so that it will be connected; O_n has two connected components.
(4) $Sp_{2n} = \{X \in GL_{2n} : X \text{ preserves } \langle,\rangle : \langle Xv, Xw\rangle = \langle v, w\rangle\ \forall v, w \in V\}$, the symplectic group. On $\mathbb{C}^{2n}$, there are non-degenerate antisymmetric bilinear forms (i.e. $\langle v, w\rangle = \langle w, v\rangle$).
(5) Exceptional ones: $G_2, F_4, E_{6,7,8}$.
(6) Relatives:
    (a) $\mu_n = \{e^{2\pi i k/n} \cdot I\}$ is a normal subgroup of $SL_n \subseteq GL_n$. Modding out by that subgroup gives $PSL_n = PGL_n = SL_n/\mu_n$.
    (b) $SO_{2n}/\{\pm 1\}$.
    (c) $Sp_{2n}/\{\pm 1\}$.

Remark 18.1. The surjection $SL_n \twoheadrightarrow PSL_n$ induces a surjection of the duals to the rings of regular functions:

$$\begin{tikzcd} \mathcal{O}(PSL_n)^* & \mathcal{O}(PSL_n)^* \arrow[l] \\ PSL_n \arrow[u] & SL_n \arrow[l] \arrow[u] \end{tikzcd}$$

Remark 18.2. Actually, the orthogonal groups are quotients of larger groups called the Spin groups Spin_n.

$$0 \to \{\pm 1\} \to \mathrm{Spin}_n \twoheadrightarrow SO_n \to 0.$$

- Spin_3 is isomorphic to $SL_2(\mathbb{C})$, which is a double cover of SO_3.
- $\mathrm{Spin}_4 \cong SL_2 \times SL_2$.
- $\mathrm{Spin}_5 \cong Sp_4$.

Eventually, we will hope to classify all of the reductive groups based on combinatorial data.

Definition 18.3. Let $X \subseteq \mathbb{C}^n$ be an algebraic variety, with $p \in X$. The tangent space T_pX. Let $f \in \mathbb{C}[z_1, \dots, z_n]$ be a function in I_X, i.e. such that $f(p) = 0$.

We say that $p + tu$ is a tangent line to X if $f(p + tu) \in t^2\mathbb{C}[t]$ for all $f \in I_X$, or u is a tangent vector.

$$T_pX = \{\text{tangent vectors to } X \text{ at } p\}.$$

In particular,

$$\frac{d}{dt}|_{t=0} f(p + tu) = \partial_u f|_p$$

depends only on $f|_X$ for $u \in T_pX$.

Given coordinates $u = (c_1, \dots, c_n)$, we set $\partial_u z_i = c_i$. This gives an injection:

$$\begin{array}{ccc} T_pX & \hookrightarrow & \mathcal{O}(X)^* \\ u & \longrightarrow & \delta : \delta(f) = \partial_u f|_p \end{array}$$

$$\delta(fg) = \delta(f)g(p) + f(p)\delta(g).$$
$$\delta(1) = 0.$$

WLOG, we may assume that p is the origin in $\mathbb{C}^n$. Then $f \in I_X$ is $\sum a_i z_i +$ higher terms.

$$\partial_u f = \sum a_i c_i \qquad \text{if } u = (c_1, \dots, c_n).$$

(∂_u is defined as $\sum c_i \partial_{z_i}$.) u will be a tangent vector if each of these are 0.

Let $\mathfrak{m}_p$ = ideal of p in $\mathcal{O}(X) = \ker \mathcal{O}(X) \to \mathbb{C}$, the evaluation map. Then, as elements of $\mathfrak{m}_p/\mathfrak{m}_p^2$, coordinates z_i satisfy $\sum a_i z_i = 0$.

The values of δ on the z_i determine δ.

Given $\lambda : \mathfrak{m}_p/\mathfrak{m}_p^2 \to \mathbb{C}$, we obtain $\delta : \mathcal{O}(X) \to \mathbb{C}$, for which $\delta|_{\mathfrak{m}_p^2} = 0$ and $\delta(1) = 0$, simply by observing

$$\mathcal{O}(X)/\mathfrak{m}_p^2 = \mathbb{C} \oplus \mathfrak{m}_p/\mathfrak{m}_p^2$$

and having λ act on the right summand, and the zero map on the left summand.

$$f = f(p) + \tilde{f} \qquad \tilde{f} \in \mathfrak{m}_p.$$
$$g = g(p) + \tilde{g} \qquad \tilde{g} \in \mathfrak{m}_p.$$

Then,

$$fg = f(p)g(p) + f(p)\tilde{g} + \tilde{f}g(p) + \tilde{f}\tilde{g}.$$

(the last term is an element of $\mathfrak{m}_p^2$)

$$\Rightarrow \delta(fg) = f(p)\delta(g) + \delta(f)g(p).$$

$$T_pX \cong \{\text{point derivations } \mathcal{O}_X \to \mathbb{C} \text{ at } p\} \cong (\mathfrak{m}_p/\mathfrak{m}_p^2).$$

We can define $D = \mathbb{C}[t]/(t^2)$ the set of "dual numbers" $= \mathcal{O}(T)$.

There is a unique homomorphism $\varepsilon : D \to \mathbb{C}$ sending $t \mapsto 0$.

The space above is also isomorphic to

$$\{\varphi \in \mathrm{Hom}_{Alg}(\mathcal{O}(X), D) : \varepsilon \circ \varphi = ev_p\}.$$

19. Friday, October 19, 2012

We saw several definitions of the tangent space T_pX. We want to define the notion of a vector field for an algebraic variety. We want a tangent vector at each point, defined in a polynomial way.

Example 19.1. On $\mathbb{C}^n$:

$$T_p\mathbb{C}^n = \mathbb{C} \cdot \{\partial z_1|_p, \dots, \partial z_n|_p\}.$$

∂z_i is a vector field. An algebraic vector field on $\mathbb{C}^n$ will be polynomial linear combination of these:

$$\sum f_i(z)\partial z_i.$$

The other way to think about this is as an operator on the ring of functions:

$$\mathcal{O}(\mathbb{C}^n) = \mathbb{C}[z_1, \dots, z_n].$$

$$\partial : \mathcal{O}(\mathbb{C}^n) \to \mathcal{O}(\mathbb{C}^n)$$

is a derivation: $\partial(fg) = \partial f \cdot g + f \cdot \partial g$. Conversely, this also gives a way to construct algebraic vector fields from derivations. So we have a correspondence:

$$\text{Algebraic Vector fields on } \mathbb{C}^n \quad \longleftrightarrow \quad \text{Derivations } \mathcal{O}(\mathbb{C}^n) \to \mathcal{O}(\mathbb{C}^n)$$

Definition 19.2. ∂ is *tangential* to X if $\partial I_X \subset I_X$. Then, we get $\partial|_X : \mathcal{O}(X) \to \mathcal{O}(X)$. Still a derivation, hence a vector field.

This defines algebraic vector fields on X by the correspondence above.

More explicitly, take $\delta = \{\delta_p \in T_pX \text{ for each } p \in X\}$, a vector field, and $f \in \mathcal{O}(X)$.

Define the function $g(p) = \delta_p f \in X^{\mathbb{C}}$.

If $\delta f \in \mathcal{O}(X) \forall,\ f \in \mathcal{O}(X)$, call it an algebraic vector field. Therefore, it gives us a derivation $\mathcal{O}(X) \to \mathcal{O}(X)$.

In the reverse direction, a derivation $\delta \mapsto \delta_p f = (\delta f)(p)$, an algebraic vector field.

