1. Friday, August 24, 2012

1.1. Course Information. Website: math.berkeley.edu/∼poirier
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Course Rubric: 25% HW every two weeks, 25% Oral Presentation, 50% Take-home Final.
First Homework Due September 17

1.2. A Soft Introduction to Algebraic Topology.

(1) What is algebraic topology (Math 215A)?
(2) Classification of Surfaces.
(3) Rough introduction to categories and functors.

One goal: Classify topological spaces (up to homeomorphism) algebraically.

Question 1.1. Given spaces X and Y, does there exist a homeomorphism $f : X \to Y$? Recall a homeomorphism is a continuous map with continuous inverse.

If there exists such an f then X and Y are topologically equivalent. Often, it is easier to answer this question in the negative. For example, if X and Y don't share some property preserved under homeomorphism, e.g. connectedness or compactness.

Definition 1.2. Let $|\pi_0(X)|$ be the number of connected components of X.

If X and Y are homeomorphic, then $|\pi_0(X)| = |\pi_0(Y)|$. Failure to be equal would thus imply failure of X and Y to be homeomorphic.

Certainly, there are X and Y such that $|\pi_0(X)| = |\pi_0(Y)|$ but X and Y are not homeomorphic, e.g. the point and the line. This means that the numerical homeomorphism invariant $|\pi_0(X)|$ is just okay.

In this class, we will examine three algebraic invariants:

(1) $\pi_1(X)$, the fundamental group.
(2) $H_\bullet(X)$, the homology.
(3) $H^\bullet(X)$, the cohomology.

Along the way, these invariants help us prove "purely topological" statements. The following theorem is an example.

Theorem 1.3 (Brouwer fixed point theorem)**.** *Let $B^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n : \sum x_i^2 \leq 1\}$ be the standard n-ball. Let $f : B^n \to B^n$ be a continuous map. Then f has a fixed point, i.e. $\exists x \in B^n$ s.t. $f(x) = x$.*

Definition 1.4 (Imprecise)**.** An n-dimensional manifold is a space that locally looks like $\mathbb{R}^n$.

Example 1.5.

- $\mathbb{R}^n$
- $S^n = \{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} : \sum x_i^2 = 1\}$, since locally it looks like $\mathbb{R}^n$. (Fig1)
- $T^n = \underbrace{S^1 \times \cdots \times S^1}_{n \text{ times}}$. ($T^2$ is often called T) (Fig2)
- $B^n = D^n$ the disk or ball. (with or without boundary)
- A disk with small disks removed.

"Classification of 0-Manifolds" – a collection of points. There is only one zero manifold, up to homeomorphsim.

Theorem 1.6 (Classification of Compact, Connected 1-Manifolds, with or without boundary)**.** *Up to homeomorphism, any 1-manifold is S^1 or $I = [0, 1]$, i.e. compact, connected 1-manifolds are completely determined by whether or not they have boundary.*

Theorem 1.7 (Classification of Surfaces)**.** *A surface is completely determined up to homeomorphism by (1) χ, the Euler characteristic, (2) the number of boundary components, and (3) orientability.*

2. Monday, August 27, 2012

Recall the following:

Definition 2.1 (Imprecise)**.** A manifold is a space which looks locally Euclidean. A surface is a 2-dimensional manifold (with boundary).

Theorem 2.2 (Classification of Surfaces)**.** *A compact, connected surface is completely determined up to homeomorphism by*

- *χ, the Euler characteristic,*
- *the number of boundary components*
- *orientability/non-orientability.*

Example 2.3. S^2 = Sphere, Cylinder Segment, M = the Möbius strip, P = pair of pants, $T = T^2 = S^1 \times S^1$ the torus, $D = B^2$ the disk.

See Figure 3 for some constructions of surfaces. In the construction of $\mathbb{RP}^2$, the surface has one face, one edge, and one vertex; we end up double wrapping that one edge + vertex around the face.

Definition 2.4 (Imprecise)**.** A Connect Sum (#) is given in Figure 4. The identity of the monoid defined by this operation is the sphere.

Definition 2.5 (Imprecise)**.** A surface is nonorientable if it contains a Möbius strip (otherwise, it is orientable).

Definition 2.6 (Imprecise)**.** A triangulation of a surface looks like Figure 5. Using this triangulation,

$$\chi(\Sigma) = V - E + F,$$

where $V = \#$ vertices, $E = \#$ edges, and $F = \#$ faces (triangles).

Theorem 2.7. *$\chi(\Sigma)$ is independent of the triangulation. For example $\Sigma(S^2) = 2$.*

Table 1. Orientable Surfaces

Theorem 2.8.

χ	*2*	*1*	*0*	*-1*	*-2*
$\#\partial$					
0	S^2		T^2		$T^2 \# T^2$
1		D		$T^2 - D$	
2			$S^1 \times S^1$		

Table 2. Non-orientable Surfaces

Theorem 2.9.

$$\sum \textit{orientable} \Rightarrow \chi(\Sigma) = 2 - 2g - \#\partial \textit{ comps.}$$

$$\sum \textit{non-orientable} \Rightarrow \chi(\Sigma) = 2 - g - \#\partial \textit{ comps.}$$

where g = genus is defined roughly as the number of handles.

3. Wednesday, August 29, 2012

I came 25 minutes late today, so my notes are incomplete. So far, Prof. Poirier has defined categories, and given examples, including groups, topological spaces, and posets.

Definition 3.1 (Imprecise)**.** A functor $\mathscr{C} \xrightarrow{F} \mathscr{D}$ is a morphism in the "category of categories." It maps the objects and morphisms of $\mathscr{C}$ to those of $\mathscr{D}$ respecting the associative structure.

In this course, we have three important functors:

(1) $\pi_1 : Top_* \to Group$.
(2) $H_* : Top \to grGroups$, the category of graded groups.
(3) $H^* : Top \to grRings$, the category of graded rings.

Remark 3.2. These functors take homeomorphism in Top to isomorphism. But in these examples, homotopy equivalence (weaker than homeomorphism) also give isomorphism. So the invariants are just okay.

Good news: they are often computable or comparable (even if they are not computable, you can find some difference between them).

By the Poincaré conjecture, π_1 distinguishes S^3 among all 3-manifolds. i.e. if $\pi_1(M^3) \cong \pi_1(S^3)$, then M^3 is homeomorphic to S^3. In fact, it generalizes to higher dimensions.

Remark 3.3. The Smooth Poincare Conjecture is false e.g. in dimension 7. Milnor's exotic spheres are homeomorphic but (not?) diffeomorphic to S^7.

4. Friday, August 31

Today: Homotopy of maps, and CW-complexes.

4.1. **Homotopy.** Fix topological spaces X and Y. Consider the space of continuous maps from X to Y, $f : X \to Y$. We use the notation:

$$Maps(X, Y) = \{f : X \to Y\}.$$

Remark 4.1. $Maps(X, Y)$ has the compact-open topology.

What is a path in $Maps(X, Y)$?

Definition 4.2. Let $X \overset{f_0}{\underset{f_1}{\rightrightarrows}} Y$ be two continuous maps from X to Y. If there exists a continuous map $X \times I \overset{F}{\Rightarrow} Y$ such that $F(x, 0) = f_0(x)$ and $F(x, 1) = f_1(x)$, then F is called a homotopy between f_0 and f_1 and f_0 and f_1 are called homotopic. [Apparently, product with X and Maps from X are adjoint functors: $X \times Z \to Y \leftrightarrow Z \to Maps(X, Y)$.]

We think of a homotopy F as a continuous family of maps $X \to Y$. $F_t = F(\bullet, t) : X \times \{t\} \cong X \to Y$. If f_0 is homotopic to f_1, write $f_0 \cong f_1$.

Exercise 4.3. Show that homotopy induces an equivalence relation on $Maps(X, Y)$.

Definition 4.4. Given

(1) $X \xrightarrow{f} Y$, $Y \xrightarrow{g} X$ is called a homotopy inverse for f if $f \circ g \cong id_Y$ and $g \circ f \cong id_X$.
(2) If $X \xrightarrow{f} Y$ has a homotopy inverse then f is called a homotopy equivalence.
(3) If there exists a homotopy equivalence $X \xrightarrow{f} Y$ then X and Y are called homotopy equivalent , write $X \cong Y$.

Exercise 4.5. Show that homotopy equivalence induces an equivalence relation on spaces.

Special case: $A \subset X$.

Definition 4.6. (1) A retraction from X to A is a map $r : X \to A$ that keeps A fixed.

(2) A map $r : X \to A$ is called a deformation retraction if $i \circ r = id_X$.

(3) If $F : X \times I \to X$ is a homotopy between $i \circ r$ and id_X and $F(a, t) = a$, for all $a \in A$, then r is a strong deformation retraction.

Usually, when one says "retraction", one means (2) or (3). (Please specify if you mean (1).)

Exercise 4.7. Write down an explicit formula for a strong deformation retraction

(1) $\mathbb{R}^2 \to \{(0, 0)\}$.
(2) $\mathbb{R}^2 - \{(0, 0)\} \to S^1$.

$\Rightarrow \mathbb{R}^2$ and $\{pt\}$ have the same homotopy type. Similarly for (2).

Definition 4.8. If X has the homotopy type of a point, then X is called contractible.

Remark 4.9. Homotopy equivalence is coarse in that non-homeomorphic spaces may be homotopy-equivalent, e.g. it does not recognize dimension ($\mathbb{R}^n$ is contractible for all n).

4.2. **CW-Complexes.** It is possible to impose a CW structure on a surface.

Example 4.10. Consider the torus T^2, in its construction via rectangle. In this form, we have vertices, edges, and faces identified in such a way that they uniquely determine the surface.

Alternatively, construct T^2 by taking 1 point, 2 edges, and 1 face. Map the boundaries of the 2 edges (i.e. vertices) onto the vertex and map the boundaries of the face (i.e. edges) and map them to the edges.

Definition 4.11. An n-cell is a space homeomorphic to D^n. A CW-complex is a space built inductively by cells and attachments.

"0-skeleton" X^0 = discrete set of points. "n-skeleton" $X^n = X^{n-1} \underset{\varphi_\alpha}{\sqcup} D^n_\alpha$, where $\varphi_\alpha : \partial D^n_\alpha \to X^{n-1}$, and α is in some indexing set.

The inclusion $X^0 \subset X^1 \subset X^2 \subset \cdots$ defines X. If $X = X^n$ then we call X an n-dimensional CW-complex.

Exercise 4.12. Determine a CW decomposition for:

(1) S^2.
(2) $S^2 \vee S^1 \vee S^1$.

5. Wednesday, September 5, 2012

Recall: A homotopy of maps $X \overset{f_0}{\underset{f_1}{\rightarrow}} Y$ is a map $F : X \times I \to Y$ such that $F(x, 0) = f_0(x)$ and $F(x, 1) = f_1(x)$ for all $x \in X$.

Definition 5.1. Let $A \overset{i}{\hookrightarrow} X$ be a subspace. We say $f_0, f_1 : X \to Y$ are homotopic relative to A if $f_0(a) = f_1(a)$ for all $a \in A$, and there exists a homotopy $F : X \times I \to Y$ such that $F(a, t) = f_0(a) = f_1(a)$ for all $a \in A$.

(Think: e.g. strong deformation retraction $r : X \to A$ such that $id_X \sim i \circ r$ and homotopy fixes A pointwise.)

