The Hurwitz Existence Problem

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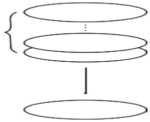
Review: Branched covers

Visual representations of...

a (compact Riemann) surface
Σ_g of genus g

a covering map
$X \longrightarrow Y$



- $f : \Sigma_g \longrightarrow \Sigma_0$ is a ***branched cover*** if it behaves like a covering map over all but a finite subset $y_1, \ldots, y_n$ of Σ_0. These points are called ***branch points***. Its ***degree*** d is $\#f^{-1}(x)$ for any $x \neq y_i$.
- As morphisms between Riemann surfaces are locally $z \mapsto z^{k_z}$, each y_i can be associated to the partition $A_i = [k_z]_{z \in f^{-1}(y_i)}$ that is necessarily a partition of d and nontrivial (i.e. not $[1, \ldots, 1]$).
- A branched cover satisfies the ***Riemann-Hurwitz formula***

$$\sum_{i=1}^{n} (d - \#A_i) = 2d - 2 + 2g.$$

The Hurwitz problem: Statement

A branched cover of degree d gives rise to a collection of partitions $A_1, \dots, A_n$ of d. We want to understand the converse of this statement.

The Hurwitz Problem (Modern Reformulation)

Fix positive integers d and g. Given a collection of partitions $A_1, \dots, A_n$ of d satisfying the Riemann-Hurwitz formula

$$\sum_{i=1}^{n} (d - \#A_i) = 2d - 2 + 2g,$$

when does it correspond to a branched cover $f : \Sigma_g \longrightarrow \Sigma_0$?

We call a collection of partitions satisfying the hypothesis of the Hurwitz problem a ***branch data***. If a branch data satisfies the conclusion of the Hurwitz problem, it is said to be ***realizable***.

Overview of some of our results

Zheng (2004) computed all non-realizable branch datum for degrees $d \leq 22$. However, his result is purely computational and does not give any insight on how to solve the Hurwitz problem in general. Below is a numerical summary.

Genus	Nonrealizable branch data of degree ≤ 22
0	27235
1	50
2	9
≥ 3	0

We found infinite families that includes all but ~ 100 of these examples, and subsumes all known results in the literature.

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We found infinite families that includes all but ~ 100 of these examples, and subsumes all known results in the literature. In this talk:

- we will give some existence results on the Hurwitz problem guaranteeing realizability when there are large cycles or many branch points,
- we look at some complementary nonexistence results in low genus that shows us the sharpness and limitations to the bounds.

Overview of some of our results

Using an algebraic formulation of the Hurwitz problem, we are able to prove the following theorem, extending results of Barański, Tomasini, and Edmonds-Kulkarni-Stong in the extreme cases.

Many Branch Point and Large Cycles Theorem

The genus $g \geq 1$ and degree $d \neq 4$ branch data

$$[a_1, 1, \ldots, 1], \ldots, [a_l, 1, \ldots, 1], A_1, \ldots, A_k \qquad (k, l \geq 0)$$

is realizable if $k + l \geq d/2 + 2 - g$, or if $\sum_{i=1}^{l} a_i > d/2 + 1 + l - g$.

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- The first inequality still holds when $g = 0$, and is sharp there by non-realizable data $[d/2 + 1, 1, \ldots, 1], [2, \ldots, 2], [2, \ldots, 2], [2, 1, \ldots, 1]^{d/2-2}$.

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- The second bound is sharp in genus 1 by non-realizable data $[(d + 2)/2, 1, \ldots, 1], [2, \ldots, 2]^3$.
- The converse is false by other infinite families results arising from our work. For example, there are realizable and nonrealizable data of the form $[\alpha, \beta, \gamma_1, \ldots, \gamma_1], [\gamma_2, \ldots, \gamma_2], [\gamma_3, \ldots, \gamma_3]$.

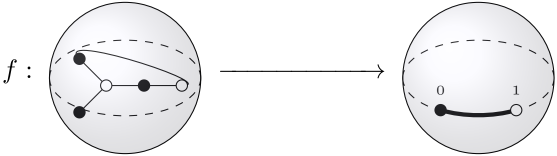
We now look more closely at the last infinite family.

Dessin d'enfants: construction

A *dessin d'enfant* associated to a branched covering $f : \Sigma_g \longrightarrow \Sigma_0$ branched over $0, 1, \infty$ (after a Möbius transformation) is the preimage $f^{-1}[0,1]$ of the line segment $[0,1]$. This is a connected bipartite graph embedded on Σ_g if:

- the preimages of 0 are colored black,
- the preimages of 1 are colored white.

Every connected component of the graph is contractible and corresponds to an element of $f^{-1}(\infty)$. Below is an example for genus zero.



$$\begin{cases} \sigma_0 = [2,2,1] \\ \sigma_1 = [3,2] \\ \sigma_\infty = [3,2] \end{cases}$$

This construction can be extended to $n > 3$ branch points by using n-vertex-coloring and imposing extra "orientable-compatibility" conditions on the edges.