Remark 19.3. $\mathcal{O}(X)^*$ is *not* a coalgebra. For example, given $\lambda \in \mathcal{O}(X)^*$,

$$\mathcal{O}(X) \otimes \mathcal{O}(X) \xrightarrow{m} \mathcal{O}(X) \xrightarrow{\lambda} \mathbb{C}.$$

$$\lambda \circ m \in (\mathcal{O}(X) \otimes \mathcal{O}(X))^* \supsetneq \mathcal{O}(X)^* \otimes \mathcal{O}(X)^*.$$

λ is a point derivation at $p \Leftrightarrow \lambda \circ m = \lambda \otimes 1 + 1 \otimes \lambda$, where $1 = ev_p$. However, this is just a rephrasing of the Leibniz rule:

$$\lambda(fg) = \lambda(f)g(p) + f(p)\lambda(g).$$

Let G be an algebraic group, which has a natural left and right action on itself (by left and right multiplication). Then it inherits a left and right action on the ring of functions, via:

$$\begin{tikzcd} G \arrow[r, bend left] \arrow[drr, dashed, no head] & G & G \arrow[l, bend right] \arrow[dll, dashed, no head] \\ G & \mathcal{O}(G) \arrow[l, bend right] \arrow[r, bend left] & G \end{tikzcd}$$

The crossed arrows is due to contravariance. Specifically, for $g \in G, f \in \mathcal{O}(G)$,

$$g \underset{R}{\cdot} f = f(- \cdot g).$$

$$gh \underset{R}{\cdot} f = f(- \cdot gh).$$

$$h \underset{R}{\cdot} f = f(- \cdot h).$$

$$g \underset{R}{\cdot} (h \underset{R}{\cdot} f) = f(- \cdot g \cdot h).$$

$$g \underset{L}{\cdot} f = f(g^{-1} \cdot -).$$

Proposition 19.4. *Both G actions on $\mathcal{O}(G)$ have coactions.*

Taking $\rho : V \to V \otimes \mathcal{O}(G)$.

$$g \cdot v = \langle \rho(v), g \rangle$$

is an action.

$$\rho(v) = \sum v_i \otimes f_i, \qquad g \cdot v = \sum f_i(g) v_i.$$

$$\Delta : \mathcal{O}(G) \to \mathcal{O}(G) \otimes \mathcal{O}(G) = \mathcal{O}(G \times G)$$

makes $\mathcal{O}(G)$ a right $\mathcal{O}(G)$ comodule, which gives an action $G \underset{R}{\curvearrowright} \mathcal{O}(G)$.

The action is $(h, g) \mapsto f(h, g)$. Take a fixed g, apply it to a function f, and get the function $f(- \cdot g)$.

Now, $\mathcal{O}(G)^*$ acts on$\mathcal{O}(G)$ and commutes with $G \underset{L}{\curvearrowright} \mathcal{O}(G)$.

For $T \in \mathcal{O}(G)^*, T : \mathcal{O}(G) \to \mathbb{C}$.

$$T \mapsto \mathcal{O}(G) \to \mathcal{O}(G) \otimes \mathcal{O}(G),$$

then have T act on the right factor, resulting in $\mathcal{O}(G) \otimes \mathbb{C} \cong \mathcal{O}(G)$. So this is an element of $\mathrm{End}_G(\mathcal{O}(G))$.

Under this map, $ev_g \mapsto g \underset{R}{\cdot} -$. Generally, $\mathcal{O}(G)^*$ maps to left invariant operators on $\mathcal{O}(G)$.

Given $S, T \in \mathcal{O}(G)^*$,

$$ST : \mathcal{O}(G) \underset{\Delta}{\to} \mathcal{O}(G) \otimes \mathcal{O}(G) \underset{S \otimes T}{\to} \mathbb{C}.$$

$$ST \curvearrowright \mathcal{O}(G) : \mathcal{O}(G) \underset{\Delta}{\to} \mathcal{O}(G) \otimes \mathcal{O}(G) \underset{1 \otimes \Delta}{\to} \mathcal{O}(G) \otimes \mathcal{O}(G) \otimes \mathcal{O}(G) \underset{1 \otimes S \otimes T}{\longrightarrow} \mathcal{O}(G).$$

(This commutes with the same diagram replacing $1 \otimes \Delta$ by $\Delta \otimes 1$.)

All of this is another way of saying that the group multiplication is associative; this is its consequence for the coordinate rings.

20. Monday, October 22, 2012

Recall that the action $G \backsim G$ gives an action $G \curvearrowright \mathcal{O}(G)$, via $g \underset{R}{\cdot} f = f(- \cdot g)$.

This makes $\mathcal{O}(G)$ an $\mathcal{O}(G)$-comodule, coaction is Δ, gives a map $\mathcal{O}(G)^* \to \operatorname{End}_{\mathbb{C}}(\mathcal{O}(G))$ extending the G action.

This identifies $\mathcal{O}(G)^*$ with left invariant operators. The inverse is $T \mapsto ev_1 \circ T$.

Let $\delta \in \mathcal{O}(G)^*$ be a point derivation at 1, i.e. $\delta \in T_1 G$. Then, $\delta \curvearrowright \mathcal{O}(G)$ acts by $\delta \cdot f = \delta_h f(gh)$, where δ_h means that δ evaluates $f(gh)$ as a function of h (Note that while $\delta(f)$ is in the field $\delta \cdot f$ is a function).

$$\delta \cdot (ef) = \delta_h ef(gh) = \delta_h e(gh) f(g) + e(g)\delta_h f(gh) = (\delta \cdot e)f + e(\delta \cdot f).$$

Therefore, $\delta \cdot f$ is a derivation.

Lemma 20.1. *The commutator of two derivations is a derivation.*

Proof. Let $X, Y \in Der\mathcal{O}$.

$$XY(fg) = X(Y(f)g + fY(g)) = XY(f) + Y(f)X(g) + X(f)Y(g) + fXY(g).$$

$$YX(fg) = YX(f)g + Y(f)X(g) + X(f)Y(g) + fYX(g).$$

$$\Rightarrow [X,Y] = (XY - YX)(fg) = [X,Y](f)g + f[X,Y](g).$$

So, the commutator is also a derivation. □

Corollary 20.2. *For any $\delta, \varepsilon \in T_1 G \subseteq \mathcal{O}(G)^* \Rightarrow [\delta, \varepsilon] \in T_1 G$.*

Definition 20.3. Define $\mathrm{Lie}(G) := T_1 G \subset \mathcal{O}(G)^*$, which is closed under $[\,,\,]$. A lie algebra satisfies:

(1) $[X,Y] = -[Y,X]$.
(2) $[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$ (left as an exercise).

20.1. Lie Algebra.

$$\varphi : G \to H \rightsquigarrow \varphi^{\#} : \mathcal{O}(H) \to \mathcal{O}(G) \rightsquigarrow \tilde{\varphi} : \mathcal{O}(G)^* \to \mathcal{O}(H)^* \rightsquigarrow d\varphi : T_1 G \to T_{\varphi(1)} H = T_1 H.$$

The final map when considered as $d\varphi : \mathrm{Lie}\ G \to \mathrm{Lie}\ H$ is a Lie algebra homomorphism.

(The functor sending $G \to \mathrm{Lie}\ G$ is faithful on $\mathbb{C}$ and for other fields of char 0, but not in char p, consider the Frobenius map for example.)

Question 20.4. What is Lie $(GL(V)) = gl(V)$?

Let $V = \mathbb{C}^n$. End V is a vector space, $\mathbb{C}^{n^2}$.

$$T_p \mathrm{End}\ V \cong \mathrm{End}\ V.$$

$$GL(V) \hookrightarrow \mathrm{End}\ (V)$$

is an open subset, i.e. complement of the vanishing set of the determinant.

$$T_1(GL(V)) \xrightarrow{\sim} T_1 \mathrm{End}\ (V) \cong \mathrm{End}\ (V).$$

Given a matrix $A \in \mathrm{End}\ V = M_n$, we have a derivation

$$\delta_A : \mathcal{O}(GL(V)) \to \mathbb{C}, \qquad \delta_A(f) = \frac{d}{dt}\Big|_{t=0} f(I + tA).$$

Remark 20.5. δ_A is the linear term of $e^{t\delta_A} \in \mathcal{O}(G)^*[[t]]$. Given the product $m : \mathcal{O}(G) \otimes \mathcal{O}(G) \to \mathcal{O}(G)$, the following map is a primitive element:

$$\delta_A \circ m = \delta_A \otimes \mathbf{1} + \mathbf{1} \otimes \delta_A,$$

where $\mathbf{1} = ev_1$ is the unit in $\mathcal{O}(G)^*$. (The above is a rephrasing of the Leibniz rule). Then,

$$\delta_A^n \circ m = \sum_{k=0}^{n} \binom{n}{k} \delta_A^k \otimes \delta_A^{n-k}.$$

This gives us

$$e^{t\delta_A} \circ m = e^{t\delta_A} \otimes e^{t\delta_A},$$

i.e. $e^{t\delta_A} : \mathcal{O}(G) \to \mathbb{C}[[t]]$ is an algebra homomorphism.

There is another homomorphism $\mathcal{O}(GL(V)) \to \mathbb{C}[[t]]$, sending

$$f \mapsto f(e^{At})$$

where e^{At} is a formal power series of matrices, or equivalently, a matrix of formal power series – $e^{At} \in M_n(\mathbb{C}[[t]])$; additionally, $e^{At} \in GL_n(\mathbb{C}[[t]])$.