5.1. **The Functor $\pi_1 : Top_* \to Groups$.** Preliminary definitions:

Definition 5.2. (1) A path in X is a map $\gamma : I \to X$.
(2) Its endpoints are $\gamma(0)$ and $\gamma(1)$.
(3) If γ' is a path in X such that $\gamma'(o) = \gamma(1)$, then γ and γ' may be concatenated by

$$(\gamma \cdot \gamma')(s) = \begin{cases} \gamma(2s) & 0 \leq s \leq \frac{1}{2} \\ \gamma'(2s-1) & \frac{1}{2} \leq s \leq 1 \end{cases}$$

(4) A loop based at $x_0 \in X$ is a path such that $\gamma(0) = \gamma(1) = x_0$.

Remark 5.3. Any two loops based at x_0 may be concatenated ("composed").

Definition 5.4. Let $\Omega(X, x_0)$ be the set of loops in X based at x_0. (This is really a space, called the based loop space of X.

Let $\cdot : \Omega(X, x_0) \times \Omega(X, x_0) \to \Omega(X, x_0)$ be given by concatenation: $(\gamma, \gamma') = \gamma \cdot \gamma'$.

Question 5.5. What kind of operation is $\cdot$? (What properties does it have?)

Associativity? Is $(\gamma \cdot \gamma') \cdot \gamma'' = \gamma \cdot (\gamma' \cdot \gamma'')$?

No, but there is a natural homotopy between these two. See Figure 1. Therefore, $\cdot$ is associative up to homotopy.

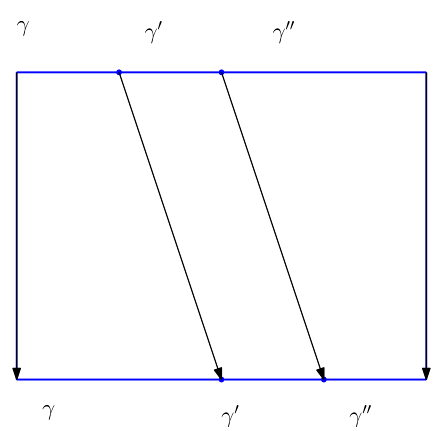


FIGURE 1. Homotopy from $(\gamma \cdot \gamma') \cdot \gamma''$ to $\gamma \cdot (\gamma' \cdot \gamma'')$

Remark 5.6. There are many homotopies between $(\gamma \cdot \gamma') \cdot \gamma''$ and $\gamma \cdot (\gamma' \cdot \gamma'')$. In fact: any two homotopies are homotopic; we can stack layers of homotopies. This leads to the study of A_∞ algebras.

Definition 5.7. Let $\pi_1(X, x_0) = \Omega(X, x_0)/\sim$, where $\sim$ is equivalence up to homotopy del x_0. In other words homotopy classes of loops in X based at x_0.
For $[\gamma]$ and $[\gamma'] \in \pi_1(X, x_0)$, let $[\gamma] \cdot [\gamma'] = [\gamma \cdot \gamma']$.

Exercise 5.8. Check that this is: 1) well defined on $\Omega(X, x_0)$, and 2) associative.

Does this have identity?
Candidate for identity in $\Omega(X, x_0)$. Let e be the constant loop at x_0 is $\gamma \cdot e = e \cdot \gamma = \gamma$?
Yes.

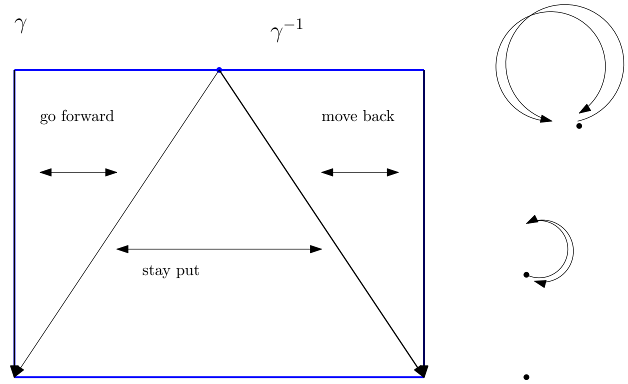


FIGURE 2. Homotopy from $(\gamma \cdot \gamma^{-1})$ the Constant Loop.

6. Friday, September 12, 2012

For the inverse element, for each γ, just take γ in the reverse orientation, i.e. $\gamma^{-1}(s) = \gamma(1-s)$. The homotopy is demonstrated in Figure 2.

Definition 6.1. $(\pi_1(X, x_0), \cdot)$ is the fundamental group of X based at x_0.

Example 6.2. $(X, x_0) = (\mathbb{R}^n, 0)$. Let $\gamma = \Omega(\mathbb{R}^n, 0)$. Let $\gamma \in \Omega(\mathbb{R}^n, 0)$. Then γ is nullhomotopic. Proof: Let $F : I \times I \to X$. $F(s,t) = (1-t)\gamma(s)$.

Therefore $\pi_1(\mathbb{R}^n, 0)$ is the trivial group.

Question 6.3. Is the fundamental group dependent on the base point?

We assume that X is path-connected. Let h be a path in X from x_1 to x_0 (exists by assumption). Then $(h \cdot \gamma) \cdot h^{-1}$ is a loop based at x_1.

Proposition 6.4.
$$\Omega(X, x_0) \longrightarrow \Omega(X, x_1). \qquad \gamma \mapsto (h \cdot \gamma) \cdot h^{-1}$$
induces a well-defined isomorphism $\pi_1(X, x_0) \xrightarrow{\beta_h} \pi_1(X, x_1)$.

Therefore, $\pi_1(X)$ is well-defined up to isomorphism (though the isomorphism is not canonical).

Definition 6.5. X is simply-connected (1-connected) if
(0) X is connected.
(1) $\pi_1(X)$ is trivial.

Example 6.6. $\mathbb{R}^n$ is simply connected for all n.

Remember that π_1 is defined as a functor $\pi_1 : Top_* \to Group$. We explained how to map the objects, but how does π_1 map the morphsims?

Given $\pi_1(X, x_0)$ and a continuous map $f : (X, x_0) \to (Y, y_0)$, what is $\pi_1(f) = f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$.

Notice:
$$(I, \partial I) \xrightarrow{\gamma} (X, x_0) \xrightarrow{f} (Y, y_0),$$
we just take the composition as a loop in Y.

Definition 6.7. $\Omega(X, x_0) \xrightarrow{\Omega f} \Omega(Y, y_0)$, sends $\gamma \mapsto f \circ \gamma$.

Proposition 6.8. *(1) Ωf induces a well-defined map $\pi_1(X, x_0) \xrightarrow{f_*} \pi_1(Y, y_0)$.*

(2) f_* is a group homomorphism. (π_1 takes morphisms to morphisms).
(3) $(X, x_0) \xrightarrow{f} (Y, y_0) \xrightarrow{g} (Z, z_0) \Rightarrow (g \circ f)_* = g_* \circ f_*$.
(4) $(X, x_0) \xrightarrow{id} (X, x_0) \Rightarrow id_* = id : \pi_1(X, x_0) \to \pi_1(X, x_0)$.
Therefore, π_1 is a bona fide functor.

Proposition 6.9. *If $f_0 : (X, x_0) \to (Y, y_0)$ and $f_1 : (X, x_0) \to (Y, y_0)$ are homotopic, then $(f_0)_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ and $(f_1)_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ are equal. (There exists an analogous statement, even if $f_0(x_0) \neq f_1(x_0)$).*

Sketch of proof: if $\gamma \in \Omega(X, x_0)$ then $(\Omega f_0)(\gamma) \sim (\Omega f_1)(\gamma)$.

7. Monday, September 10, 2012

Remark 7.1. Recall that $\pi_1 : Top_* \to Group$ is a functor taking continuous maps to group homomorphisms.

$$\begin{tikzcd} (X, x_0) \arrow[r, bend left, "f_0"] \arrow[r, bend right, "f_1"'] \arrow[r, phantom, "\downarrow"] & (Y, y_0) \arrow[d, "g"] \\ & (Z, z_0) \end{tikzcd} \longrightarrow \begin{tikzcd} \pi_1(X, x_0) \arrow[r, "(f_0)_* = (f_1)_*"] & \pi_1(Y, y_0) \arrow[d, "g_*"] \\ & \pi_1(Z, z_0) \end{tikzcd}$$

Specifically, it takes pointed spaces to groups, maps of pointed spaces to group homomorphism, homotopic maps of spaces to the *same* group homomorphism.

Remark 7.2. Recall also that $\mathbb{R}^n$ is simply connected (connected and $\pi_1(\mathbb{R}^n)$ is trivial.)

Proposition 7.3. *If $f : X \to Y$ is a homotopy equivalence, then $f_* : \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$ is an isomorphism.*

Proof. Let $g : Y \to X$ be a homotopy inverse for f. $f \circ g \sim id_Y$ and $g \circ f \sim id_X$. Usually this means that $g_* \circ f_* =$ "id" (where we need to be careful about basepoints).

$$\begin{tikzcd} \pi_1(X, x_0) \arrow[r, "f_*"] \arrow[drr, "id"'] & \pi_1(Y, y_0) \arrow[r, "g_*"] & \pi_1(X, g(f(x_0))) \\ & & \pi_1(X, x_0) \arrow[u, "\beta_h \cong"'] \end{tikzcd}$$

$\Rightarrow g_* \circ f_*$ is an isomorphism $\Rightarrow f_*$ is injective. $\Rightarrow f_* \circ g_*$ is an isomorphism $\Rightarrow f_*$ is surjective. $\square$

Corollary 7.4. *If $i : A \hookrightarrow X$ is a deformation retract, $x_0 \in A$. Then $\pi_1(X, x_0) \cong \pi_1(A, x_0)$.*

Proposition 7.5. *If $r : X \to A$ is any retraction (not necessarily deformation retraction), then $i_* : \pi_1(A, x_0) \to \pi_1(X, x_0)$ is injective.*

The proof uses the fact that $r \circ i = id_A$...

Proposition 7.6. *X path-connected. Then, $\pi_1(X)$ is trivial if and only if $\forall x_0, x_1 \in X$ any two paths from x_0 to x_1 are homotopic.*

Theorem 7.7. $\pi_1(S^1, 1) \cong \mathbb{Z}$. *Specifically, if we define*

$$\omega_n : I \to S^1, \text{ by } s \mapsto (\cos 2\pi ns, \sin 2\pi ns),$$

then

$$\Phi : \mathbb{Z} \to \pi_1(S^1, 1) \qquad n \mapsto [\omega_n]$$

is an isomorphism.

Corollary 7.8 (Brouwer's Fixed Point Theorem for $n = 2$)**.** *Any continuous map $f : D^2 \to D^2$ has a fixed point.*

Proof. Assume $f(x) \neq x$ for all $x \in D^2$, then $x, f(x)$ determine a line for all $x \in D^2$. Let $r : D^2 \to S^1$ map x to the point where the ray from $f(x)$ to x hits the circle.

Check: r is continuous, fixes S^1.

