Examples of spectral sequences

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Spectral sequences are a collection of useful tools that helps compute the cohomology of various algebraic or topological objects. In this note we will only give examples of first-quadrant cohomological spectral sequences. Dual notions exists for homology and for other quadrants, and one can see [7, Chapter 5] and [3] for definitions and examples.

1 Introduction

We briefly introduce the notion of a spectral sequence in this section. Since the construction of spectral sequences is not very illuminating, we have chosen to spend more time on examples; see the next few sections for some of them. Here is an abstract definition of a spectral sequence.

Definition 1. Let R be a ring, and let r_0 be a fixed nonnegative integer. A (first-quadrant cohomological) *spectral sequence* E consists of the following data:

- a collection of R-modules $E_r^{p,q}$, with integers $p, q \geq 0$ and $r \geq r_0$,
- and a collection of differentials $d_r = d_r^{p,q} : E_r^{p,q} \longrightarrow E_r^{p+r,q-r+1}$ such that $(d_r^{p,q})^2 = 0$ and $E_{r+1}^{p,q}$ is the homology at $E_r^{p,q}$, i.e.

$$E_{r+1}^{p,q} = \frac{\ker\left(d_r \colon E_r^{p,q} \longrightarrow E_r^{p+r,q-r+1}\right)}{\operatorname{im}\left(d_r \colon E_r^{p-r,q+r-1} \longrightarrow E_r^{p,q}\right)}.$$

The collection $E_r = \{(E_r^{p,q}, d_r) : p, q \geq 0\}$ for a fixed r is called the r^{th}*-page*. The spectral sequence *converges* if $E_s = E_{s+1} = E_{s+2} = \cdots$ for a large enough s, and we denote this page by E_∞.

Remark. We may sometimes assign an algebra structure to a page E_r of a spectral sequence. In this case, the algebra structure $\smile$ needs to respect the grading, i.e. $\smile \colon E_r^{p,q} \times E_r^{p',q'} \longrightarrow E_r^{p+p',q+q'}$.

The reason why spectral sequences are useful is because of the following definition, which characterizes results that are related to spectral sequences. The examples in this note will make clear of this definition.

Definition 2. A *filtered chain complex* is a chain complex $\mathcal{C} = (C_i, \delta_i)$ of R-modules together with a differential-preserving filtration on each C_i, i.e. for each C_i, there is an increasing sequence of R-modules

$$F_0C_i \subset F_1C_i \subset \cdots \subset F_{p-1}C_i \subset F_pC_i \subset F_{p+1}C_i \subset \cdots$$

such that

$$\bigcup_p F_pC_i = C_i, \qquad \bigcap_p F_pC_i = 0, \qquad \delta_i(F_pC_i) \subset F_pC_{i+1}.$$

A filtration on the chain complex induces a filtration $F_pH^*(\mathcal{C})$ on the cohomology groups $H^*(\mathcal{C})$. The associated graded pieces are defined by

$$G_pH^n = F_pH^n(\mathcal{C})/F_{p+1}H^n(\mathcal{C}).$$

A spectral sequence E *converges to* $H^*(\mathcal{C})$, written $E_{r_0}^{p,q} \Rightarrow H^{p+q}(\mathcal{C})$, if E converges and

$$E_\infty^{p,q} \cong G_pH^{p+q}.$$

As the definition implies, a convergent spectral sequence does not compute the cohomology groups we want, but rather a filtration of it. Hence we have the following question, called the *lifting problem*.

$$\text{How is } H^n(\mathcal{C}) \text{ related to } \bigoplus_{p+q=n} E_\infty^{p,q}?$$

There are many instances where the two terms are equal. An instance is when the diagonal $p + q = n$ contains exactly one nontrivial term, which we will see a lot of in the next section. If a nontrivial algebra structure is involved, the following easy proposition will guarantee equality.

Proposition 3. *If the E_∞-page is a free graded commutative algebra, then $Tot(E_\infty)$ equals $H^n(\mathcal{C})$.*

We end the introduction by stating a consequence of the definition of spectral sequences, which follows by diagram chasing.

Proposition 4. *Given a spectral sequence starting from the E_2-page and converging to the cohomology of $\mathcal{C}$ (so $E_2^{p,q} \Rightarrow H^{p+q}(\mathcal{C})$ in the notation of Definition 2), there is an associated five-term exact sequence*

$$0 \longrightarrow E_2^{1,0} \longrightarrow H^1(\mathcal{C}) \longrightarrow E_2^{0,1} \xrightarrow{d_2} E_2^{2,0} \longrightarrow H^2(\mathcal{C}).$$

2 Three examples of spectral sequences

In this section we state three examples of spectral sequences, and give examples for each of them. One can refer to the references for proofs of these spectral sequences.