Proposition 20.6. *The two homomorphisms $\mathcal{O}(GL(V)) \to \mathbb{C}[[t]]$ are the same. Explicitly,*

$$e^{t\delta_A}(f) = f(e^{At}) \qquad \in \mathbb{C}[[t]].$$

Proof. It is sufficient to check for $f = x_{ij}$. The matrix with (i,j) entry $\delta_A(x_{ij})$ is (based on the formula from before)

$$\frac{d}{dt}\Big|_{t=0} (I + tA) = A$$

$$\delta_A^k x_{ij} = \delta_A^{\otimes k}(x^{(1)} \cdots x^{(k)})_{ij} = \frac{d}{dt_1} \cdots \frac{d}{dt_k}\Big|_{t=0} (I + t_1 A) \cdots (I + t_k A) = A^k.$$

So the matrix with entries $e^{t\delta_A}(x_{ij})$ is e^{tA} = the matrix with entries $x_{ij}(e^{tA})$. □

Then we send $\delta_A \to D_A \in \text{End}_{\mathbb{C}}(\mathcal{O}(G))$, via $e^{t\delta_A} \mapsto e^{tD_A}$, which acts by $e^{tD_A} f = f(- \cdot e^{tA})$.

21. Wednesday, October 24, 2012

Recall: We want to define Lie GL_n. GL_n has an obvious map to M_n, so $T_1 GL_n = T_1 M_n = M_n$. Given $A \in M_n$, we have a derivation:

$$\delta_A f = \frac{d}{dt}\Big|_{t=0} f(I + tA).$$

Then given that $\delta_A \in \mathcal{O}(G)^*$,

$$e^{t\delta_A} \in \mathcal{O}(G)^*[[t]] \text{ or } e^{t\delta_A} : \mathcal{O}(G) \to \mathbb{C}[[t]].$$

Alternatively, we can send $f \in \mathcal{O}(GL_n) \to f(e^{tA})$ which gives a map $\mathcal{O}(GL_n) \to \mathbb{C}[[t]]$. Recall that both homomorphisms were the same.

$$e^{tX} e^{tY} = e^{t(X+Y+\cdots)}.$$

X and Y can be linear operators, or they can be elements of any algebra.

Proposition 21.1.

$$e^{-tX} e^{t(X+Y)} e^{-tY} \equiv 1 - \frac{t^2}{2}[X,Y]) \mod (t^3).$$

Proof. The following statement is equivalent:

$$e^{-t(X+Y)} \overset{?}{\equiv} e^{tX}(1-\frac{t^2}{2}[X,Y])e^{tY} \mod (t^3).$$

$$RHS = (1+tX+\frac{t^2}{2}X^2)(1-\frac{t^2}{2}[X,Y])(1+tY+\frac{t^2}{2}Y^2)$$
$$= 1+t(X+Y)+\frac{t^2}{2}(X^2+Y^2-[X,Y]+2XY)$$

This last summand is $X^2+Y^2+YX+XY=(X+Y)^2 \Rightarrow$ the desired equation is true. □

Corollary 21.2.

$$[\delta_A,\delta_B]=\delta_{[A,B]}.$$

Definition 21.3.

$$\text{Lie } GL(V) \underset{def}{=} gl(V) = (\text{End } V, [\ ,\ ]).$$

Our convention in these examples is that X is an element of GL_n as a group, while A is an element of gl_n as a Lie group.

Example 21.4. Consider the injection $SL(V) \hookrightarrow GL(V)$. This is isomorphic to $SL_n \hookrightarrow GL_n$. The condition that $\det X = 1$ means that $\det(I+A)=1$, where

$$\det(I+A) = 1 + \text{tr } A + \text{ quadratic terms.}$$

So we need $\text{tr } A = 0$. So,

$$\text{Lie } (SL_n) = sl_n \subset gl_n = \{A : \text{tr } A = 0\}.$$

Example 21.5. Consider the injection $SO_n \hookrightarrow GL_n$, the special orthogonal group. These are the matrices

$$\{X : XX^T = I\}.$$

An element of the group $X = e^{tA}$ satisfying

$$e^{tA}e^{tA^T} = e^0 \Rightarrow e^{(A+A^T)} = I \Rightarrow A + A^T = 0.$$

Therefore,

$$\text{Lie } (SO_n) = so_n = \{A : A + A^T = 0\}.$$

Note that this relation is preserved by Lie bracket. Suppose $A=-A^T, B=-B^T$.

$$[A,B]^T = (AB-BA)^T = B^TA^T - A^TB^T = -[A^T,B^T] = -[A,B].$$

Incidentally, the fact that applying the exponential to so_n keeps us in SO_n confirms that the relation we chose was sufficient to define the variety.

Example 21.6. Now let us look at the symplectic group Sp_{2n}. Let

$$\{e_1,\dots,e_n\}, \qquad \{e_{2n},\dots,e_{n+1}\} = \{f_1,\dots,f_n\}.$$

be a dual basis, i.e.

$$\langle e_i, f_i\rangle = 1, \qquad \langle f_i, e_i\rangle = -1.$$

We need to hold on to the labeling up to $2n$, because we want to define an order to write matrices.

Given these relations, the inner product is described by the matrix:

$$J = \langle e_i, e_j \rangle = \begin{pmatrix} & & & & & 1 \\ & 0 & & & \iddots & \\ & & & 1 & & \\ & & -1 & & & \\ & \iddots & & & 0 & \\ -1 & & & & & \end{pmatrix}$$

The symplectic group is then

$$Sp_{2n} = \{X : \langle Xv, Xw \rangle = \langle v, w \rangle\}$$

Take $X = I + tA$.

$$\langle v + tAv, w + tAw \rangle - \langle v, w \rangle = 0.$$

$$\Rightarrow t(\langle Av, w \rangle + \langle v, Aw \rangle).$$

$$\Rightarrow \text{Lie } Sp_{2n} = \{A : \langle Av, w \rangle = -\langle v, Aw \rangle \forall v, w\}.$$

Since the bilinear form is given by $\langle v, w \rangle = v^T Jw$, this condition is:

$$v^T A^T Jw = -v^T JAW.$$

Observe that $-J^{-1}A^T J = A \Rightarrow A^T J = -JA$.

Define A^R to be the "Rong" transpose over the anitdiagonal. Then the set of matrices satisfying the desired property are those of the form:

$$\left(\begin{array}{c|c} A & B = B^R \\ \hline C = C^R & -A^R \end{array}\right)$$

22. Friday, October 26, 2012

22.1. **Representations of $SL_2(\mathbb{C})$.** Recall from last time that

$$sl_2 = \left\{ \begin{bmatrix} a & b \\ c & -a \end{bmatrix} \right\}$$

So, we can take a basis for the group:

$$E = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad F = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \qquad G = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

These have commutators:

$$[E, F] = H, \qquad [H, E] = 2E, \qquad [H, F] = -2F.$$

SL_2 has defining representation $\mathbb{C}^2$. So acting on the basis e_1, e_2

The Lie algebra acts as in Figure 2

SL_2 acts on $S^n \cdot \mathbb{C}^2$, polynomials of degree n in 2 variables in x, y by changing the variables via a linear transformation

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

with determinant 1.

In this case,

$$\begin{aligned} E &: \quad y \mapsto x, x \mapsto 0 \\ F &: \quad x \mapsto y, y \mapsto 0 \\ H &: \quad x \mapsto x, y \mapsto -y. \end{aligned}$$

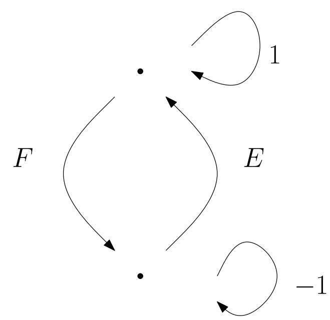


FIGURE 2. Action of the Lie Group on the Vector Space

Viewing these as derivations:

$$E = x\partial y, \qquad F = y\partial x, \qquad H = x\partial x - y\partial y.$$

So, given a polynomial $f(x,y) \in S\mathbb{C}^2 \mapsto g(x,y,a,b,c,d) \in S\mathbb{C}^2 \otimes \mathcal{O}(SL_2)$.

Example 22.1. Consider the second symmetric power $S^2\mathbb{C}^2$. The action of the Lie group on these monomials is pictured in Figure **??**.