So, r is a retraction $\Rightarrow i : S^1 \hookrightarrow D^2$ induces $i_* : \pi_1(S^1, 1) \hookrightarrow \pi_1(D^2, 1)$. But $\mathbb{Z}$ cannot inject into the trivial group, contradiction! Therefore, there must be a fixed point where r is not defined. $\square$

Ingredient for the proof of Theorem 7.7:

Definition 7.9. The map $(\pi : \tilde{X} \to X)$ has the homotopy lifting property if given

(1) $F : Y \times I \to X$
(2) a lift $\tilde{f}$ of $f = F(\cdots, 0)$.

there exists a unique lift $\tilde{F}$ of F inducing $\tilde{f}$.

$$\begin{tikzcd} & \tilde{X} \arrow[d, "\pi"] \\ Y \arrow[ur, "\tilde{f}"] \arrow[r, "f"'] & X \end{tikzcd} \quad \Rightarrow \quad \begin{tikzcd} & \tilde{X} \arrow[d, "\pi"] \\ Y \times I \arrow[ur, "\exists ! \tilde{F}"] \arrow[r, "F"'] & X \end{tikzcd}$$

Example 7.10. (1) $Y = \{pt\}$, then $(\pi : \tilde{X} \to X)$ has the path lifting property: given any path $\gamma \in X$ and a lift $\tilde{\gamma}(0)$ of $\gamma(0)$, then there exists a unique lift of the whole path γ to $\tilde{X}$ agreeing with $\tilde{\gamma}(0)$

(2) $Y = I$, then $(\pi : \tilde{X} \to X)$ has the "homotopy of paths" lifting property: given a homotopy of paths in X and a lift of one end to $\tilde{X}$, then there exists a unique lift f the whole homotopy to $\tilde{X}$.

The map we use is $\pi : \tilde{X} \to X$ where $\tilde{x}$ is $\mathbb{R}^3$ and we project onto the plane. If we map the interval by $s \mapsto (\cos 2\pi ns, \sin 2\pi ns, s)$, then we have the homotopy lifting property, and we know some things about this space that we can use.

8. Wednesday, September 12, 2012

I arrived late to class. Prof. Poirier is proving the isomorphism of fundamental groups asserted in the last class. She went in three steps: homomorphism, surjectivity, and injectivity, using paths "upstairs" in the preimage spiral, and paths "downstairs" on the circle.

Theorem 8.1. *Let X, Y be connected spaces, and let the product have projections as shown. Then,*

$$\pi_1(X \times Y, (x_0, y_0)) \longrightarrow \pi_1(X, x_0) \times \pi_1(Y, y_0), \qquad [\gamma] \mapsto ((p_1)_*[\gamma], (p_2)_*[\gamma])$$

is an isomorphism.

Exercise 8.2. γ is completely determined by $p_1(\gamma), p_2(\gamma)$.

$$\begin{tikzcd} X \times Y \arrow[r, "p_2"] \arrow[d, "p_1"] & Y \\ Y \end{tikzcd}$$

Example 8.3. The n-torus,

$$T^n := \underbrace{S^1 \times \cdots \times S^1}_{n \text{ times}} \Rightarrow \pi_1(T^n) \cong \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}.$$

The homotopy class of the curve for T^2 is completely determined by two integers.

9. Wednesday, September 19, 2012

[Absent for Monday, September 17 due to Rosh Hashanah]

Theorem 9.1 (Van Kampen Theorem)**.** *$X = U \cup V$, such that U, V are open, path connected, $U \cap V$ path connected, and $x_0 \in U \cap V$. There are canonical maps*

$$\begin{tikzcd} & U \cap V \arrow[dl, "i"'] \arrow[dr, "j"] & \\ U \arrow[dr, "i'"'] & & V \arrow[dl, "j'"] \\ & U \cup V & \end{tikzcd}$$

(1) $\pi_1(X)$ is the pushout of:

$$\begin{tikzcd} & \pi_1(U \cap V) \arrow[dl, "i_*"'] \arrow[dr, "j_*"] & \\ \pi_1(U) & & \pi_1(V) \end{tikzcd}$$

(2) $\pi_1(X) \cong [\pi_1(U) * \pi_1(V)]/N$, where N is the normal subgroup generated by $i_*[\gamma] \cdot (j_*[\gamma])^{-1}$, for $\gamma \in \pi(U \cap V)$.

Example 9.2. Take $X = \mathbb{R}P^2$. Let

$$U = \mathbb{R}P^2 - \text{ pt } \cong S^1.$$

$$V = \text{ nbhd of pt } \cong \text{ pt }.$$

$$U \cap V = (\text{ nbhd of pt }) - \text{ pt } \cong S^1.$$

$\pi_1(U) \cong \mathbb{Z}$, so take a generator a. Then the map of spaces induces the following map of fundamental groups:

$$\begin{tikzcd} & \pi_1(U \cap V) \cong \mathbb{Z} \arrow[dl, "c \mapsto 2a"'] \arrow[dr] & \\ \pi_1(U) \cong \mathbb{Z} & & \pi_1(V) \cong 1 \end{tikzcd}$$

So the pushout will satisfy $\pi_1(X) \cong [\langle a \rangle * 1]/\langle 2a \cdot 1 \rangle \cong \mathbb{Z}/2\mathbb{Z}$.

$$
\begin{tikzcd}
& \pi_1(U \cap V) \arrow[dl, "i_*"'] \arrow[dr, "j_*"] & \\
\pi_1(U) \arrow[r] \arrow[dr, "i'_*"'] & \frac{\pi_1(U)*\pi_1(V)}{N} \arrow[d, "\exists!\Phi"] & \pi_1(V) \arrow[l] \arrow[dl, "j'_*"] \\
& \pi_1(X) &
\end{tikzcd}
$$

Idea of Proof. Want to show that Φ is an isomorphism, in the following diagram: Method:

- Define a map $\Phi : \pi_1(U) * \pi_1(V) \to \pi_1(X)$ a surjective homomorphism.
- Show that ker $\Phi = N$.
- By the first isomorphism theorem Φ induces the desired isomorphism.
- $\pi_1(U) * \pi_1(V)$ is generated by $[\gamma]_u$ and $[\gamma]_v$, i.e. γ is a loop in U and $[\gamma]_u$ is its homotopy class in U (V resp).
- Define Φ on generators by $\Phi([\gamma]_u) = [\gamma]_X$, resp. V. Then extend Φ as a homomorphism.
- This Φ is surjective: Take any $\gamma \in X$, look at where it crosses back and forth between U and V. Define base points in each transition section. Then these will be base points for a sequence of loops entirely in U or entirely in V.
- $N \leq$ ker Φ is easy. $i_*[\gamma]_{U\cap V} = [\gamma]_U$, and $j_*[\gamma]_{U\cap V} = [\gamma]_V$. Then $\Phi(i_*[\gamma](j_*[\gamma])^{-1}) = [\gamma]_X([\gamma]_X)^{-1} = 1 \in \pi_1(X)$.
- Ker $\Phi \leq N$ is trickier.

□

10. Friday, September 21, 2012

Proof Continued. To show that the ker $\Phi = N$, we have the following idea:

- We have $N <$ ker $\Phi < \pi_1(U) * \pi_1(V)$.
- Consider the cosets of N and $ker\Phi$.
- The coset $\Phi^{-1}([\gamma])$ = all possible factorizations of $[\gamma]$.

Strategy:

(1) Define equivalence relation $\sim$ on $\Phi^{-1}([\gamma])$ so that if two factorizations are equivalent, they lie in the same N coset.
(2) Show that any two factors are equivalent.

Definition 10.1. Let the following generate $\sim$ on $\Phi^{-1}([\gamma])$

(1) $[\gamma_1][\gamma_2]\cdots[\gamma_i][\gamma_{i+1}]\cdots[\gamma_N] \sim [\gamma_1][\gamma_2]\cdots[\gamma_i \cdot \gamma_{i+1}]\cdots[\gamma_N]$ where $[\gamma_i]$ and $[\gamma_{i+1}]$ are both in U or V.
(2) $[\gamma_1][\gamma_2]\cdots[\gamma_i]_U\cdots[\gamma_N] \sim [\gamma_1][\gamma_2]\cdots[\gamma_i]_V\cdots[\gamma_N]$ where $[\gamma_i]$ and $[\gamma_{i+1}]$ are both in U or V.

□

11. Monday, September 24, 2012

11.1. **Covering Spaces.** Recall that in proving the fundamental group of S^1, we had a map from $\mathbb{R}$ to S^1 via P.

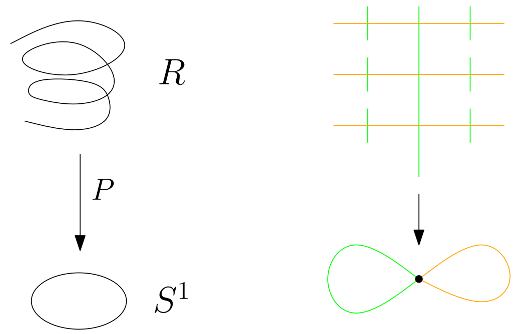


FIGURE 3. Examples of Covering Spaces

We used covering spaces and covering maps to compute $\pi_1(S^1, 1)$ and $\pi_1(S^1 \vee S^1, *)$ in another example.

Definition 11.1. Let X be a topological space. A covering space for X is a space $\tilde{X}$, together with a covering map $p : \tilde{X} \to X$ such that $\forall x \in X$ there is a neighborhood U_x of x such that $p^{-1}(U_x)$ is homeomorphic to $U_x \times S$, where S is some discrete set.

In other words, there is a finite set of pre-image neighborhoods. (If $S = F$, some topological space, see "fiber bundle.")

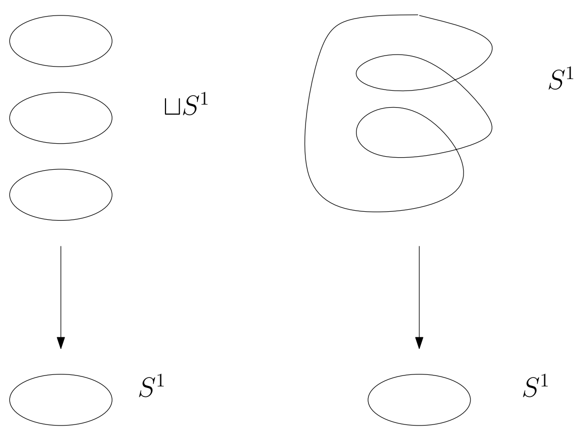


FIGURE 4. Examples (1) and (2) of covers of the circle.

In Example (2), the map sends z to z^n, where n is the number of pre-image neighborhoods. In Example (3), the map sends z on the boundary of the Möbius strip to z^2.

Exercise 11.2. If $p : \tilde{X} \to X$ and $p' : \tilde{X}' \to X'$ are two covering spaces then so is $p \times p' : \tilde{X} \times \tilde{X}'$.

Corollary 11.3. *Since $\mathbb{R}$ is a covering space of S^1, $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ is also a cover of $S^1 \times S^1 = T^2$.*

Recall:

Definition 11.4. The map $p : \tilde{X} \to X$ satisfies the homotopy lifting property if for all Y, and for all maps $F : Y \times I \to X$ and for all lifts $\tilde{f}_0 : Y \to \tilde{X}$ of $f_0 := F(\bullet, 0) : Y \to X$, there exists a unique lift $\tilde{F} : Y \times I \to \tilde{X}$ of F that completes the diagram in Figure 6.

Recall:

(1) If $Y = \{\text{pt}\}$. This is called the path lifting property.

