Math 371 Homework#7

Due on 4/3 11:59pm EST on canvas

1. **Artin, Chapter 11, 1.7 (a)**
   Let U be an arbitrary set and R be the set of subsets in U. Addition and multiplication of elements of R are defined by $A + B = A \cup B - A \cap B$ and $A \cdot B = A \cap B$. Prove that R is a ring.

2. Determine whether the division with reminder $g(x) = f(x)q(x) + r(x)$ exists in $R[x]$ for the following $R, f(x), g(x)$. If it exists, find the $q(x), r(x)$.

   (a) $R = \mathbb{Z}, f(x) = 2x^2 + x + 1, g(x) = 2x^3 + 7x^2 + 4x + 8$.

   (b) $R = \mathbb{Z}/6\mathbb{Z}, f(x) = 2x+1, g(x) = 2x^2+2x$. (Here the leading coefficient of $f(x)$ is 2 and is not a unit in R, but it is still possible to find such $q(x), r(x)$, but maybe not unique.)

   (c) $R = \mathbb{Z}/6\mathbb{Z}, f(x) = 5x + 1, g(x) = 2x^2 + 2x$.

3. Let R be a ring and I, J ideals of R. Prove the following

   (a) $I \cap J$ is an ideal of R,

   (b) $I + J = \{a + b | a \in I, b \in J\}$ is an ideal of R,

   (c) $IJ = \{\sum_{i=0}^{n} a_i b_i | a_i \in I, b_i \in J, n \in \mathbb{N}\}$ is an ideal of R.

4. Let $I = (a)$ and $J = (b)$ be two ideals of R. Prove that $I \subset J$ if and only if b divides a. Use this fact and correspondence theorem to classify all the ideals in $\mathbb{Z}/12\mathbb{Z}$.

5. **Artin Chapter 11, 3.4** Let $\phi\colon \mathbb{C}[x, y] \to \mathbb{C}[t]$ defined by $x \mapsto t + 1$ and $y \mapsto t^3 - 1$. Determine the kernel K of ϕ and prove that every ideal I of $\mathbb{C}[x, y]$ that contains K can be generated by two elements.

6. **Artin Chapter 11, 3.2** Prove that any ideal of Gaussian integers $\mathbb{Z}[i]$ must contain an integer.

7. Identify the ring $\mathbb{Z}[\sqrt{-3}]/(2 + \sqrt{-3})$ with $\mathbb{Z}/n\mathbb{Z}$ for some n. Here $\mathbb{Z}[\sqrt{-3}] = \{a + b\sqrt{-3} | a, b \in \mathbb{Z}\}$

8. **Artin Chapter 11, 3.3 a) b)** Find generators of the kernels of the following maps:

   (a) $\mathbb{R}[x, y] \to \mathbb{R}$ by $f(x, y) \mapsto f(0, 0)$,

   (b) $\mathbb{R}[x] \to \mathbb{C}$ by $f(x) \mapsto f(2 + i)$.