Double complexes

Theorem 5 (Double complex spectral sequence [6, Section 1.7])**.** *Let $\mathcal{C} = C^{p,q}$ be a double complex over an abelian category. Then there exists spectral sequences*

$$E_0^{p,q} = C^{p,q}, \qquad E_0^{p,q} = C^{q,p},$$

both converging to $H^{p+q}(Tot(\mathcal{C}))$. The differentials in the respective $E_0^{p,q}$-pages are induced from the maps in the double complex $\mathcal{C}$.

Example 6. Let us use the above theorem to prove the Snake Lemma, which says that the following commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & 0 \\
0 \arrow[r] & A \arrow[r] \arrow[u, "f"] & B \arrow[r] \arrow[u, "g"] & C \arrow[r] \arrow[u, "h"] & 0
\end{tikzcd}$$

gives rise to the exact sequence

$$0 \longrightarrow \ker f \longrightarrow \ker g \longrightarrow \ker h \xrightarrow{\delta} \operatorname{coker} f \longrightarrow \operatorname{coker} g \longrightarrow \operatorname{coker} h \longrightarrow 0.$$

To do this, we first use the second spectral sequence, with E_0-page as follows.

$$\begin{array}{|ccc}
0 & 0 & 0 \\
A & A' & 0 \\
B & B' & 0 \\
C & C' & 0 \\
0 & 0 & 0 \\
\hline
\end{array}$$

By exactness, this spectral sequence converges to zero. Let us now look at the first spectral sequence, with the following E_0-page.

$$\begin{array}{|ccccc}
0 & 0 & 0 & 0 & 0 \\
0 & A' & B' & C' & 0 \\
0 & A & B & C & 0 \\
\hline
\end{array}$$

The E_1-page will look like

$$\begin{array}{|ccccc}
0 & 0 & 0 & 0 & 0 \\
0 & \operatorname{coker} f & \operatorname{coker} g & \operatorname{coker} h & 0 \\
0 & \ker f & \ker g & \ker h & 0 \\
\hline
\end{array}$$

and the E_2-page is as follow.

$$\begin{array}{|ccccc}
0 & 0 & 0 & 0 & 0 \\
0 & \ker(\operatorname{coker} f \longrightarrow \operatorname{coker} g) & 0 & 0 & 0 \\
0 & 0 & 0 & (\ker h)/(\ker g) & 0 \\
\hline
\end{array}$$

Since the spectral sequence converges to zero, we have an isomorphism

$$d : \ker(\operatorname{coker} f \longrightarrow \operatorname{coker} g) \longrightarrow (\ker h)/(\ker g).$$

Therefore $\delta := d^{-1}$ induces the connecting homomorphism in the Snake Lemma, while the other maps are the natural ones (which also corresponds to the differentials in the E_1-page above).

Fibrations

The next spectral sequence we will look at is associated to a fibration, and is usually given as the first example of a spectral sequence.

Theorem 7 (Serre spectral sequence, [3, Theorem 5.1])**.** *Let $F \longrightarrow E \longrightarrow B$ be a fibration with B simply connected, and let R be a ring. Then there is a first quadrant spectral sequence of algebras*

$$E_2^{p,q} = H^p(B; H^q(F; R)) \Rightarrow H^{p+q}(E; R).$$

The left hand side is the cohomology of B with local coefficients in the cohomology of F with respect to R. Furthermore, the multiplication structure on E_2 is compatible with the cup product structure on cohomology.

Example 8. Let $K(\mathbb{Z}, n)$ be the Eilenberg-Maclane space with $\pi_n(K(\mathbb{Z}, n)) = \mathbb{Z}$ and $\pi_i(K(\mathbb{Z}, n)) = 0$ otherwise. We now show by induction that the cohomology ring

$$H^*(K(\mathbb{Z}, n), \mathbb{Q}) = \begin{cases} \mathbb{Q}[z], & \text{if } n \text{ is even,} \\ \mathbb{Q}[z]/z^2 & \text{if } n \text{ is odd,} \end{cases}$$

where $|z| = n$. (This is still true if we replace $\mathbb{Q}$ by a field of characteristic 0.)