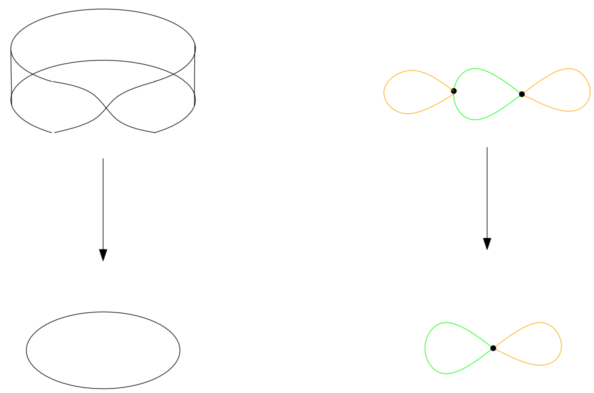


FIGURE 5. Examples (3) and (4) of covering spaces.

$$\begin{tikzcd} Y \times \{0\} \arrow[r, "\tilde{f}_0"] \arrow[d] & \tilde{X} \arrow[d, "p"] \\ Y \times I \arrow[r, "F"'] \arrow[ur, "\tilde{F}"] & X \end{tikzcd}$$

FIGURE 6. Homotopy Lifting Property

(2) If $Y = I$, we called this the "homotopy of paths" – lifting property.
We used these two to prove that $\pi_1(S^1, 1) \cong \mathbb{Z}$.

This was okay because the first map in Figure 3 is a covering space.

Proposition 11.5. *Covering spaces and covering maps satisfy the homotopy lifting property.*

We will prove that covering spaces satisfy the path lifting property in Figure 7.

$$\begin{tikzcd} \{0\} \arrow[r, "\tilde{\gamma}(0)"] \arrow[d] & \tilde{X} \arrow[d, "p"] \\ Y \times I \arrow[r, "\gamma"'] \arrow[ur, "\tilde{\gamma}"] & X \end{tikzcd}$$

FIGURE 7. Path Lifting Property

Intuition: the neighborhood of our base point has multiple pre-images. We choose one of them: by definition, the neighborhood we have chosen is homeomorphic to the neighborhood in the image, so we can "invert" the map into this neighborhood, and get the pre-image of the path in the image space.

We observe that $\bigcup_{x \in \mathrm{Im}\ \gamma} \gamma^{-1}(U_x)$ covers I.

I is compact, therefore there is a finite subcover to any cover. So there is a partition of I into $\cup_{i=1}^{N} I_i$ such that $\gamma(I_1)$ lies in one of the U_x's.

Consider the component $V_{\tilde{\gamma}(0)}$ of $p^{-1}(U_{\gamma(0)})$ containing $\tilde{\gamma}(0)$.

$$p|_{V_{\tilde{\gamma}(0)}} : V_{\tilde{\gamma}(0)} \to U_{\gamma(0)}.$$

Lift $\gamma|_{I_1}$ by $\tilde{\gamma}|_{I_1} = p^{-1} \circ \gamma|_{I_1}$.

Proceed analogously.

Exercise 11.6. (1) Check that the definition of $\tilde{\gamma}$ doesn't change if you choose another finite sub cover of I and partition.
(2) The "same" method works to prove the general homotopy lifting property for covering spaces.

12. Friday, October 5, 2012

[Three classes missed, for seminar and the holiday of Sukkot.]

Today, we are finishing up covering spaces, and discussing deck transformations (see Hatcher).

Example 12.1. Consider the covering space $\mathbb{R} \times \mathbb{R} \to S^1 \times S^1 = T^2$. The fundamental group $\pi_1(T^2, x_0) = \mathbb{Z} \oplus \mathbb{Z} = \langle a, b | [a, b] \rangle$, where a and b are the generating cycles.

Specifically $\pi_1(T^2, x_0)$ acts on $\mathbb{R}^2$ by a, b, which send $x \mapsto x + m\tilde{a} + n\tilde{b}$. In words, generator $[a]$ of $\pi_1(T^2, x_0)$ acts by translation by $\tilde{a}$.

$$\{ \text{ isomorphism classes of covering spaces } \} \quad \Leftrightarrow \quad \{ \text{ subgroups of } \pi_1(X, x_0) \}$$

Definition 12.2. The (isomorphism class of) simply connected cover(s) is called the "universal cover of X."

A simply connected cover is unique up to isomorphism and the path-space construction of $\tilde{X}$ gives us a model for the universal cover.

12.1. Homology Cartoon.

We want to generalize π_1 to higher dimensions.

Example 12.3 (Higher Homotopy Groups)**.** Consider $I^n = [0, 1]^n$. Let $\pi_n(X, x_0)$ = set of homotopy class of $(I^n, \partial I^n) \to (X, x_0)$.

$n = 0$. $I^0 = \{ \text{ pt } \}$, and $\partial I^0 = \varnothing$. Therefore, $\pi_0(X, x_0)$ = set of path components of X.

$n \geq 1$. We introduce a group operation (the analogue of concatenation of loops):

$$(f + g)(s_1, \ldots, s_n) = \begin{cases} f(2s_1, s_2, \ldots, s_n) & s_1 \in [0, 1/2] \\ g(2s - 1, s_2, \ldots, s_n) & s_1 \in [1/2, 1] \end{cases}.$$

This is well-defined on homotopy classes.

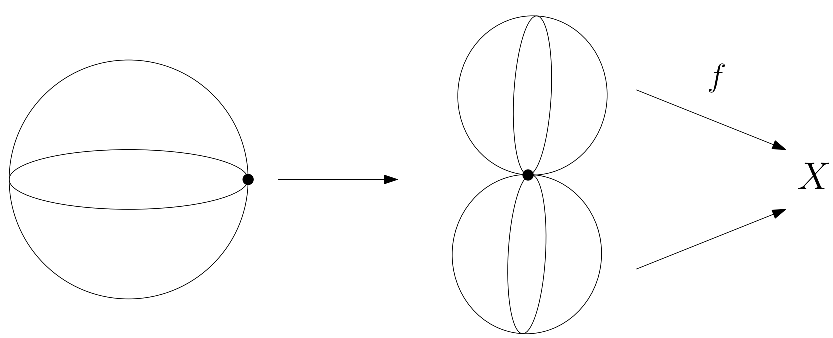


Figure 8. Higher Homotopy Concatenation.

In fact, $\pi_n(X, x_0)$ for $n \geq 2$ is abelian. See Figure 9 for a visual proof.

Theorem 12.4 (Whitehead's Theorem)**.** *Let X, Y be connected CW complexes. Let $f : X \to Y$ be a continuous map such that $f_* : \pi_n(X) \to \pi_n(Y)$ is an isomorphism for all n. (i.e. f is a weak homotopy), then f is a homotopy equivalence.*

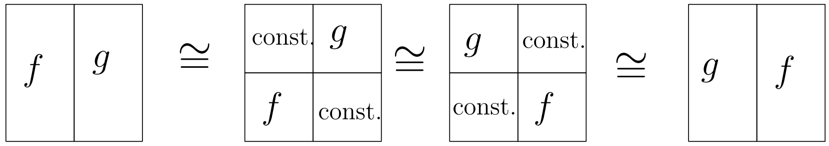


Figure 9. Visual Proof that Higher Homotopy Groups are Abelian.

Problem: π_n are notoriously hard to compute, even for simple spaces. (Covered in Math 215B).

Another idea: (Co)Bordism. For π_n, S^n is a manifold. Homotopy of $S^n \to X$ is $S^n \times I \to X$. $S^n \times I$ is an $n+1$-dimensional manifold with boundary.

We generalize by replacing S^n with M^n an n-dimensional manifold, and W^{n+1} is $n+1$-dimensional manifold with boundary.

Two n-dimensional manifolds M_1 and M_2 are (co)bordant if $\exists W^{n+1}$ such that

$$\partial W^{n+1} = M_1^n \sqcup M_2^n.$$

Example 12.5. The pair of pants is the cobordism of two copies of S^1 and one copy of S^1. A hemisphere is a cobordism of S^1 and $\varnothing$.

13. Monday, October 15, 2012

Definition 13.1. A *homology theory* is a (covariant) functor

$$(\text{pairs of topological spaces } A \subset X) \quad \to \quad (\text{graded Abelian groups})$$

satisfying four "Eilenberg-Steenrod" axioms.

(0) (Dimension)

$$H_*(\text{point}) = \begin{cases} \text{"coefficients"} \in \mathbb{Z} & * = 0. \\ 0 & * \neq 0. \end{cases}$$

(1) (Homotopy) Let f, g be homotopic maps from $(X, A) \to (Y, B)$. Then, $H_*(f) = H_*(g)$ as maps from $H_*(X, A) \to H_*(Y, B)$.

(2) (Exactness) There exists a map δ such that the inclusion $i : A \to X$ induces a long exact sequence:

$$\begin{tikzcd}
H_{i+1}(A) \arrow[r] & H_{i+1}(X) \arrow[r] & H_{i+1}(X,A) \arrow[dll, "\delta"] \\
H_i(A) \arrow[r] & H_i(X) \arrow[r] & H_i(X,A) \arrow[dll, "\delta"] \\
\cdots & &
\end{tikzcd}$$

(3) (Excision) If $Z \subset A \subset X$ and $\bar{Z} \subset \text{int}(A)$, then $i : (X - Z, A - Z) \hookrightarrow (X, A)$ induces an isomorphism $H_*(i) : H_*(X - Z, A - Z) \xrightarrow{\sim} H_*(X, A)$.

Theorem 13.2 (Eilenberg-Steenrod)**.** *There is only one homology theory.*

Definition 13.3. An *exotic homology theory* is a functor satisfying (1),(2),(3).

Definition 13.4. A functor is covariant if it preserves the directions of arrows. A functor is contravariant if it reverses directions of arrows.

Definition 13.5. A contravariant functor satisfying versions of (0), (1), (2), and (3) is called a cohomology theory.

13.1. **Simplicial Homology of Simplicial Complexes.**

Definition 13.6. An n-dimensional simplex $[v_0, \dots, v_n]$ is the convex hull of $n+1$ ordered points $v_0, \dots, v_n$ in $\mathbb{R}^m$ that do not lie in an affine subspace of dim $< n$.

Definition 13.7. The *standard n-simplex* Δ^n is the convex hull of the endpoints of the standard basis vectors in $\mathbb{R}^{n+1}$:

$$(1, 0, \dots, 0), \dots, (0, \dots, 0, 1).$$

In "barycentric coordinates",

$$\Delta^n = \{(t_0, \dots, t_n) : \sum t_i = 1, 0 \leq t_i \leq 1.$$

Example 13.8. $\Delta^0 \subset \mathbb{R}^1$ is a point.
$\Delta^1 \subset \mathbb{R}^2$ is a line segment.
$\Delta^2 \subset \mathbb{R}^3$ is a triangle.

Remark 13.9. There exists a canonical linear homeomorphism from any n-simplex to Δ^n.

Remark 13.10. Sometimes we only care about the combinatorial structure of an n-simplex, i.e. the vertices. Other times we'll care about the underlying space.

Definition 13.11. A face of $[v_0, \dots, v_n]$ is the sub simplex given by any (nonempty) (ordered) subset of the v_i.

Remark 13.12. Sometimes empty is okay. Hatcher only uses the $n-1$ dimensional faces.