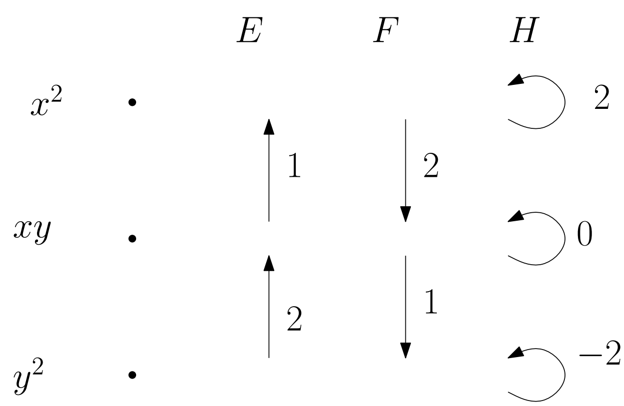


FIGURE 3. Action of the Lie Group on $S^2\mathbb{C}^2$

The diagrams for the general symmetric power $S^n\mathbb{C}^2$ is similar.

Proposition 22.2. *$S^n\mathbb{C}^2$ is irreducible.*

Proof. x^n generates it (see F). x^n is in every nonzero submodule (E). □

These are all the finite-dimensional irreducible representations.

Remark 22.3. Why is this so? We have an action $sl_2 \curvearrowright V$, i.e. we have a map $sl_2 \to gl(V)$.
Observe that $[H,E] = 2E$. Suppose that $Hv = \lambda v$.

$$\Rightarrow HEv = [H,E]v + EHv = (2+\lambda)Ev.$$

So for any eigenvalue λ of H, either $(2+\lambda)$ is an eignevalue *or* the eigenvector is killed by E. If we are finite dimensional, this implies that you will eventually find the eigenvector of H killed by E.

Claim: Applying powers of F to that original vector gives you all vectors in the basis, i.e. all the irreps
Start with that eigenvector v. Apply F until you arrive at a vector killed by F, i.e. $F^n v \neq 0$ such that $F^{n+1}v = 0$. The eigenvalues corresponding to H for each vector are $\{\lambda, \lambda-2, \ldots, \lambda-2n\}$.

$$EFv_0 = ([E,F] + FE)v_0 = Hv_0 = \lambda v_0.$$

$$EF^2 v_0 = [E,F]Fv_0 + FEFv_0 HFv_0 + \lambda F v_0 = (2\lambda - 2).$$

These calculations continue until our general formula.

$$EF^{k+1}v_0 = (C_k)F^k v_0.$$

$$LHS = ([E,F] + FE)F^k v_0 = (\lambda - 2k + C_{k-1})F^k v_0 ...$$

We find that $C_k = k\lambda - 2\binom{k+1}{2}$. This forces n to be a multiple of 2? I need to clarify this last part.

The fact that we get the full span of values by applying E and F means we have a submodule, but this is irreducible.

23. Monday, October 29, 2012

Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2$ such that the determinant is 1. This acts on polynomials in two variables by

$$g \cdot f(x,y) = f(ax + cy, bx + dy),$$

giving an action of $SL_2 \curvearrowright \mathbb{C}[x,y]_n$.

In particular, $SL_2 \curvearrowright \mathbb{C}[x,y]_1 = V$ is the defining representation matrix of g in basis (x,y) is $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

The coaction

$$\begin{aligned} \mathbb{C}[x,y] &\longrightarrow \mathbb{C}[x,y] \otimes \mathcal{O}(SL_2) = \mathbb{C}[x,y,a,b,c,d]/(ad - bc - 1) \\ f(x,y) &\longmapsto f(ax + cy, bx + dy) \end{aligned}$$

leads to a Lie algebra action defined by derivations on $\mathbb{C}[x,y]$. Specifically,

$$E = x\partial y, \qquad F = y\partial x, \qquad H = x\partial x - y\partial y.$$

$$E = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad F = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \qquad H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

On the symmetric algebra $S^n V$, one can list the monomials $x^n y^0, \ldots, x^0 y^n$ and the actions send each monomial up or down, with H having each monomial as an eigenvector.

23.1. Complete Reducibility of $\mathcal{O}(SL_2)$-Comodules. First we need to explore the general properties of coalgebras with completely reducible comodules.

Let $\mathcal{O}$ be a coalgebra, with $V \to V \otimes \mathcal{O}$. This All comodules are 'locally finite dimensional', i.e. every $v \in V$ is contained in a finite dimensional subcomodule.

$$v_j \xmapsto{\rho} \sum_{i=1}^n v_i \otimes f_i \underset{id\otimes\Delta}{\overset{\rho\otimes id}{\rightrightarrows}} \sum_{i,k} v_{ik} \otimes f_i \otimes f_k \qquad f_i \in \mathcal{O}.$$

These maps must be the same to respect the coalgebra structure, so we have an expression in terms of finitely many basis vectors. This means that all irreducible comodules are finite dimensional.

Let V be a finite-dimensional $\mathcal{O}$-comodule. The action is:

$$\begin{array}{ccc} \rho : V & \longrightarrow & V \otimes \mathcal{O} \\ & & \updownarrow \\ (\text{End } V)^* = & & V \otimes V^* \to \mathcal{O} \\ & & \downarrow \\ & & \mathcal{O}^* \to \text{End } V \end{array}$$

This gives a surjection from $\mathcal{O}^* \twoheadrightarrow A \hookrightarrow \text{End } V$ For $\mathbb{C} = k = \bar{k}$, V irreducible implies that A is the endomorphism ring of V. Why? Because if the field is contained in a finite dimensional division ring, then taking one element generates an algebraic extension which must already be contained in the algebraically closed field.

This means that the surjection $\mathcal{O}^* \twoheadrightarrow \text{End } V$ is a consequence of the injection $(\text{End } V)^* \hookrightarrow \mathcal{O}$, which is a two-sided subcomodule. It is two-sided in that when you apply the diagonal map $\mathcal{O} \to \mathcal{O} \otimes \mathcal{O}$ you can define the action on the left or right factor.

In general, you cannot turn a left comodule into a right comodule or vice versa; groups are special in this regard because the inverse is an antipode. But because $(\text{End } V)^* \cong V \otimes V^*$, we pick up the left module structure of one and the right module structure of the other to obtain a bimodule.

Moreover, $(\text{End } V)^*$ is a finite direct sum of copies of V as a right comodule. If $(V_i)_{i \in I}$ are non-isomorphic irreducibles. Then $(\text{End } V_i)^* \subseteq \mathcal{O}$ are linearly independent.

Reason: Given $(\text{End } V_1)^* + \cdots + (\text{End } V_n)^* = \bigoplus \subseteq \mathcal{O}$ it is a direct sum of V_i's for $i = 1, \ldots, n$. Then End $V_{n+1})^*$ has zero intersection with it.

We get a canonical map:

$$\bigoplus_{i \in I} (\text{End } V_i)^* \hookrightarrow \mathcal{O}.$$

Example 23.1.

$$\mathcal{O}(G_m) = \mathbb{C}[t, t^{-1}].$$

Each V_m is (t^m). Applying the coproduct, $\Delta 1 = 1 \otimes 1$, $\Delta t = t \otimes t$ and $\Delta t^m = t^m \otimes t^m$.

Then $(\text{End } V_m)^* = \mathbb{C} \cdot t^m \subset \mathbb{C}[t, t^{-1}]$ is a subcomodule.

$$\mathcal{O}(G_m) = \bigoplus (\text{End } V_i)^*.$$

On the other hand, we had only one semisimple module. We will see that the property of a coalgebra being semisimple over itself is equivalent ot the category of comodules over the coalgebra being semisimple.

24. Wednesday, October 31, 2012

Review:

We have a coalgebra, defined by the following maps:

$$\Delta : \mathcal{O} \to \mathcal{O} \otimes \mathcal{O}.$$

$$\mathbf{1} : \mathcal{O} \to k.$$

Then $\mathcal{O}^*$ is an algebra. The irreducible comodules are finite-dimensional. Specifically, for V irreducible, $\mathcal{O}^* \twoheadrightarrow \text{End } (V)$, when $k = \bar{k}$ algebraically closed.

This comes from the injection

$$\text{End } (V)^* \hookrightarrow \mathcal{O}.$$

$$\text{End } V = V \otimes V^*.$$

The images of these maps for each irreducible are irreducible, so we get $\bigoplus_{i \in I} (\text{End } V_i)^* \subset \mathcal{O}$.

Theorem 24.1. *The following are equivalent:*

(1) $\mathcal{O} = \bigoplus_i (End\ V_i)^*$. *[i.e. satisfies Peter-Weyl Theorem]*
(2) $\mathcal{O}$ *is completely reducible as a (right)* $\mathcal{O}$*-comodule. (We always mean right comodules and left modules unless otherwise specified.)*
(3) Every $\mathcal{O}$*-comodule is completely reducible.*

Proof. $(1) \Rightarrow (2), (3) \Rightarrow (2)$ are trivial. Given (2), we want to prove (1) and (3).