An infinite family of the form $[\alpha, \beta, \gamma_1^*], [\gamma_2^*], [\gamma_3^*]$

We used dessin d'enfants to classify genus one branch data of the form

$$P = \{[\alpha, \beta, 2, \ldots, 2], [2, \ldots, 2], [r, \ldots, r]\}.$$

We proved that P is not realizable if and only if $\alpha - \beta = 2$ with α odd.

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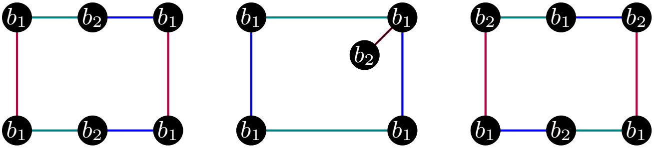
- The case $(\alpha, \beta) = (3, 5)$ was already discovered by Petronio-Pervova.
- The idea of proof is by delicate counting of the lengths of faces, and then doing induction on the degree.
- This is hard to do on the surface. Hence we use surgery theory to translate our study into the combinatorics of line configurations on the following three rectangles (vertices/edges with the same label/color are identified).



There are six possible configurations of the black vertices α and β if we position them at b_1 or b_2, and the white vertices can be ignored in a dessin d'enfant drawing as they are of degree two.

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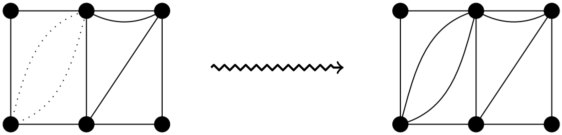
- The key is to use *two edge iteration*: if two "neighboring" Jordan paths with fixed endpoints b_1 or b_2 are homotopic, then adding/deleting them gives another dessin of the form P.



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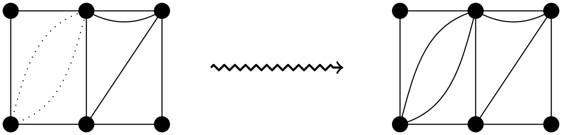

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- As for the remaining cases, some estimates on r using the Riemann-Hurwitz formula gives us 32 base cases, after which we do induction by some construction based on two edge iterations.

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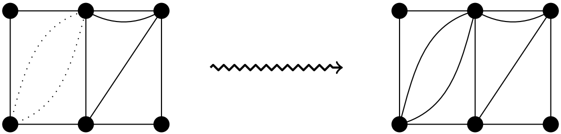

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- As for the remaining cases, some estimates on r using the Riemann-Hurwitz formula gives us 32 base cases, after which we do induction by some construction based on two edge iterations.
- We can use a similar idea with extended dessin d'enfants to show non-realizability of $[(d+2)/2, 1, \ldots, 1], [2, \ldots, 2]^3$.

Another infinite family of the form $[\alpha, \beta, \gamma_1^*], [\gamma_2^*], [\gamma_3^*]$

There is another hard case of non-realizable genus one infinite families to study.

$$[2, 4, 3, \ldots, 3], [3, \ldots, 3], [3, \ldots, 3]$$
$$[5, 7, 6, \ldots, 6], [2, \ldots, 2], [3, \ldots, 3]$$
$$[3, 5, 4, \ldots, 4], [2, \ldots, 2], [4, \ldots, 4]$$

They are the genus one branch data of the form

$$[p-1, p+1, p, \ldots, p], [m, \ldots, m], [n, \ldots, n]$$

with m and n dividing p.

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They are the genus one branch data of the form

$$[p-1, p+1, p, \ldots, p], [m, \ldots, m], [n, \ldots, n]$$

with m and n dividing p. Our analysis of these was as follow.

- Assume realizability to get a dessin $\mathcal{D}$ for contradiction.
- Make general topological statements of $\mathcal{D}$ that we can assume, using surgery theory to avoid possible topological obstructions.
- Create an algorithm that reduces the branch data to a smaller degree one that is still inside the same infinite family to get a contradiction by induction.
- Make sure the base case is small enough so we can analyze directly or by using Zheng's data.

Conjecture from REU work

"Master" Conjecture

Every nonrealizable branch data $\mathcal{A} = \{A_1, \ldots, A_n\}$ *satisfy one of the following properties:*

- $\gcd(A_i) > 1$ *for some partition* A_i*;*
- $\mathcal{A}$ *is one of the three genus zero families:*
  1. $\mathcal{A} = \{[1, 3, \ldots, 3], [1, 3, \ldots, 3], [1, 3, \ldots, 3]\}$;
  2. $\mathcal{A} = \{[1, 2, \ldots, 2], [1, 4, \ldots, 4], [1, 4, \ldots, 4]\}$;
  3. $\mathcal{A} = \{[1, 2, \ldots, 2], [1, 3, \ldots, 3], [1, 6, \ldots, 6]\}$.

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  3. $\mathcal{A} = \{[1,2,\ldots,2],[1,3,\ldots,3],[1,6,\ldots,6]\}$.

- This conjecture is consistent with the 27294 nonrealizable branch data computed by Zheng, and is also consistent with all our results.
- It is also stronger than the *prime degree conjecture*, that asks if every branch data of prime degree is realizable.
- Suffices to prove for genera zero and one if we have the *higher genus conjecture*, which asserts every branch data of genus ≥ 2 is realizable except for two infinite families and an isolated example.