Example 13.13. Consider the 3-simplex. [123] is a 2-dimensional face of the 3-simplex [0123]. [23] is a 1-dimensional face of [0123].

14. Wednesday, October 17, 2012

14.1. **Simplicial Complexes.** Roughly, a simplicial complex is a space formed from a collection of simplifies *identified* along faces.

Definition 14.1. A simplicial complex K is a collection of simplifies in $\mathbb{R}^n$ such that
(1) K is closed under taking faces.
(2) Simplices intersect only in faces.

Remark 14.2. A space may have a number of simplicial decompositions. For example, Δ^2 can be divided using barycentric subdivision.

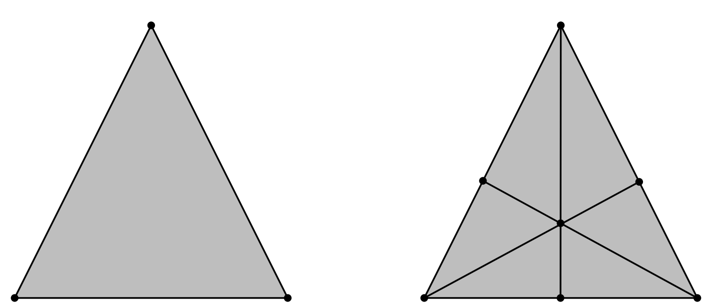


Figure 10. Barycentric Subdivision of the 2-simplex.

Remark 14.3. There is a purely combinatorial definition - the *abstract simplicial complex*:
A (nonempty) collection of subsets of $\{1, \dots, n\}$ which is closed under taking subsets.

Remark 14.4. In a simplicial complex (as defined), a simplex is completely determined by its vertices.

Hatcher uses a more general notion: Δ-complexes where faces of simplices may be identified.

Example 14.5. Spaces which are Δ-complexes, but not simplicial complexes:

- A vertex with a loop.
- A triangle with two edges identified, i.e. a cone.
- A pair of triangles with the same set of vertices, embedded in a cone.

14.2. **Simplicial Homology of a Simplicial Complex.** Let K be a Δ-complex.

Definition 14.6. The Δ-chain complex of K,

$$C_*^{\Delta}(K), \partial)$$

is given by:

- $C_n^{\Delta}(K)$ is the free abelian group generated by the n-simplices of K.
- $\partial : C_n^{\Delta}(K) \to C_{n-1}^{\Delta}(K)$ is defined on generators $[v_0, \dots, v_n]$ by

$$\partial_n[v_0, \dots, v_n] = \sum_{i=0}^{n} (-1)^i [v_0, v_1, \dots, \hat{v}_i, \dots, v_n].$$

i.e. ∂ is given by taking a signed sum of codimension 1 faces.

Lemma 14.7.

$$(\partial_{n-1} \circ \partial_n) = 0.$$

Proof.

$$(\partial_{n-1} \circ \partial_n)[v_0, \dots, v_n] = \partial_{n-1} \left(\sum_{i=0}^{n} (-1)^i [v_0, v_1, \dots, \hat{v}_i, \dots, v_n] \right).$$

$$= \sum_{i=0}^{n} (-1)^i \partial_{n-1} \left([v_0, v_1, \dots, \hat{v}_i, \dots, v_n] \right)$$

$$= \sum_{j<i} (-1)^j (-1)^i [v_0, v_1, \dots, \hat{v}_j, \dots, \hat{v}_i, \dots, v_n]$$

. ∎

15. Friday, October 19, 2012

Recall:

- Simplicial chains of a Δ-complex K: $(C_*^{\Delta}(K), \partial)$.
- $C_n^{\Delta}(K)$ = the free abelian group generated by the n-simplices of K.
- $\partial_n : C_n^{\Delta} \to C_{n-1}^{\Delta}(K)$, acting like $\partial_n[v_0, \dots, v_n] = \sum (-1)^i [v_0, \dots, \hat{v}_i, \dots, v_n]$.

Recall: "Definition": A Δ-complex K is a space $|K|$ together with maps $\sigma_\alpha : \Delta^n \to |K|$ such that

(1) $\sigma_\alpha|_{\text{int}(\Delta^n)}$ is injective and each point of $|K|$ is in exactly one "open simplex."
(2) $\sigma_\alpha|_{\text{face of } \Delta^n}$ is one of the σ_β.
(3) $A \subset |K|$ is open if and only if $\sigma_\alpha^{-1}(A)$ is open in $\Delta^n \forall \alpha$.

In this setting, the $\sigma_\alpha : \Delta^n \to |K|$ generate $C_n^{\Delta}(K)$ and

$$\partial_n(\sigma_\alpha^n) = \sum (-1)^i \sigma_\alpha|_{[v_0, \dots, \hat{v}_i, \dots, v_n]}.$$

Definition 15.1.

$$H_*^\Delta(K) := H_*(C_*^\Delta(K), \partial)$$

is the simplicial homology of K.

Example 15.2. Let $K = \text{pt}$.

$$C_*(\text{pt}) = \cdots \to 0 \overset{\partial_1}{\to} \mathbb{Z} \overset{\partial_0}{\to} 0.$$

All ∂_i are 0. $H_0^\Delta(\text{pt}) = \ker \partial_0 / \text{Im } \partial_1 = \mathbb{Z}/0 = \mathbb{Z}$.

So $H_*^\Delta(\text{pt}) = \begin{cases} \mathbb{Z} & * = 0 \\ 0 & * > 0 \end{cases}$.

Example 15.3. Let $K =$ the 2-simplex. The faces are:

$$\begin{array}{cc} 0 & [v_0], [v_1], [v_2] \\ 1 & [v_0, v_1], [v_0, v_2], [v_1, v_2] \\ 2 & [v_0, v_1, v_2] \\ n(n > 2) & \varnothing \end{array}$$

We know already that $C_n^\Delta(K) = 0$, for $n > 2 \Rightarrow H_n^\Delta(K) = 0$, for $n > 2$.

$$\partial_2[v_0, v_1, v_2] = [v_1, v_2] - [v_0, v_2] + [v_0, v_1] \neq 0. \Rightarrow \partial_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

$$\partial_1[v_0, v_1] = [v_1] - [v_0]. \qquad \partial_1[v_0, v_2] = [v_2] - [v_0]. \qquad \partial_1[v_1, v_2] = [v_2] - [v_1].$$

$$\Rightarrow \partial_1 = \begin{pmatrix} -1 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}$$

Now to compute homology groups:

$$H_2^\Delta(K) = \ker \partial_2 / \text{Im } \partial_3 = 0/0 = 0.$$

$$H_1^\Delta(K) = \ker \partial_1 / \text{Im } \partial_2 = \langle [v_1, v_2] - [v_0, v_2] + [v_0, v_1] \rangle / \langle [v_1, v_2] - [v_0, v_2] + [v_0, v_1] \rangle = 0.$$

$$H_0^\Delta(K) = \ker \partial_0 / \text{Im } \partial_1 = \langle [v_0], [v_1], [v_2] \rangle / \langle [v_1] - [v_0], [v_2] - [v_0] \rangle$$
$$= \langle [v_0], [v_1] - [v_0], [v_2] - [v_0] \rangle / \langle [v_1] - [v_0], [v_2] - [v_0] \rangle = \langle [v_0] \rangle \cong \mathbb{Z}.$$

Therefore,

$$H_*^\Delta(K) = \begin{cases} \mathbb{Z} & * = 0 \\ 0 & * \neq 0 \end{cases}.$$

Example 15.4. Consider the torus as a Δ-complex as in Figure 11.

$$C_*^\Delta(T) = \cdots \to 0 \overset{\partial_3}{\to} \mathbb{Z}_A \oplus \mathbb{Z}_B \overset{\partial_2}{\to} \mathbb{Z}_a \oplus \mathbb{Z}_b \oplus \mathbb{Z}_c \overset{\partial_1}{\to} \mathbb{Z}_v \overset{\partial_0}{\to} 0.$$

$$\partial_2(A) = a - c + b, \partial_2(B) = b - c + a \Rightarrow \ker \partial_2 = \langle A - B \rangle, \text{ Im } \partial_2 = \langle a + b - c \rangle.$$

$$H_*^\Delta(T) = \begin{cases} \mathbb{Z} & * = 0 \\ \mathbb{Z} \oplus \mathbb{Z} & * = 1 \\ \mathbb{Z} & * = 2 \\ 0 & * > 2, * < 0 \end{cases}.$$

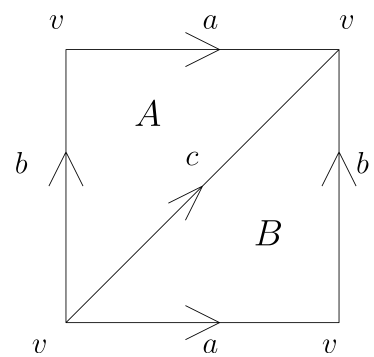


FIGURE 11. Delta Complex for the Torus.

16. Monday, October 22, 2012

Recall: Simplicial homology of a Δ-complex–

- $H_*^\Delta(K) := H_*(C_*^\Delta(K), \partial)$.
- $H_*^\Delta(\text{pt}) = \begin{cases} \mathbb{Z} & * = 0 \\ 0 & * \neq 0 \end{cases}$.
- $H_*^\Delta(T) = \begin{cases} \mathbb{Z} & * = 0 \\ \mathbb{Z} \oplus \mathbb{Z} & * = 1 \\ \mathbb{Z} & * = 2 \\ 0 & * > 2, * < 0. \end{cases}$.

Remark 16.1. Note that H_1 is the abelianization of π_1. (The homology groups have to be abelian.)

Question 16.2. (1) Does $H_*^\Delta(K)$ depend on the Δ-decomposition of $|K|$?
(2) What about homology for a general topological space X?

16.1. **Singular Homology of a Space X.** Program:

(1) Build a complex $S(X)$ from X, called the "singular complex."
(2) Define $C_*(X) := C_*^\Delta(S(X))$, the "simplicial chain complex."
(3) Define $H_*(X) := H_*(C_*(X)) = H_*^\Delta(S(X))$, "singular homology."