Let $\mathcal{O} \cong \bigoplus_{i \in I} F_i \otimes V_i$, where F_i is some (possibly infinite-dimensional) vectorspace. (Including the F_i allows us to have $V_i \ncong V_j$ for $i \neq j$, all irreducible.)

These vectorspaces have a natural isomorphism

$$F_i \cong \mathrm{Hom}_{\mathcal{O}-comod}(V_i, \mathcal{O})$$

for $k = \bar{k}$ algebraically closed.

$\mathcal{O}^*, (\mathcal{O}^*)^{op}$ act on $\mathcal{O}$. $a \in \mathcal{O}^*$ acts as

$$\mathcal{O} \xrightarrow{\Delta} \mathcal{O} \otimes \mathcal{O} \xrightarrow{1 \otimes a} \mathcal{O}.$$

$a \in (\mathcal{O}^*)^{op}$ acts as

$$\mathcal{O} \xrightarrow{\Delta} \mathcal{O} \otimes \mathcal{O} \xrightarrow{a \otimes 1} \mathcal{O}.$$

Claim: $(\mathcal{O}^*)^{op} \to \mathrm{End}_{\mathcal{O}^*}(\mathcal{O})$ is an isomorphism with inverse

$$\varphi \in \mathrm{End}_{\mathcal{O}^*}(\mathcal{O}) \mapsto \mathbf{1} \circ \varphi \in \mathcal{O}^*.$$

□

25. Friday, November 2, 2012

Picking up from where we were cut off last time by the fire alarm:

Proof of Theorem 24.1. $(2) \Rightarrow (1)$. Let $\mathcal{O} \cong \bigoplus_{i \in I} F_i \otimes V_i$, where F_i is some (possibly infinite-dimensional) vectorspace. (Including the F_i allows us to have $V_i \ncong V_j$ for $i \neq j$, all irreducible.)

These vectorspaces have a natural isomorphism

$$F_i \cong \mathrm{Hom}_{\mathcal{O}-comod}(V_i, \mathcal{O})$$

for $k = \bar{k}$ algebraically closed.

$\mathcal{O}^*, (\mathcal{O}^*)^{op}$ act on $\mathcal{O}$. $a \in \mathcal{O}^*$ acts as

$$\mathcal{O} \xrightarrow{\Delta} \mathcal{O} \otimes \mathcal{O} \xrightarrow{\mathbf{1} \otimes a} \mathcal{O}.$$

$a \in (\mathcal{O}^*)^{op}$ acts as

$$\mathcal{O} \xrightarrow{\Delta} \mathcal{O} \otimes \mathcal{O} \xrightarrow{a \otimes \mathbf{1}} \mathcal{O}.$$

Claim: $(\mathcal{O}^*)^{op} \to \mathrm{End}_{\mathcal{O}^*}(\mathcal{O})$ is an isomorphism with inverse

$$\varphi \in \mathrm{End}_{\mathcal{O}^*}(\mathcal{O}) \mapsto \mathbf{1} \circ \varphi \in \mathcal{O}^*.$$

Therefore,

$$(\mathcal{O}^*)^{op} \cong \prod_i \mathrm{End}_k(F_i).$$

We get $e_i \in \mathcal{O}^*$ acting as projections on the summands $F_i \otimes V_i$.

- $e_i e_j = \delta_{ij} e_i$.
- e_i is in the center of $\mathcal{O}^*$.
- "$\sum_i e_i = 1$" in the sense that given any operator $f, \langle e_i, f \rangle$ is $\mathbf{1}$ (projection of f onto $F_i \otimes V_i$). Since f is non-zero for only finite number of indices i.

The map $\mathcal{O}^* \to \text{End }(V_i)$ is dual to $\text{End }(V_i)^* \hookrightarrow \mathcal{O}$. This endomorphism ring has

$$\text{End }(V_i)^* \cong V_i^* \otimes V_i \subset F_i \otimes V_i.$$

This implies that e_j acts as zero on V_i for all $j \neq i$.

On the other hand, if we map e_i into $\mathcal{O}$, we note that it merely projects onto itself then comultiplying we obtain the counit of the coalgebra (I think?). Therefore, e_i acts as 1 on V_i.

This means that $\mathcal{O}^* \to \text{End }(V_i)$, where $\mathcal{O}^* = \prod_i \text{End }(F_i)^{op}$ and this map factors:

$$\mathcal{O}^* \to (\text{End } F_i)^{op} \twoheadrightarrow \text{End } V_i.$$

The surjectivity comes from the "Density Theorem."

Even if it were infinite, the endomorphism ring would be simple in the sense that it has no proper two-sided ideal. However, because the kernel of the second map needs to be a two-sided ideal, this means that the map is an isomorphism.

$$\Rightarrow (\text{End } F_i)^{op} \cong \text{End }(V_i).$$

This implies that the dimension of $F_i = \dim V_i^*$, which further implies that $\text{End }(V_i)^* \subset \mathcal{O}$ in fact gives $\bigoplus_i \text{End }(V_i)^* = \mathcal{O}$. Proving

(Note that even if we are not in k algebraically closed, this argument could be easily corrected by paying attention to what division rings we land in)

(2) $\Rightarrow$ (3). Let M be an $\mathcal{O}$-comodule. Then,

$$\mathcal{O}^* \curvearrowright \Rightarrow M = \bigoplus M_i.$$

with $e_i \in mcO^*$ acting as the projection on M_i. e_j, when $j \neq i$, kills M_i. Hence,

$$\begin{tikzcd}
\mathcal{O}^* \arrow[r] \arrow[dr] & \text{End }(F_i)^{op} \arrow[r] \arrow[d, equal] & \text{End }(M_i) \\
& \text{End }(V_i) &
\end{tikzcd}$$

Then $\text{End }(M_i)$ is a matrix algebra over $\text{End }(V_i)$, a semisimple algebra, which means that it too is semisimple. Therefore, M_i is a direct sum of copies of V_i. □

Example 25.1 (Peter-Weyl Theorem for SL_2)**.** SL_2 is the group of matrices $\begin{matrix} a & b \\ c & d \end{matrix}$, where $ad-bc = 1$. The ring of functions

$$\mathcal{O}(SL_2) = \mathbb{C}[a,b,c,d]/(ad-bc-1)$$

has a degree filtration $\mathbb{C} = F_0 \subset F_1 \subset \cdots$, where F_d = image of $\mathbb{C}[a,b,c,d]_{\deg \leq d}$. $F_k F_l \subseteq F_{k+l}$.

This gives an associated graded ring

$$gr_F \mathcal{O}(SL_2) = \bigoplus_d F_d/F_{d-1}.$$

with the map

$$F_k/F_{k-1} \otimes F_l/F_{l-1} \to F_{k+l}/F_{k+l-1}.$$

We define a map

$$\mathbb{C}[a,b,c,d]/(ad-bc) \twoheadrightarrow gr_F \mathcal{O}(SL_2).$$

Lemma 25.2.

$$\dim F_m/F_{m-1} \leq (m+1)^2.$$

Proof. Spanned by monomials of degree m, but due to the relation $ad = bc$, we only need monomials *not* divisible by ad.

We can separately consider $a^{m-r-s}b^r c^s$ and $b^r c^s d^{m-r-s}$ which overlap when there are no factors of a or d. Together these fill up a square of side length $m+1$. □

26. Monday, November 5, 2012

We will pick up with demonstrating the Peter-Weyl Theorem for SL_2.

For complete reducibility of the $\mathcal{O}$-comodules, it is sufficient to show that $\mathcal{O}$ itself is completely reducible as an $\mathcal{O}$-comodule.

Recall the map

$$\mathcal{O}(SL_2) \twoheadleftarrow \mathbb{C}[a,b,c,d]/(ad-bc-1)$$

We define the filtration as we did last time with F_m = image of $\mathbb{C}[a,b,c,d]_{\deg \le m}$, and

$$gr_F\mathcal{O}(SL_2) = \bigoplus_d F_d/F_{d-1}.$$

The ring map

$$\mathbb{C}[a,b,c,d]/(ad-bc) \twoheadrightarrow gr_F\mathcal{O}(SL_2)$$

is a surjection. Note that we quotient by $ad-bc$ since $ad-bc$ is quotiented out as an element of smaller degree, since it is equal to 1. This is a general strategy for non homogeneous ideals – make a graded filtration, and obtain a homogeneous ideal as a result.

We also saw last time that

$$\dim F_m/F_{m-1} \le (m+1)^2.$$

$$\bigoplus_m (\text{End } V_m)^* \subset \mathcal{O}(SL_2)$$

$$V_m = S^m\mathbb{C}^2$$

which has matrix coefficients polynomials of degree m in a,b,c,d.