For $n = 1$ one has $K(\mathbb{Z}, 1) = S^1$, so this is clearly true. For $n \geq 2$ we consider

$$\Omega K(\mathbb{Z}, n) \longrightarrow PK(\mathbb{Z}, n) \longrightarrow K(\mathbb{Z}, n),$$

where $\Omega K(\mathbb{Z}, n) \simeq K(\mathbb{Z}, n-1)$ is the loop space and $PK(\mathbb{Z}, n) \simeq *$ is the path space of $K(\mathbb{Z}, n)$. Note that $\pi_1(K(\mathbb{Z}, n)) = 1$, so by the Serre spectral sequence

$$E_2^{p,q} = H^p(K(\mathbb{Z}, n); \mathbb{Q}) \otimes_{\mathbb{Q}} H^q(K(\mathbb{Z}, n-1); \mathbb{Q}) \Rightarrow H^{p+q}(*, \mathbb{Q}).$$

This implies that the E_∞ page of the spectral sequence is $\mathbb{Z}$ at coordinate $(0, 0)$ and zero everywhere.

Let us first consider the case where $n \geq 2$ is even. Then $E_2^{0,n-1}$ is generated by some fixed 2-torsion element $z \in H^{n-1}(K(\mathbb{Z}, n-1))$, and for the page to converge to 0 along this diagonal we require $d_n(z) = x$ with d_n an isomorphism. Thus $E_2^{n,n-1}$ is generated by xz, and by the Leibniz rule

$$d_n(xz) = d_n(x)z + (-1)^n x d_n(z) = x^2,$$

since $d_n(x) = 0$. Inductively, we get the following diagram on the E_2-page.

$$\begin{tikzcd}[row sep=small, column sep=small]
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
z \arrow[ddddrrrr, "d_n"] & 0 & \dots & 0 & xz \arrow[ddddrrrr, "d_n"] & 0 & \dots & 0 & x^2z & 0 \\
0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\
1 & 0 & \dots & 0 & x & 0 & \dots & 0 & x^2 & 0
\end{tikzcd}$$

This gives us what we want for $n \geq 2$ even. Next we consider the case where $n \geq 2$ is odd. Then $E_2^{0,k(n-1)}$ must be isomorphic to $E_2^{n,(k-1)(n-1)}$ for any positive integer n, and the rest of the E_2 page is zero except at $(0,0)$. Similar to before, $E_2^{0,n-1}$ is generated by $z \in H^{n-1}(K(\mathbb{Z}, n-1))$, and $d_n(z) = x$ with $d_n : E_2^{0,n-1} \longrightarrow E_2^{n,0}$ an isomorphism. Now

$$d_n(z^2) = d_n(z)z + (-1)^{n-1} z d_n(z) = 2xz,$$

and so $E_2^{n,n-1}$ is generated by xz (since $\mathbb{Q}$ has characteristic zero).

$$\begin{tikzcd}[row sep=small, column sep=small]
0 & 0 & \dots & 0 & 0 & 0 & \dots \\
\frac{1}{2}z^2 \arrow[ddddrrrr, "d_n"] & 0 & \dots & 0 & xz^2 & 0 & \dots \\
0 & 0 & \dots & 0 & 0 & 0 & \dots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & \dots \\
z \arrow[ddddrrrr, "d_n"] & 0 & \dots & 0 & xz & 0 & \dots \\
0 & 0 & \dots & 0 & 0 & 0 & \dots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \dots & 0 & 0 & 0 & \dots \\
1 & 0 & \dots & 0 & x & 0 & \dots
\end{tikzcd}$$

Since the $2n^{th}$ column of the E_2 page is zero, we conclude that x must be 2-torsion, as desired.

Example 9. We show that

$$H^*(\mathbb{CP}^n) = \mathbb{Z}[x]/(x^{n+1}), \qquad |x| = 2.$$

To do this, we use the Hopf fibration

$$S^1 \longrightarrow S^{2n+1} \longrightarrow \mathbb{CP}^n.$$

Explcitly, the map $S^{2n+1} \longrightarrow \mathbb{CP}^n$ is defined by the projection, where we view both S^{2n+1} and $\mathbb{CP}^n$ as the usual subspace and subquotient of $\mathbb{C}^{n+1}$.