Definition 16.3. $S(X)$ has one n-simplex for each continuous map

$$\sigma : \Delta^n \to X \qquad (\text{"singular simplex of } X).$$

The $(n-1)$-faces of σ are given by

$$\sigma|_{[v_0,\dots,\hat{v}_i,\dots,v_n]} : \Delta^{n-1} \to X.$$

Equivalently, we can start by saying:

Definition 16.4. $C_*(X)$ is generated by all continuous $\sigma^n : \Delta^n \to X$. ∂ is defined on the generators by

$$\partial(\sigma^n) = \sum_{i=0}^{n} (-1)^i \sigma^n|_{[v_1,\dots,\hat{v}_i,\dots,v_n]}.$$

Example 16.5. Let X be a point. There is one continuous map $\sigma^n : \Delta^n \to X$ for each n. Therefore, $S(X)$ has one simplex in each dimension.

$$\Rightarrow C_*(X) = \cdots \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z} \to 0 \to \cdots .$$

What is the differential?

$$\partial(\sigma^n) = \sum (-1)^i \sigma^n|_{[\dots,\hat{v_i},\dots]} = \sum (-1)^i \sigma^{n-1} = \begin{cases} 0 & n \text{ odd} \\ \sigma^{n-1} & n \text{ even } > 0 \\ 0 & n = 0 \end{cases}.$$

$$\ker \partial_n = \begin{cases} \mathbb{Z} & n \text{ odd} \\ 0 & n \text{ even } > 0 \\ \mathbb{Z} & n = 0 \end{cases}.$$

$$\mathrm{Im}\ \partial_{n+1} = \begin{cases} \mathbb{Z} & n \text{ odd} \\ 0 & n \text{ even } > 0 \\ 0 & n = 0 \end{cases}.$$

$$\Rightarrow H_n(\mathrm{pt}) = \begin{cases} \mathbb{Z}/\mathbb{Z} = 0 & n \text{ odd} \\ 0/0 = 0 & n \text{ even } > 0 \\ \mathbb{Z}/0 = \mathbb{Z} & n = 0 \end{cases}.$$

16.2. **Relative Homology / Chains.**

Definition 16.6. If $f : X \to Y$ is a continuous map, then we define $f_\# : C_*(X) \to C_*(Y)$ with $(\sigma : \Delta^n \to X) \mapsto (f \circ \sigma : \Delta^n \to Y)$.

Exercise 16.7. Check that $f_\#$ is a chain map, i.e. it commutes with ∂.

Denote the map induced by $f_\#$ on the homology groups by

$$f_* : H_*(X) \to H_*(Y).$$

Definition 16.8. Let $i : A \hookrightarrow X$ be a subspace. Then $i_\# : C_*(A) \hookrightarrow C_*(X)$ is an injection. Denote the quotient $C_*(X)/C_*(A)$ by $C_*(X, A)$.

17. Friday, October 26, 2012

Recall:

$$H_* : \{\text{ Pairs of Top. Spaces } (X, A), A \subset X.\} \to \text{ graded abelian groups.}$$

satisfying

(0) (Dimension)

$$H_*(\text{point}) = \begin{cases} \text{“coefficients”} \in \mathbb{Z} & * = 0. \\ 0 & * \neq 0. \end{cases}$$

(1) (Homotopy) Let f, g be homotopic maps from $(X, A) \to (Y, B)$. Then, $H_*(f) = H_*(g)$ as maps from $H_*(X, A) \to H_*(Y, B)$.

(2) (Exactness) There exists a map δ such that the inclusion $i : A \to X$ induces a long exact sequence:

$$\begin{tikzcd}
H_{i+1}(A) \arrow[r] & H_{i+1}(X) \arrow[r] & H_{i+1}(X, A) \arrow[dll, "\delta"] \\
H_i(A) \arrow[r] & H_i(X) \arrow[r] & H_i(X, A) \arrow[dll, "\delta"] \\
\cdots & &
\end{tikzcd}$$

(3) Excision Theorem/ Axiom:
Let $Z \subset A \subset X$ be a subspace such that $\bar{Z} \subset \operatorname{int}(A)$. Then,

$$(X - Z, A - Z) \overset{i}{\hookrightarrow} (X, A)$$

induces an isomorphism:

$$H_*(X - Z, A - Z) \overset{i_*}{\hookrightarrow} H_*(X, A)$$

Proof. The proof is in two steps:

Definition 17.1. Let $\mathcal{U}$ be a collection of sets whose interiors cover X.
Let $C_n^{\mathcal{U}}(X) \subset C_n(X)$ be generated by $\sigma : \Delta^n \to U_i$, for $U_i \in \mathcal{U}$. Call those $\mathcal{U}$-small chains.

Easy Claim 1:

$$\partial : C_n^{\mathcal{U}}(X) \to C_{n-1}^{\mathcal{U}}(X).$$

so $(C_*^{\mathcal{U}}(X), \partial)$ is a chain complex with homology $H_*^{\mathcal{U}}(X)$.
Easy Claim 2:

$$C_*^{\mathcal{U}}(X) \xrightarrow{\mathcal{I}} C_*(X).$$

is a chain map by inclusion $\sigma \mapsto \sigma$.

Proposition 17.2.

$$C_*^{\mathcal{U}}(X) \xrightarrow{\mathcal{I}} C_*(X)$$

is a quasi-isomorphism, i.e. it induces an isomorphism

$$\mathcal{I}_* : H_*^{\mathcal{U}}(X) \xrightarrow{\sim} H_*(X).$$

A picture of the previous proof is given in Figure 12. This allows us to get pieces of a subspace

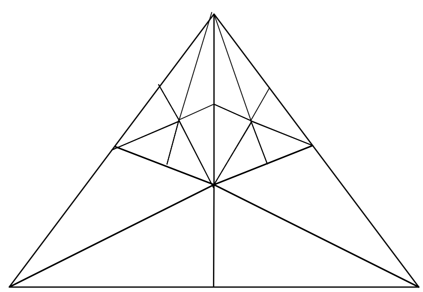


Figure 12. Iterated Barycentric Subdivision.

small enough so that each piece fits in an open set in the cover.
Why does the proposition prove the theorem?
Let $\mathcal{U} = \{A, X - Z\}$. Write: $C_*^{\mathcal{U}}(X) = C_*(A) + C_*(X - Z)$, i.e. the subgroup of $C_*(X)$ generated by $C_*(A)$ and $C_*(X - Z)$.

Definition 17.3. $C_*^{\mathcal{U}}(X, A)$ is defined as $C_*^{\mathcal{U}}(X)/C_*^{\mathcal{U}}(A)$.

Consider the maps

$$\begin{tikzcd} C_n(X - Z, A - Z) \arrow[rr] \arrow[dr] & & C_n(X, A) \\ & C_n^{\mathcal{U}}(X, A) \arrow[ur] & \end{tikzcd}$$

The left map is induced by inclusion. The right map is quasi-isomorphism by the proposition. We want to show that the top map is a quasi-isomorphism.

$$C_n(X-Z, A-Z) = C_n(X-Z)/C_n(A-Z) = C_n(X-Z)/(C_n(X-Z)\cap C_n(A).$$
$$C_n^{\mathcal{U}}(X,A) := C_n^{\mathcal{U}}(X)/C_n^{\mathcal{U}}(A) = (C_n(A)+C_n(X-Z))/C_n(A).$$

is an isomorphism for all n by the second isomorphism theorem for groups. This forces the top map in terms of the homology to also be an isomorphism. □

18. Monday, October 29, 2012

We are still discussing the proof of the excision theorem for singular homology.

Remark 18.1. Suppose instead we were talking about simplicial homology, with X an A-complex, and A, Z subcomplexes of X. Consider $C_*^{\Delta}(X-Z, A-Z)$ and $C_*^{\Delta}(X,A)$. These are given by simplicial complexes that are already broken up into pieces.

With the singular picture on the other hand, we have to define these subcomplexes by hand.

We use barycentric subdivision as pictured above. The barycenter of $\Delta^n \subseteq \mathbb{R}^{n+1}$ can be defined as the point $(\frac{1}{n+1}, \dots, \frac{1}{n+1})$.

Our goal right now is to build a chain homotopy inverse ρ to $\mathcal{I}$.

$$C_*^{\mathcal{U}}(X) \underset{\mathcal{I}}{\overset{\rho}{\leftrightarrows}} C_*(X).$$

Point of Barycentric Subdivision: n simplices in barycentric subdivisions are strictly smaller than $[v_0, \dots, v_n]$. Iterate barycentric subdivision until the diameter of an n-simplex approaches 0.

Idea: Subdivision map:

$$S : C_*(X) \longrightarrow C_*(X)$$
$$(\sigma : \Delta^n \to X) \longmapsto (\text{Subdivide } \Delta \text{ and take sum of restrictions.})$$

This is a chain map, is chain homotopic to

$$id : C_*(X) \to C_*(X),$$

via chain homotopy

$$T : C_*(X) \to C_{*+1}(X).$$

Therefore, S induces an isomorphism $H_*(X) \to H_*(X)$.

Given $\sigma : \Delta^n \to X$, there is $m(\sigma) \in \mathbb{Z}_{\geq 0}$ such that

$$S^{m(\sigma)}(\sigma) \in C_n^{\mathcal{U}}(X).$$

We would like to define

$$\rho : C_n(X) \to C_n^{\mathcal{U}}(X)$$

by $\rho(\sigma) = S^{m(\sigma)}(\sigma)$.

Problem: This might not be a chain map. For example, let τ be a face of σ and $m(\tau) < m(\sigma)$ (we always have $m(\tau) \leq m(\sigma)$).

For example, see Figure 13

The solution to this problem is to add a correction term. (Capturing the failure of ρ to be a chain map.) This is illustrated in Figure 14.

Notation:

$$D_m = \sum_{0\leq i\leq m} Ts^i : C_*(X) \to C_{*+1}(X)$$

is a chain homotopy between S^m and id.

I missed some details of this proof.

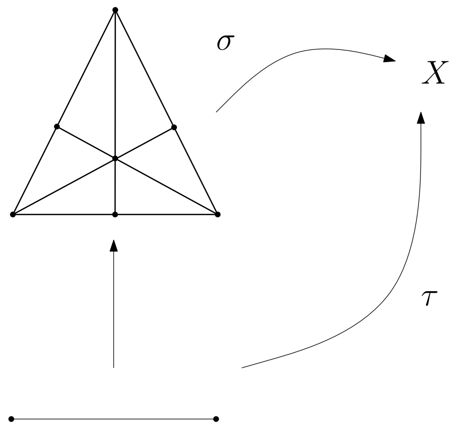


FIGURE 13. Failure to be a Chain Map.

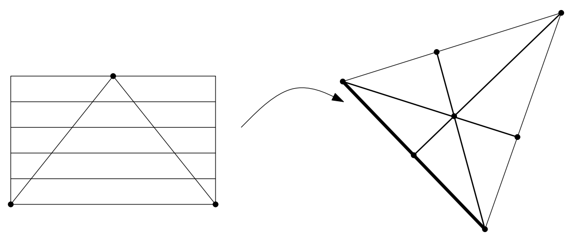


FIGURE 14. Correction to the Chain Map.

19. Friday, November 2, 2012

[Arrived late to class.]

For CW complexes X, we want a simpler way of computing homology – *cellular homology*. We will

(1) define $H_*^{CW}(X)$, and
(2) show $H_*^{CW}(X) \cong H_*(X)$.

The definition uses a lot of things.

(1) First,

$$H_*(S^n) = \begin{cases} \mathbb{Z} & * = 0, n \\ 0 & \text{else.} \end{cases}$$

$H_0(X)$ is generated by the path components of X. We will use LES of $(D^n, \partial D^n)$ by induction on n. For $n = 1$,

$$C_*^{\Delta}(S^1) = (0 \to \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0.$$

$$\Rightarrow H_*(S^1) = \begin{cases} \mathbb{Z} & * = 0, 1 \\ 0 & \text{else.} \end{cases}$$

For $n > 1$ we have the exact sequence:

$$\to H_{k+1}(D^n) \to H_{k+1}(D^n, \partial D^n) \to H_k(\partial D^n) \to \cdots$$

which is equivalent to:

$$0 \to \tilde{H}_{k+1}(S^n) \xrightarrow{\sim} H_k(S^{n-1}) \to 0 \to \tilde{H}_k(S^n) \to \cdots.$$

Therefore, by induction and the argument for $k = 1$, we have $\tilde{H}_k(S^n) \cong H_{k-1}(S^{n-1})$ by induction and argument for $k = 1$.