This means that

$$\Rightarrow \bigoplus_{j\le m} (\text{End } V_j)^* \subseteq F_m.$$

The dimension of this direct sum of representations is given by:

$$\dim = \sum_{j\le m} (j+1)^2.$$

This is because V_m as defined has dimension $m+1$.

Since $\dim F_m \le \sum_{j\le m}(j+1)^2$, this implies

$$\Rightarrow \bigoplus_{j\le m} (\text{End } V_j)^* = F_m, \text{ and } (\text{End } V_m)^* \xrightarrow{\sim} F_m/F_{m-1}.$$

This in turn implies that the monomials not containing ad are a basis for the graded ring, which in turn means that they are a basis for the non-graded version.

Remark 26.1. There is a grading in which the original ideal is in fact homogeneous, specifically $\mathbb{Z}/2\mathbb{Z}$ grading where the variables have odd grading. In this grading the generator of the ideal is contained in the even graded piece.

$$\mathcal{O}(SL_2)_{\text{even}} = \mathbb{C}[a^2, ab, ac, \ldots]/(ad-bc-1, a^2b^2-(ab)(ab), \ldots).$$

[The coproduct of elements in this algebra sends an entry to what it would be in the product of two matrices, e.g. $\Delta a = a\otimes a + b\otimes c$.]

Actually this subring corresponds to the coordinate ring of another algebraic group, specifically:

$$\mathcal{O}(SL_2)_{\text{even}} = \mathcal{O}(SL_2)^{\{\pm I\}} = \mathcal{O}(SL_2/\{\pm I\}) = \mathcal{O}(PSL_2).$$

26.1. **Classical Reductive Groups.** We have examined $GL_n, SL_n, PGL_n, SO_n, Sp_{2n}, \dots$ There are structure theorems about reductive groups that they look like these, or one of the exceptional groups, e.g. $E_6, E_7, E_8, \dots$

We want to calculate characters, cocharacters, roots, Weyl groups, etc.

First we look for the maximal torus in every classical group.

Let the algebraic torus $T \subset GL_n$ denote the set of matrices:

$$T = \left\{ \begin{pmatrix} z_1 & & 0 \\ & \ddots & \\ 0 & & z_n \end{pmatrix} \right\} = (\mathbb{C}^\times)^n.$$

For SO_n, take (x, y) on $\mathbb{C}^n$ to be $x^T \begin{pmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{pmatrix} y$, so $(e_i, e_{n+i-j}) = \delta_{ij}$.

Then $SO_n = \{X \in SL_n : X^T J X = J\}$. In other words,

$$X^T J = J X^{-1} \Leftrightarrow J X^T J = X^{-1}$$

Multiplying the transpose on either side by J is the same as transposing over the antidiagonal, which we described before as X^R.

Therefore,

$$so_n = \{A : A^R = -A\}.$$

Let T be the set of diagonal matrices in here, we have

$$t = \begin{pmatrix} x_n & & & & & \\ & \ddots & & & & \\ & & x_1 & & & \\ & & & -x_1 & & \\ & & & & \ddots & \\ & & & & & -x_n \end{pmatrix}.$$

Sending this back to the group by exponentiation, we have:

$$T = \begin{pmatrix} z_n & & & & & \\ & \ddots & & & & \\ & & z_1 & & & \\ & & & z_1^{-1} & & \\ & & & & \ddots & \\ & & & & & z_n^{-1} \end{pmatrix}.$$

On the other hand, for Sp_{2n} we multiply by a different matrix J:

$$J = \begin{pmatrix} & & & & & 1 \\ & & & & \iddots & \\ & & & 1 & & \\ & & -1 & & & \\ & \iddots & & & & \\ -1 & & & & & \end{pmatrix}$$

We get the same torus as we did in SO_{2n}.

26.2. **Tori.** Let $T = (\mathbb{C}^\times)^n$. One natural way to get coordinates is:
$$\mathcal{O}(T) = \mathbb{C}[z_1^{\pm 1}, \dots, z_n^{\pm 1}].$$

We want a more coordinate-free approach, so we look at characters:

Characters: T is reductive, irreps are 1-dimensional,
$$\begin{array}{ccc} T & \to & \mathbb{C} \\ \underline{z} & \mapsto & z_1^{m_1} \cdots z_n^{m_n} \end{array}$$
where $m_i \in \mathbb{Z} \Leftrightarrow \underline{m} \in \mathbb{Z}^n$. These are the endomorphism rings $(\text{End } V_{\underline{m}})^*$.

These form an abelian group $X \cong \mathbb{Z}^n$ under tensor product of representations. Then,
$$\mathcal{O}(T) = \mathbb{C} \cdot X.$$
Then X is its character lattice. The coproduct in this coalgebra is $\Delta x = x \otimes x$.

27. Wednesday, November 7, 2012

We have been examining the action of the torus on classical groups.

This whole construction is functorial. Suppose you have an algebraic group homomorphism $T \to T'$, then we have a ring homomorphism $\mathcal{O}(T) \leftarrow \mathcal{O}(T')$. Because we have a map $T \to T' \xrightarrow{\lambda} \mathbb{C}^\times$, this ring map sends the character lattice X' to X.

This implies:
$$\text{Hom}_{AlgGp}(T, T') = \text{Hom}_{\mathbb{Z}-mod}(X', X).$$
In terms of the cocharacters,
$$\{\mathbb{C}^\times \to T\} \leftrightarrow \text{Hom}_{\mathbb{Z}}(X, \mathbb{Z}) \cong X^V.$$
since each map takes $t \mapsto t^m$, for some $m \in \mathbb{Z}$.

How do we get a character and a cocharacter and get an integer? $\mathbb{C}^\times \xrightarrow{\varphi} T \xrightarrow{\lambda} \mathbb{C}^\times$. Specifically, $\langle \lambda, \varphi \rangle$ = the integer m associated to $\lambda \circ \varphi$.

[This may be connected to Langlands duality.]

Now let us move to the specific classical groups, $G = GL_n, SL_n, PGL_n, SO_n$ or Sp_{2n}, with its torus subgroup.

$G \curvearrowright G$ by conjugation, via $\theta_g(h) = ghg^{-1}$. Then $\theta_g : G \to G$ has corresponding map of Lie groups $(d\theta_g)_1 : \mathcal{G} \to \mathcal{G}$. This gives an action of G on $\mathcal{G} = LieG$. For GL_n, this is just matrix conjugation, since
$$g(I + tA)g^{-1} = I + tgAg^{-1}.$$
Hence this also holds for the rest of the classical groups (as subgroups/quotients of GL_n).

T preserves $\mathcal{T} = Lie(T) \subset Lie(G) = \mathcal{G}$ and acts trivially on $\mathcal{T}$.

$T \subset G$ acts on $\mathcal{G}$, which decomposes as $\mathcal{G} = \bigoplus_{\alpha \in R \cup \{0\} \subset X} \mathcal{G}_\alpha$. The Lie group $\mathcal{T}$ is in $\mathcal{G}_0$ in this decomposition.

Example 27.1. Let $G = GL_n$. Then the torus group is
$$T = \begin{pmatrix} z_1 & & \\ & \ddots & \\ & & z_n \end{pmatrix}.$$
The coordinate ring is $\mathcal{O}(T) = \mathbb{C}[z_1^{\pm 1}, \dots, z_n^{\pm 1}$. The character lattice is $\mathbb{Z}^n$.

Conjugating A by T gives the matrix $\left(\frac{z_i}{z_j} a_{ij}\right)_{i,j}$. So, for the Lie group $\mathcal{G} = \mathcal{GL}_\backslash$, we have $\mathcal{G}_0 = t =$ the diagonal matrices. The set of roots
$$\mathcal{R} = \{z_i z_j^{-1} : j \neq i\} = \{e_i - e_j : i \neq j\} \subseteq \mathbb{Z}^n.$$

Therefore, taking $\alpha = e_i - e_j$, $\mathcal{G}_\alpha = \mathbb{C} \cdot E_{ij}$, where E_{ij} = the matrix with 1 in the (i,j)-th position.

Example 27.2. Now, let $G = SL_n$. Then we have the sequence

$$0 \to T_{SL_n} \to T_{GL_n} \to \mathbb{C}^X \to 0.$$

Translating to character lattices,

$$\mathbb{Z} \xrightarrow{e_1 + \cdots + e_n} X_{GL_n} = \mathbb{Z}^n \to X_{SL_n} = \mathbb{Z}^n / \mathbb{Z} \cdot (1, \ldots, 1)$$

in which $(m_1, \ldots, m_n) \mapsto z_1^{m_1} \cdots z_n^{m_n}$.

