COMBINATORIAL HOPF ALGEBRAS - ECCO'12
EXERCISES LECTURE 3

(1). Prove that $Sym^* \cong Sym$ by showing that the correspondance

$$h^*_\lambda \mapsto m_\lambda$$

is an isomorphism.

(2). Prove that

$$h_\lambda = \sum_\mu K_{\lambda,\mu} s_\mu$$

where $K_{\lambda,\mu}$ is the number of semistandard Young tableaux of shape λ and content (or filling) μ. Compare this coefficient with the ones from question (2) from yesterday.

(3). Write a formula for the product and coproduct of the basis $\{M_\gamma\}$ of $QSym$.

(4). Define $SSym := \bigoplus_{n\geq 0} kS_n$. A basis at degree n is given by $\{F_\sigma\}_{\sigma\in S_n}$. This basis multiplies and comultiplies as follows:

$$F_\sigma F_\mu = \sum_{\nu=\sigma\cdot\mu} F_\nu \qquad \Delta(F_\sigma) = \sum_{\sigma=\tau\cdot\pi} F_{st(\tau)} \otimes F_{st(\pi)}$$

Can you realize this (Hopf) algebra as a subspace of $k\langle\langle x_1, x_2, \ldots\rangle\rangle$?

(5). Compute the dimension (as a vector space) of

$$k[x_1, x_2, \ldots, x_n]/\langle QSym^+\rangle$$

for $n = 1, 2, 3, \ldots$