(2) Let $X = \vee_\alpha X_\alpha$. If $x_\alpha \in X_\alpha$ corresponding to the wedge point gives a good pair (X_α, x_α), then

$$\tilde{H}_*(X) \cong \bigoplus_\alpha \tilde{H}_*(X_\alpha).$$

Proof. Idea: $(\sqcup_\alpha X_\alpha, x_\alpha)$ is a good pair and $X = \sqcup_\alpha X_\alpha / \sqcup_\alpha x_\alpha$. We use the idea that : if $X = \sqcup_\alpha X_\alpha$, then

$$C_*(X) = \bigoplus_\alpha C_*(X_\alpha) \Rightarrow H_*(X) = \bigoplus H_*(X_\alpha).$$

$$H_*(\wedge S^n) = \begin{cases} \mathbb{Z}^{\#\text{ copies of } S^n} & * = n \\ \mathbb{Z} & * = 0 \\ 0 & \text{else} \end{cases}$$

□

(3) Let X be a CW complex with X^n its n-skeleton, then (X^n, X^{n-1}) is a good pair and

$$X^n / X^{n-1} \cong \vee^{\#\text{ of n-cells}} S^n.$$

(4) $H_*(X^n) = 0$ if $* > n$. Proof uses LES of (X^n, X^{n-1}) and 1.

(5) $X^n \hookrightarrow X$ induces an isomorphism $H_k(X^n) \xrightarrow{\sim} H_k(X)$ if $k < n$.

Definition 19.1 (Cellular Chain Complex)**.** Let X be a CW complex.

$$C_n^{CW}(X) := H_n(X^n, X^{n-1}).$$

(i.e. generated by the n-cells of X.)

The differential $d_n : C_n^{CW}(X) \to C_{n-1}^{CW}(X)$, i.e. $H_n(X^n, X^{n-1}) \to H_{n-1}^{CW}(X^{n-1}, X^{n-2})$.

20. Monday, November 4, 2012

Theorem 20.1.

$$H_*^{CW}(X) \cong H_*(X).$$

Proof. Recall: $H_*(X^n) = 0$ if $* > n$.

X^{n+1}/X induces an isomorphism $H_*(X^{n+1}) \to H_*(X)$ if $* < n + 1$.

This allows us to construct the diagram:

$$\begin{tikzcd}
0 = H_n(X^{n+1}) \arrow[dr] & & H_n(X^{n+1}) \cong H_n(X) \xrightarrow{j_n} H_n(X^{n+1}, X^n) = 0 & & \\
& H_n(X^n) \arrow[ur, "i_n"] \arrow[dr, "j_n"] & & & \\
H_{n+1}(X^{n+1}, X^n) \arrow[rr, "d_{n+1}"'] & & H_n(X^n, X^{n+1}) \arrow[rr, "d_n"] \arrow[dr, "\delta_n"'] & & H_{n-1}(X^{n-1}, X^{n-2}) \\
& & & H_{n-1}(X^{n-1}) \arrow[ur, "j_{n-1}"'] & \\
& & 0 \arrow[ur] & &
\end{tikzcd}$$

□

Example 20.2. Let $X = \mathbb{CP}^n$ be a cell structure with cells in dimension $0, 2, 4, \ldots, 2n$. Inductively,

- $n = 1$. $\mathbb{CP}^1 \sim S^2$.
- $n > 1$. $\mathbb{CP}^n \sim \mathbb{CP}^{n-1} \cup e^{2n}$.

$$\Rightarrow C_*^{CW}(\mathbb{CP}^n) = \left(\cdots \to 0 \to \overset{2n}{\mathbb{Z}} \to \overset{2n-1}{0} \to \overset{2n-2}{\mathbb{Z}} \to 0 \to \cdots \to \overset{1}{0} \to \overset{0}{\mathbb{Z}} \to 0 \to \cdots \right)$$

This means $d_n = 0$ for all n.

$$\Rightarrow H_*(\mathbb{CP}^n) = \begin{cases} \mathbb{Z} & * = 2k, k = 0, \ldots, n. \\ 0 & \text{otherwise.} \end{cases}.$$

Remark 20.3. In general, d_n is hard to work with as defined. We want a more geometric definition, one that uses the *degree* of a map from $S^n \to S^n$.

Definition 20.4. Given a continuous map $f : S^n \to S^n$, f induces

$$f_* : H_n(S^n) \to H_n(S^n),$$

i.e. a homomorphism $\mathbb{Z} \overset{\cdot d}{\to} \mathbb{Z}$, sending $\alpha \mapsto d\alpha$. d is called the degree of f.

Example 20.5. The degree of id is 1, since the induced homomorphism of groups is the identity.

If f is not surjective then f factors through $D^n \hookrightarrow S^n$ (the 1-point compactification of D^n). This implies that f is nullhomotopic, which means that the degree f.

Definition 20.6. Cellular Boundary Formula.

$$\begin{array}{ccc} d_n : H_n(X^n, X^{n-1}) & \longrightarrow & H_{n-1}(X^{n-1}, X^{n-2}) \\ \text{(generated by } n\text{-cells)} & & \text{(generated by } (n-1)\text{-cells)} \end{array}$$

is defined by

$$d_n(e_\alpha^n) = \sum_\beta d_{\alpha\beta} e_\beta^{n-1}$$

where $d_{\alpha\beta}$ is the degree of the map

$$S^n \overset{\varphi_\alpha}{\to} X^{n-1} \to X^{n-1}/(X^{n-1} - e_\beta^{n-1}) \cong S^{n-1}.$$

21. Wednesday, November 7, 2012

Example 21.1. Let $X = \mathbb{R}P^n$. X has a cell structure with a cell in each dimension up to n.

$$\mathbb{R}P^n \cong S^n / \text{ antipodal map.}$$

So, we use the cell structure on S^n with 2 cells in each dimension up to n. After quotienting, $\mathbb{R}P^n$ inherits the cell structure.

$$\Rightarrow C_*^{CW}(\mathbb{RP}^n) = \left(\cdots \to 0 \to \overset{n}{\mathbb{Z}} \to \overset{n-1}{\mathbb{Z}} \to \overset{n-2}{\mathbb{Z}} \to \mathbb{Z} \to \cdots \to \overset{1}{\mathbb{Z}} \to \overset{0}{\mathbb{Z}} \to 0 \to \cdots \right)$$

$$S^{k-1} \overset{\varphi_\alpha}{\to} X^{k-1} \to X^{k-1}/(X^{k-1} - e_\beta^{k-1}) \cong S^n.$$

φ_α is a two-sheeted cover.

Exercise 21.2. Show: $q \circ \varphi_\alpha$ has degree $\begin{cases} 2 & \text{if } k \text{ is even} \\ 0 & \text{if } k \text{ is odd} \end{cases}$. Read about local degree.

$$\Rightarrow C_*^{CW}(\mathbb{RP}^n) = \begin{cases} 0 \to \overset{n}{\mathbb{Z}} \overset{\times 2}{\to} \overset{n-1}{\mathbb{Z}} \overset{\times 0}{\to} \overset{n-2}{\mathbb{Z}} \overset{\times 2}{\to} \mathbb{Z} \overset{\times 0}{\to} \cdots \overset{\times 2}{\to} \overset{1}{\mathbb{Z}} \overset{\times 0}{\to} \overset{0}{\mathbb{Z}} \to 0 \to \cdots & n \text{ even} \\ 0 \to \overset{n}{\mathbb{Z}} \overset{\times 0}{\to} \overset{n-1}{\mathbb{Z}} \overset{\times 2}{\to} \overset{n-2}{\mathbb{Z}} \overset{\times 0}{\to} \mathbb{Z} \overset{\times 2}{\to} \cdots \overset{\times 2}{\to} \overset{1}{\mathbb{Z}} \overset{\times 0}{\to} \overset{0}{\mathbb{Z}} \to 0 \to \cdots & n \text{ odd} \end{cases}$$

Now we can compute the homology:

$$\Rightarrow H_*^{CW}(\mathbb{R}P^n) = \begin{cases} \mathbb{Z} & * = 0 \\ \mathbb{Z} & * = n \text{ odd} \\ \mathbb{Z}/2\mathbb{Z} & *\text{odd}, * < n \\ 0 & \text{otherwise} \end{cases}.$$

21.1. **Mayer-Vietoris Sequence.** (Homology analogue of Van Kampen's Theorem for π_1)

The Van Kampen's Theorem told us that if we have a decomposition for a space, we can compute the fundamental group in terms of the groups of each space.

(1) Let $A, B \subset X$ be subsets such that $int(A) \cup int(B) = X$.
  Let $C_*(A+B) = C_*^{\mathcal{U}}(X)$ where $\mathcal{U} = \{A, B\}$.
  [We saw that $C_*^{\mathcal{U}}(X) \hookrightarrow C_*(X)$ is a quasi-isomorphism.]

(2) Inclusions

$$\begin{tikzcd} & A \cap B \arrow[dl, "i_A"'] \arrow[dr, "i_B"] & \\ A \arrow[dr, "j_A"'] & & B \arrow[dl, "j_B"] \\ & X & \end{tikzcd}$$

induce a short exact sequence (exercise) of chain complexes:

$$\begin{array}{c} 0 \longrightarrow C_*(A \cap B) \to C_*(A) \oplus C_*(B) \to C_*(A+B) \to 0. \\ x \mapsto (i_{A_*}(x), -i_{B_*}(x)) \qquad (x,y) \mapsto j_{A_*}(x) + j_{B_*}(y) \end{array}$$

(3) The short exact sequence of chain complexes induces a long exact sequence of homology

$$\begin{tikzcd} H_n(A \cap B) \arrow[r, "\varphi_*"] & H_n(A) \oplus H_n(B) \arrow[r, "\psi_*"] & H_n(X) \arrow[dll, "\delta"] \\ H_{n-1}(A \cap B) \arrow[r] & H_{n-1}(A) \oplus H_{n-1}(B) \arrow[r] & H_{n-1}(X) \arrow[dll, "\delta"] \\ \cdots & & \end{tikzcd}$$

and we have a similar statement for reduced / relative homology. This is called the Mayer-Vietoris Sequence.

22. Wednesday, November 14, 2012

[Class missed on Friday 11/9 due to travel.]

22.1. **Homology with Coefficients.** Fix an abelian group G.

Definition 22.1. Singular chains with coefficients in G are written as:

$$C_n(X; G) = \left\{ \sum g_i \sigma_i \mid \sigma_i : \Delta^n \to X, g_i \in G \right\}.$$

$$\partial \left(\sum g_i \sigma_i \right) = \sum g_i (\partial \sigma_i) = \sum g_i (\pm \sigma_i|_{v_0 \cdots \hat{v_i} \cdots v_n}).$$

Remark 22.2.

$$(C_*(X; \mathbb{Z}), \partial) = (C_*(X), \partial).$$

Again $\partial^2 = 0 \Rightarrow (C_*(X; G), \partial)$ is a chain complex with homology $H_*(X; G)$.