Thinking about the action on the Lie group, we have $\mathcal{G} = \mathcal{SL}_{\backslash}$. Again, $\mathcal{G}_0 = \mathcal{T}$ = the diagonal matrices in $\mathcal{SL}_{\backslash}$.

The root system is

$$\mathcal{R} = \{\bar{e}_i - \bar{e}_j\} \subseteq \mathbb{Z} / \mathbb{Z} \cdot (1, \ldots, 1),$$

where the bar indicates the image of $\mathbb{Z} \cdot (1, \ldots, 1) \oplus \{(m_1, \ldots, m_n) : m = 0\}$ in $\mathbb{Z}^n$. Because these subspaces are orthogonal, the image of each root is still distinct.

$\mathcal{G}_{e_i e_j} = \mathbb{C}\bar{t}_{ij}$.

Example 27.3. Now consider $PGL_n = GL_n$/scalars. The torus is:

$$T = \left\{ \begin{pmatrix} z_1 & & \\ & \ddots & \\ & & z_n \end{pmatrix} \right\} / \left\{ \begin{pmatrix} t & & \\ & \ddots & \\ & & t \end{pmatrix} \right\}.$$

We have an exact sequence of groups:

$$0 \to \mathbb{C}^\times \to T_{GL_n} \to T_{PGL_n} \to 0.$$

which gives us an exact sequence of character lattices in the opposite direction:

$$0 \leftarrow \mathbb{Z} \leftarrow \mathbb{Z}^n \leftarrow X_{PGL_n} \leftarrow 0.$$

Specifically,

$$\begin{tikzcd}
 & PGL_n = PSL_n & \{(\underline{m}) : \sum m_i = 0\} \arrow[d] \arrow[dd, bend left] & e_i - e_j \arrow[d] \\
GL_n \arrow[ur] & & \mathbb{Z}^n \arrow[d] & e_i - e_j \arrow[d] \\
 & SL_n \arrow[uu] \arrow[ul] & \mathbb{Z}^n / \mathbb{Z} \cdot (1, \ldots, 1) & e_i - e_j \\
 & \mu_n = \{e^{2\pi i k n}\} \arrow[u] & &
\end{tikzcd}$$

We have a surjection of tori, which gives a discrete kernel determined by the degree of the covering map. In general the character lattice of a torus X is a weight lattice, the sublattice generated by roots is Q. (Many details here were missed)

28. Wednesday, November 14, 2012

[Class missed on Friday, November 9 for travel]

Given G, with Lie group

$$\mathfrak{g} = t \oplus \bigoplus_{\alpha \in \Omega} \mathfrak{g}_\alpha.$$

$$\begin{tikzcd} \varphi_\alpha: & SL_2 \arrow[r] & G \\ & \mathfrak{sl}_2 \arrow[r] & \mathfrak{g} \\ & E \arrow[r, mapsto] & \chi_\alpha \\ & F \arrow[r, mapsto] & \chi_{-\alpha} \\ & H \arrow[r, mapsto] & \text{“}\alpha\text{”} \end{tikzcd}$$

$$s_\alpha(\beta) = \beta - \langle \beta, \alpha^\vee \rangle \alpha.$$

$$s_\alpha : X \to X.$$

s_α is the reflection fixing $\langle -, \alpha^\vee \rangle = 0$, sending α to $-\alpha$.

W = the Weyl group is the group of symmetries of X generated by s_α's. It acts on R (and on $R^\vee \subset X^\vee$).

Take $s \in SL_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. $\varphi_\alpha(s)$ normalizes $T \subset G$. Then $\to T$.

$$\begin{tikzcd} \mathbb{C}^\times \subset SL_2 \arrow[d] & \mathfrak{sl}_2 \arrow[d] & H \arrow[d] \\ T & \mathfrak{g} & h \end{tikzcd}$$

We find $\mathfrak{sl}_2 \cong g_\alpha \oplus \mathbb{C}h \oplus g_{-\alpha}$.

The element $\varphi_\alpha(s)$ defines an element $s_\alpha^\bullet \in N(T) \subset G$ the normalizer of the torus, which maps to $s_\alpha \in N(T)/T$. Note $s_\alpha^2 = 1$.

From the action $G \curvearrowright V$, the action of the subtorus T gives a weight space decomposition

$$V = \bigoplus_{\lambda \in X} V_\lambda, \qquad X = X(T).$$

$$s_\alpha^\bullet(V_\lambda) = V_{s_\alpha(\lambda)}.$$

Notation 28.1. We have a map from 2×2 matrices into the group by selecting α off-diagonal, and takes the 2×2 matrix whose diagonal coincides with the torus, and let everything else be identity.

$$\varphi_\alpha : SL_2 \to G$$

The image in the Lie algebra is only the off-diagonal entry.

The lie group of the subtorus is

$$\mathfrak{t} = \mathfrak{t}' \oplus \mathbb{C}h$$

$$\begin{tikzcd} \alpha^\vee: & \mathbb{C}^\times \arrow[r] \arrow[d, hook] & T \arrow[d, hook] \\ & SL_2 \arrow[r] & G \end{tikzcd}$$

$$\mathfrak{sl}_2 \cong \begin{pmatrix} & & 0 & & \\ & a & & c & \\ 0 & & 0 & & 0 \\ & b & & -a & \\ & & 0 & & \end{pmatrix} \subset \mathfrak{gl}_n$$

$S_\alpha^\bullet \curvearrowright T$ acts on the character lattice by the same reflection as the s_α that we formally defined earlier.

28.1. **Examples.** Consider GL_n.

- The torus T = diagonal matrices $\begin{pmatrix} z_1 & & \\ & \ddots & \\ & & z_n \end{pmatrix} \cong (\mathbb{C}^\times)^n$.
- The Lie algebra $\mathfrak{gl}_n = n \times n$ matrices.
- $\mathfrak{t}$ is the subgroup of diagonal matrices.
- $\mathfrak{g}_\alpha = \mathbb{C} \cdot E_{ij}, i \neq j$, where $\alpha = e_i - e_j$ in $\mathfrak{g}$, or $z_i z_j^{-1}$, in G.
- φ_α maps $SL_2 \to GL_n$ by sending:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} 1 & & & & & & 0 \\ & a & & \cdots & & b & \\ & & 1 & & & & \\ & \vdots & & \ddots & & \vdots & \\ & & & & 1 & & \\ & c & & \cdots & & d & \\ 0 & & & & & & 1 \end{pmatrix}, \qquad ad - bc = 1.$$

- We have $s_\alpha^\bullet = \varphi_\alpha$ applied to the matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

$$s_\alpha^\bullet = \begin{pmatrix} 1 & & & & & & 0 \\ & 0 & & \cdots & & 1 & \\ & & 1 & & & & \\ & \vdots & & \ddots & & \vdots & \\ & & & & 1 & & \\ & -1 & & \cdots & & 0 & \\ 0 & & & & & & 1 \end{pmatrix}$$

- Given character lattice $X = \mathbb{Z}^n$, then $s_\alpha = (ij)$ fixes those vectors which have $m_i = m_j$, and sends the vector that has $m_i = -m_j$ to its opposite.

29. Friday, November 16, 2012

Recall that the root system of GL_n is given by looking at the generic torus. The root system is $X = \mathbb{Z}^n$. Mapping $\mathbf{z}^{\mathbf{m}} \mapsto z_i/z_j$.

Then the elements $\alpha = e_i - e_j$ where e_i is the basis for $\mathbb{Z}^n = X$. $e_i^\vee$ is a basis for $X^\vee \cong \mathbb{Z}^n$. The action $S_\alpha(m) = (ij)$ describe an action on the roots which generates an action of S_n. Alternatively, you can think about an element of SL_2 in the normalizer of the torus.

The Weyl group can be defined as the group of symmetries on the lattice X; alternatively, it can be described by its corresponding action on the torus – specifically, the normalizer of the torus mod the torus $N(T)/T$.

29.1. **Root System of SO_{2n+1}.** Let us try a different example. We take the special orthogonal group on an odd number of elements, since it comes out different depending on parity.

Recall we can define these as matrices that fix a bilinear form $\langle , \rangle$, where the bilinear form is defined by

$$\langle x, y \rangle = x^T J y, \qquad J = \begin{pmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{pmatrix}.$$

This is a generic element of the subtorus:

$$\begin{pmatrix} z_n & & & & & & \\ & \ddots & & & & & \\ & & z_1 & & & & \\ & & & 1 & & & \\ & & & & z_1^{-1} & & \\ & & & & & \ddots & \\ & & & & & & z_n \end{pmatrix}.$$

The Lie group $\mathfrak{so}_{2n+1}$ is then given by matrices such that the antitranspose is the negative $A^R = -A$.