Since $\mathbb{CP}^n$ is simply connected, the Serre spectral sequence in this case is

$$E_2^{p,q} = H^p(\mathbb{CP}^n, H^q(S^1)) \Rightarrow H^{p+q}(S^{2n+1}).$$

The E_2-page has nontrivial terms only on the first two rows.

$$\begin{array}{|ccccc}
0 & 0 & 0 & 0 & \dots \\
H^0(\mathbb{CP}^n) & H^1(\mathbb{CP}^n) & H^2(\mathbb{CP}^n) & H^3(\mathbb{CP}^n) & \dots \\
H^0(\mathbb{CP}^n) & H^1(\mathbb{CP}^n) & H^2(\mathbb{CP}^n) & H^3(\mathbb{CP}^n) & \dots \\
\hline
\end{array}$$

Since the spectral sequence converges after one page to $H^{p+q}(S^{2n+1})$, which equals $\mathbb{Z}$ when $p+q \in \{0, 2n+1\}$ and is 0 otherwise, it is easily seen that

$$H^i(\mathbb{CP}^n) \cong \begin{cases} \mathbb{Z} & \text{for } i = 0, 2, 4, \dots, 2n, \\ 0 & \text{otherwise.} \end{cases}$$

As the cup product structure is compatible with the product on E_2, the cohomology ring is as claimed.

Groups

We now look at the spectral sequence associated to group cohomology. One can either view this purely algebraically, or relate this to topology via the isomorphism $H^n(G, A) \cong H^n(BG, A)$, where BG is the classifying space of G, and A is a trivial G-module (see [7, Theorem 6.10.5]). For example,

$$H^n(\mathbb{Z}^{\oplus n}, \mathbb{Z}) \cong H^n(\mathbb{T}^n),$$

where $\mathbb{T}^n$ is the n-torus. We assume the language of group cohomology; a reference is [2, Chapter 4].

Theorem 10 (Lyndon-Hochschild-Serre spectral sequence, [7, Theorem 6.8.2])**.** *Let*

$$0 \longrightarrow N \longrightarrow G \longrightarrow Q \longrightarrow 0$$

be a short exact sequence of groups, with N normal, and let M be a ring that is also a G-module. Then there is a first quadrant spectral sequence of algebras

$$E_2^{p,q} = H^p(Q, H^q(N, M)) \Rightarrow H^{p+q}(G, M).$$

The left hand side is the cohomology of B with local coefficients in the cohomology of F with respect to R.

The Lyndon-Hochschild-Serre spectral sequence can be viewed as a special case of the *Grothendieck spectral sequence*, which states the following: if $F : \mathcal{A} \longrightarrow \mathcal{B}$ and $G : \mathcal{B} \longrightarrow \mathcal{C}$ are two additive left-exact functors between abelian categories such that F takes acyclic objects to acyclic objects and $\mathcal{B}$ has enough injectives, then there is a spectral sequence

$$E_2^{p,q} = (R^pG \circ R^qF)(A) \Rightarrow R^{p+q}(G \circ F)(A)$$

for every object A of $\mathcal{A}$. To see this, we view $H^*(G, -)$ as the the derived functor of $(-)^G$, and note that the composition of $(-)^N$ and $(-)^{G/N}$ is $(-)^G$.

Remark. If we identify Q with G/N, then the associated five-term exact sequence is the inflation-restriction sequence

$$0 \longrightarrow H^1(G/N, M^N) \longrightarrow H^1(G, M) \longrightarrow H^1(N, M)^{G/N} \longrightarrow H^2(G/N, M^N) \longrightarrow H^2(G, M).$$

In fact, the differential $d_2^{0,1}$ from $H^1(N, M)^{G/N} = E_2^{0,1}$ to $H^2(G/N, M^N) = E_2^{0,2}$ is the transgression map, which we recall here. For the short exact sequence of groups in Theorem 10 one gets an extension

$$0 \longrightarrow N^{ab} \longrightarrow G/[N, N] \longrightarrow G/N \longrightarrow 0,$$

where we identify Q with G/N. Let $\xi \in H^2(G/N, N^{ab})$ be the element corresponding to this extension. Now suppose $f \in H^1(N, M)^{G/N}$. Then $f \in \mathrm{Hom}(N, M) = \mathrm{Hom}(N^{ab}, M)$ and f is G/N-invariant, so we have an induced homomorphism

$$f^* : H^2(G/N, N^{ab}) \longrightarrow H^2(G/N, M).$$

The map $d_2^{0,1}$ is then defined to be $d_2^{0,1}(f) = f^*(\xi)$.