Definition 22.3 (Informal)**.** Analogous definitions, for $A \subset X$,

$$C_*(X, A; G) \to H_*(X, A; G).$$

similar for $C_*^{\Delta}(X; G)$ and $C_*^{CW}(X; G)$, and the same proof shows

$$H_*^{\Delta}(X; G) \cong H_*(X; G) \cong H_*^{CW}(X; G).$$

Remark 22.4.

$$C_n(X; \mathbb{Z}) \cong \bigoplus_{\text{singular n-simplices of } X} \mathbb{Z}.$$

$$C_n(X; G) \cong \bigoplus G.$$

$$C_n(X; G) \cong C_n(X) \otimes G.$$

$$C_n(X; G) \overset{\partial \otimes id}{\to} C_{n-1}(X; G).$$

We need not have $H_n(X; G) \cong H_n(X) \otimes G$! Universal Coefficient Theorem.

Sometimes calculations over $G \neq \mathbb{Z}$ reveal something new.

Example 22.5. Let $X = \mathbb{R}P^n$. Take $G = F$ a field. Recall: $C_*^{CW}(\mathbb{R}P^n)$ is:

$$\cdots \to 0 \to \overset{n}{\mathbb{Z}} \underset{\times 2 \ \ n \text{ even}}{\overset{\times 0 \ \ n \text{ odd}}{\longrightarrow}} \overset{n-1}{\mathbb{Z}} \to \cdots \overset{\times 2}{\longrightarrow} \overset{1}{\mathbb{Z}} \overset{\times 0}{\longrightarrow} \overset{0}{\mathbb{Z}} \to 0.$$

which implies that:

$$H_*(\mathbb{R}P^n) = \begin{cases} \mathbb{Z} & * = 0, n,\ n \text{ odd} \\ \mathbb{Z}/2\mathbb{Z} & 0 < * < n \text{ odd} \\ 0 & \text{otherwise} \end{cases}$$

Over F, $C_*^{CW}(\mathbb{R}P^n; F)$ is given by

$$\cdots \to 0 \to \overset{n}{F} \underset{\times 2 \ \ n \text{ even}}{\overset{\times 0 \ \ n \text{ odd}}{\longrightarrow}} \overset{n-1}{F} \to \cdots \overset{\times 2}{\longrightarrow} \overset{1}{F} \overset{\times 0}{\longrightarrow} \overset{0}{F} \to 0.$$

if char $F = 2$, then

$$H_*(\mathbb{R}P^n; F) = \begin{cases} F & 0 \leq * \leq n \\ 0 & \text{otherwise} \end{cases}$$

If char $F \neq 2$, then every $\times 2$ is an isomorphism which means that

$$H_*(\mathbb{R}P^n; F) = \begin{cases} F & * = 0, * = n \text{ odd} \\ 0 & \text{otherwise} \end{cases}$$

22.2. **Cohomology.**

Definition 22.6. A cochain complex (C^*, δ) is a sequence of abelian groups (R-modules/F-vectorspaces...) C^i together with $\delta^{i+1} : C^i \to C^{i+1}$ such that $\partial^{i+1} \circ \partial^i = 0$.

$$\cdots \leftarrow C^{i+2} \overset{\delta}{\leftarrow} C^{i+1} \overset{\delta}{\leftarrow} C^i \overset{\delta}{\leftarrow} C^{i-1} \leftarrow \cdots$$

Definition 22.7. The i-th cohomology group of a cochain complex (C^*, δ)

$$H^i(C^*, \delta) := \ker \delta^{i+1} / \text{im } \delta^i.$$

Definition 22.8. Given a chain complex (C_*, ∂) and an abelian group G, one can build a cochain complex

$$C^i := \text{Hom}(C_i, G) = \{\varphi : C_i \to G\}.$$

$$\delta^i : \text{Hom}(C_i, G) \to \text{Hom}(C_{i+1}, G), \qquad \varphi \mapsto \varphi \circ \partial.$$

Proposition 22.9. *Given a chain map*

$$f_{\#} : C_* \to D_*,$$

there exists an induced cochain map

$$f^{\#} : D^* \to C^*.$$

Proposition 22.10. *A cochain map* $f^{\#} : D^* \to C^*$ *induces a map on cohomology*

$$f^* : H^*(D^*) \to H^*(C^*).$$

23. Friday, November 16, 2012

Everything said last class about normal cohomology is true also for G-valued cohomology.

Definition 23.1. An *ordinary cohomology theory* is a contravariant functor.

$$(\text{Pairs of topological spaces } A \subset X, (X,A)) \xrightarrow{H^*(-;G)} (\text{graded abelian groups})$$

satisfying the following four axioms:

(1) "dimension".

$$H^*(pt;G) = \begin{cases} G & * = 0 \\ 0 & * \neq 0 \end{cases}$$

(2) "homotopy". If

$$\begin{tikzcd} (X,A) \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] & (Y,B) \end{tikzcd}$$

are homotopic, then

$$H^*(X,A;G) \xleftarrow[f=g]{} H^*(Y,B;G)$$

(3) "exactness". Maps induce Long Exact Sequence

$$\begin{tikzcd} & & \cdots & H^{n+1}(X,A;G) \arrow[l] \\ H^n(A;G) \arrow[urrr, "\delta"] & H^n(X;G) \arrow[l] & & H^n(X,A;G) \arrow[ll] \\ H^{n-1}(A;G) \arrow[urrr, "\delta"] & \cdots \arrow[l] & & \end{tikzcd}$$

(4) "excision". Given $Z \subset A \subset X$ such that $\bar{Z} \subset int(A)$, then inclusion $(X-Z, A-Z) \hookrightarrow (X,A)$ induces an isomorphism

$$H^*(X-Z, A-Z;G) \xleftarrow[i^*]{\sim} H^*(X,A;G)$$

Definition 23.2. An *extraordinary/exotic cohomology theory* satisfies the last three axioms.

Theorem 23.3 (Eilenberg-Steenrod)**.** *Any two cohomology theories satisfying the four axioms are equivalent.*

Theorem 23.4. *If X has a Δ-structure or CW-structure, then*

$$H^*(X;G) \cong H^*_{\Delta}(X;G) \cong H^*_{CW}(X;G).$$

Theorem 23.5. *Singular cohomology satisfies the four axioms.*

Theorem 23.6 (de Rham)**.** *If X is a smooth $\mathbb{R}$-manifold, its differential forms form a cochain complex ($\delta =$ exterior derivative).*

$$H^*_{dR}(X;\mathbb{R}) \cong H^*(X;\mathbb{R}).$$

Remark 23.7. All "co-proofs" work. For example, M-V sequence for cohomology.

Remark 23.8. In general,
$$H^n(X;G) \ncong \mathrm{Hom}(H_n(X);G),$$
but the universal coefficient theorem provides conditions when they are.

It looks like $H^*(X;G)$ and $H_*(X;G)$ carry the same info about X but if $G = R$ a commutative ring with 1, then
$$\bigoplus H^i(X;G) = H^*(X;G)$$
can be given a ring structure.

Plan:

(1) We will define $C^k(X;R) \times C^l(X;R) \xrightarrow{\cup} C^{k+l}(X;R)$.
(2) We will prove an algebraic statement about when a product on a cohcain complex $C^k \times C^l \to C^{k+l}$ induces a product on the cohomology $H^k \times H^l \to H^{k+l}$.
(3) We will show (1) satisfies the property in (2).
(4) We will see an example where this structure distinguishes two spaces with isomorphic homology groups.

24. Monday, November 19, 2012

We will begin with step (2) of the plan from last class. The algebraic statement will motivate the definition of the cup product.

Question 24.1. Let (C^*, δ) be a cochain complex and let $\cdot : C^k \times C^l \to C^{k+l}$ be an associative product for all k, l (i.e. a map $C^* \times C^* \to C^*$).

What must $(\cdot, \delta)$ satisfy to induce a well-defined product
$$H^k \times H^l \xrightarrow{\cdot} H^{k+l} \qquad \forall k, l,$$
$$H^* \times H^* \to H^*?$$
We want $[\varphi] \cdot [\psi] \in H^{k+l}$ for $[\varphi] \in H^k, [\psi] \in H^l$ for cocycles $\varphi \in C^k, \psi \in C^l$.

We want: $\varphi \cdot \psi$ cocyle to satisfy $[\varphi] \cdot [\psi] := [\varphi \cdot \psi]$. We need:

- product of cocycles = cocycle.
- coboundary times cocycle = coboundary.
- cocycle times coboundary = coboundary.

Lemma 24.2. *If $\delta(\varphi \cdot psi) = (\delta\varphi) \cdot \psi + (-1)^k \varphi \cdot (\delta\psi)$, for all $\varphi, \psi \in C^*$ where $\varphi \in C^k$, then the chain map induces a well-defined map on homology.*

Proof. Let $\delta\varphi = \delta\psi = 0$. Then $\delta(\varphi \cdot \psi) = 0$.

Assuem $\varphi = \delta\phi$, and $\delta\psi = 0$, then $\delta(\phi \cdot \psi) = \delta\phi \cdot \psi \pm \phi \cdot \delta\psi$. Then $\delta(\phi \cdot \psi) = \varphi \cdot \psi$. □

Definition 24.3. If δ satisfies the hypothesis of the Lemma, then it is called a *derivation* of $\cdot$.

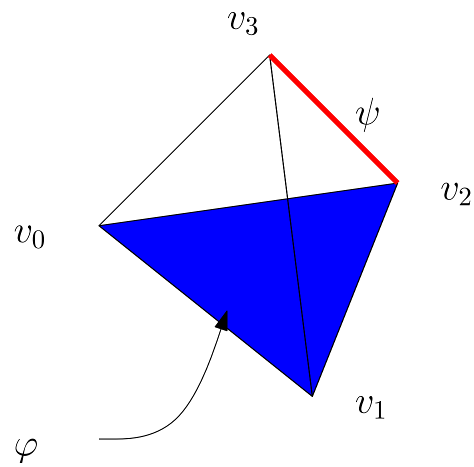
Now we return to step (1), defining the cup product for singular cochains of a space $C^*(X;R)$, with X a space, and R a commutative ring with 1.

Definition 24.4. We define the map on *singular cochains*
$$C^k(X;R) \times C^l(X;R) \xrightarrow{\cup} C^{k+l}(X;R)$$
by taking $\varphi \in C^k(X;R)$ and $\psi \in C^l(X;R)$, and defining a function acting on $k+l$-chains.

Let the $k+l$-simplex be denoted by $\Delta^{k+l} = [v_0, \ldots, v_k, v_{k+1}, \ldots, v_{k+1}]$. Then,
$$(\varphi \cup \psi)(\sigma) = \varphi(\sigma|_{[v_0,\ldots,v_k]}) \cdot \psi(\sigma|_{[v_k,\ldots,v_{k+l}]}).$$
where $\cdot$ is the product in R.

This defines a product $\cup$ on $(C^*(X;R),\delta)$.

Example 24.5. The following image has $k=2$, $l=1$, and shows which faces of the 3-simplex φ and ψ act on.



Exercise 24.6. Show that the cup product satisfies the lemma, so that $\cup$ induces a well-defined product on homology.

Now we proceed to computing an example.
[Recall: $C_*^{\Delta}(X) \hookrightarrow C_*(X)$ is a quasi-isomorphism. Likewise for $C^*_{\Delta}(X) \leftarrow C^*(X)$.]
Let $X = T = S^1 \times S^1$, and $Y = S^1 \vee S^1 \vee S^2$.
We claim that the cup product for homology distinguishes these two spaces.