Now we want to decompose: $\mathfrak{so}_{2n+1} = \mathfrak{t} \oplus ?$.

Consider $\mathfrak{t}$ acting on A by conjugation:

$$tAt^{-1} = \chi(t)A. \Rightarrow t^{-1}A^T t = \chi(t)A^T.$$

$$\Rightarrow tA^T t^{-1} = \chi(t^{-1})A^T \Rightarrow \chi(t^{-1}) = \chi(t)^{-1}.$$

$T \subset B = \{\text{upper triangular matrices in } G\}$.

Lie $B = \mathfrak{t} \oplus \bigoplus_{\alpha \in R^+} \mathfrak{g}_\alpha$.

$e_i - e_j$, such that $i < j$ is an element of R^+, the positive root system.

The $\alpha = e_i - e_j$ is given by a zero matrix with 1 in the (ij) position and -1 so that its reverse transpose will be its negative, i.e. $(2n+2-j, 2n+2-i)$.

The basis for the standard representation of SO_{2n+1} given by $v_1, \dots, v_{2n+1} \cong \mathbb{C}^{2n+1}$ can be thought of as $\mathbb{C}^{2n+1} = \mathbb{C}^n \oplus \mathbb{C} \oplus (\mathbb{C}^n)^*$.

To preserve the inner product, we only need to act on the dual space with the transpose of whatever action we use on the first $\mathbb{C}^n$.

Again we can take the action of $\varphi_\alpha(SL_2) \to GL_n \to SO_{2n+1}$, where the action is defined by putting the matrix as the ij 2×2 submatrix in the first $\mathbb{C}^n$, and then its dual on the dual space.

$\alpha^\vee$ gives a map from $\mathbb{C}^\times$ to the torus T, by

$$\begin{pmatrix} 1 & & & & & & \\ & t & & & & & \\ & & t^{-1} & & & & \\ & & & 1 & & & \\ & & & & t & & \\ & & & & & t^{-1} & \\ & & & & & & 1 \end{pmatrix}.$$

i.e. $z_i = t, z_j = t^{-1}$, and the antitranspose property is the inverse.

This is only one block of the group action. Instead of having $j \le n$, what if we have $j = n+1$ (i.e. in the same column as the central 1)?

SL_2 acts on $\mathfrak{sl}_2 \cong \mathbb{C}^3$. The group of isometries on $\mathbb{C}^3$ is SO_3. We can use this to map SL_2 to SO_3. Since E, F, and H are a basis for the action on $\mathfrak{sl}_2$, we know that H has matrix in SO_3:

$$\begin{pmatrix} 2 & & \\ & 0 & \\ & & 2 \end{pmatrix}.$$

This map carries

$$\begin{pmatrix} t & \\ & t^{-1} \end{pmatrix} \mapsto \begin{pmatrix} 2 & & \\ & 0 & \\ & & 2 \end{pmatrix}.$$

$\alpha^\vee$ is the map $z_i \mapsto t^2, z_j \mapsto 1$.

Changing the basis from SL_2 to SO_3, and carrying over the bilinear form $tr(XY)$ defines a symmetric, non degenerate inner product on $\mathfrak{sl}_2$.

This gives us root $2e_i$.

30. November 19, 2012

Recall we were discussing the group SO_{2n+1} defined by matrices whose inverse are their reverse transpose (transpose over the anti diagonal).

We have a map

$$SL_2 \overset{\varphi_\alpha}{\to} GL_n \hookrightarrow SO_{2n+1}$$

by sending an invertible 2×2 matrix A mapping to the $n \times n$ matrix M with submatrix A in the index α and identity elsewhere, and then sending that to the orthogonal matrix with three blocks: M, 1, and $(M^{-1})^R$, where ^{R} indicates the reverse transpose.

This was for the case $\alpha = e_i - e_j$ where $1 \le i < j \le n$.

In the case of $i = n+1$, we can consider SL_2 as a subgroup of SO_3 in that both act on $\mathfrak{sl}_2 \cong \mathbb{C}^3$. SO_3 also acts on $\mathbb{C}\{v_i, v_{n+1}, v_{2n+2-i}\}$ Bilinear form which $\mathfrak{sl}_2$ preserves is $\langle X, Y\rangle = trXY$.

The map from $SL_2 \to SO_3$ actually factors through $PSL_2 = SL_2/\{\pm 1\}$.

Then the root is $\alpha = e_i$. But $\alpha^\vee = 2e_i^\vee$.

Now we have one final type of root, where $1 \le i < n+1 < j$, i.e. something in the northeast quadrant.

$$\left(\begin{array}{cc|c|cc} z_i & & & 1 & \\ & z_j & & & -1 \\ \hline & & 1 & & \\ \hline & & & z_j^{-1} & \\ & & & & z_i^{-1} \end{array}\right).$$

Here we have $SL_2 \curvearrowright \mathbb{C} \cdot \{v_i, v_{2n+2-j}\}$ and dually on $\mathbb{C} \cdot \{v_j, v_{2n+2-i}\}$.

So we have $\alpha = e_i + e_j$, and $\alpha^\vee = e_i^\vee + e_j^\vee$.

So, our root system $R_+ = (I) \cup (II) \cup (III)$, where these are the three types listed.

As for the Weyl group:

(1) switch $m_i \leftrightarrow m_j$ in $(m_1, \ldots, m_n) \in \mathbb{Z}^n = X$.
(2) $m_i \leftrightarrow -m_i$.
(3) $m_i \leftrightarrow -m_j$.

This leaves us with the Weyl group W = the group of signed permutations $B_n = S_n \ltimes \{\pm 1\}^n$.

Exercise 30.1. Work out Sp_{2n}:

$$T = \begin{pmatrix} z_n & & & & & \\ & \ddots & & & & \\ & & z_1 & & & \\ & & & z_1^{-1} & & \\ & & & & \ddots & \\ & & & & & z_n \end{pmatrix}.$$

$X = \mathbb{Z}^n$. Root system is the same as SO_{2n+1}, except we get $2e_i, e_i^\vee$ instead. In particular, this implies SO_{2n+1}, Sp_{2n} are Langlands dual.

$Sp_{2n}/\{\pm 1\}$ has the same root system as Sp_{2n} except X becomes

$$Y \subseteq \mathbb{Z}^n = \{(m_1, \dots, m_n) : \sum m_i \text{ even}\},$$

and $X^\vee$ becomes

$$Y^\vee = \mathbb{Z}^n \cup (\mathbb{Z}^n + (\frac{1}{2}, \dots, \frac{1}{2})).$$

Then what is the Langlands dual?

It is the odd-dimensional $Spin_{2n+1}$ defined as a central extension of our group, as illustrated by the sequence:

$$0 \to \mathbb{Z}/2\mathbb{Z} \to Spin_{2n+1} \twoheadrightarrow SO_{2n+1} \to 0.$$

The smallest representation of the Spin group that *does not* factor through SO_{2n+1} acts on a Clifford algebra of dimension $\approx 2^{2n+1}$. (huge.)

Let V be a G-module. Then it is also a T-module (on the subtorus $T \subset G$). Then, $V|_T$ is a direct sum of 1-dimensional submodules. The isotypic components are called *weight spaces*.

This gives the decomposition $V = \bigoplus_{\lambda \in X} V_\lambda$.

Definition 30.2. The character of V,

$$ch(V) := \sum_\lambda \dim V_\lambda x^\lambda \in \mathcal{O}(T) = \mathbb{C} \cdot X,$$

where x^λ is a way of denoting the character $\lambda : T \to \mathbb{C}^\times$.

Example 30.3. Consider SL_3 and the SL_3-module $\mathfrak{sl}_3$. This decomposes as $\mathfrak{t} \oplus \bigoplus_{\alpha \in R} g_\alpha$. Then,

$$ch(V) = 2 \cdot 1 + z_1/z_2 + z_1/z_3 + z_2/z_3 + z_2/z_1 + z_3/z_1 + z_3/z_2.$$

This is because $\alpha = e_i - e_j$ and $x^\alpha = z_i/z_j$.

Remark 30.4.

$$ch(V)|_1 = \dim V.$$

This sum is $tr_V(t)$ as a function of $t \in T$.

$W = N(T)/T$ acts on the set of weight spaces. $\dot{w} \in N(T)$ will determine how the torus maps into the weight spaces, but only w determines which weight spaces it goes to.