Example 11. Consider the integral Heisenberg group

$$G = \left\{ \begin{bmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} : a, b, c \in \mathbb{Z} \right\}.$$

We wish to compute $H^*(G, A)$, where A is a trivial G-module. Note that the Heisenberg group is a semidirect product

$$G = N \rtimes K \cong (\mathbb{Z} \oplus \mathbb{Z}) \rtimes \mathbb{Z}$$

where

$$N = \left\{ \begin{bmatrix} 1 & 0 & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} : b, c \in \mathbb{Z} \right\}, \qquad K = \left\{ \begin{bmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} : a \in \mathbb{Z} \right\}$$

and the structure map $\varphi : \mathbb{Z} \longrightarrow \mathrm{Aut}(\mathbb{Z} \oplus \mathbb{Z})$ is given by

$$\varphi(k) = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}^k.$$

Thus we have a split extension

$$0 \longrightarrow \mathbb{Z} \longrightarrow G \longrightarrow \mathbb{Z} \oplus \mathbb{Z} \longrightarrow 0.$$

The associated E_2-page looks like

$$\begin{array}{|llll}
0 & 0 & 0 & 0 \\
A & A \oplus A & A & 0 \\
A & A \oplus A & A & 0 \\
\hline
\end{array}$$

with one nontrivial differential $d_2^{0,1} : A \longrightarrow A$. Let us show that $d_2^{0,1} = 1$. By definition (see above remark)

$$d_2^{0,1}(f) = f^*(\xi)$$

where $\xi \in H^2(\mathbb{Z} \oplus \mathbb{Z}, \mathbb{Z})$ is the class corresponding to the extension above. We have seen that the extension is split, so $\xi = 0$ and $d_2^{0,1}$ is the identity. This implies that E_3 looks like the diagram below, and the spectral sequence degenerates after one step.

$$\begin{array}{|llll}
0 & 0 & 0 & 0 \\
0 & A \oplus A & A & 0 \\
A & A \oplus A & 0 & 0 \\
\hline
\end{array}$$

By Proposition 3 one sees that

$$H^n(G, A) = \begin{cases} A & \text{if } n = 0, 3, \\ A \oplus A & \text{if } n = 1, 2, \\ 0 & \text{otherwise.} \end{cases}$$

3 A sketch of the lifting problem

As we mentioned in the introduction, the lifting problem is the problem where the total cohomology of the E_∞-page of a spectral sequence is not equal to the cohomology of the object one is trying to compute. This is because the terms defining the spectral sequence are quotients of predetermined filtrations of cohomology groups, and in general a group is not isomorphic to the direct sum of its filtration.

Example 12. Let C_n be the cyclic group of order n. It is well-known (see Chapter 4.8 of [2]) that

$$H^*(C_n, Z) = \mathbb{Z}[z]/(nz)$$

where $|z| = 2$. We now demonstrate the lifting problem via the short exact sequence

$$0 \longrightarrow C_2 \longrightarrow C_4 \longrightarrow C_2 \longrightarrow 0.,$$

which is a central extension. Using the Lyndon-Hoschschild-Serre spectral sequence the corner of the E_2-page is

$$\begin{array}{|ccccccc}
y^2 & 0 & y^2x & 0 & y^2x^2 & 0 & y^2x^3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
y & 0 & yx & 0 & yx^2 & 0 & yx^3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & x & 0 & x^2 & 0 & x^3 \\
\hline
\end{array}$$

It is clear that $E_2 = E_\infty$, and $Tot(E_\infty) = \mathbb{Z}[x,y]/(2x, 2y)$. However, this is not the cohomology of C_4.

Although the spectral sequence may not tell us much about the cohomology groups, a little more can be said if we assume finite generation.

Theorem 13 ([1, Theorem 2.1])**.** *If $Tot(E_\infty)$ is a finitely generated bigraded commutative algebra, then so is the cohomology ring H^* that we are trying to compute. Moreover, if $Tot(E_\infty)$ is presented by (double) generators and homogeneous relations, then the generators and relations of H^* are lifts of those of $Tot(E_\infty)$, and they have the same total degree.*

If we work over finite fields, the above theorem tells us that knowledge of $Tot(E_\infty)$ gives us only finitely many possibilities for H^*. See [5] for more examples of the lifting problem.

4 Application: homotopy groups of spheres[1]

Our goal now is to prove that $\pi_i(S^n)$ is finite if n is odd and $i \neq n$. The idea is to be more flexible and only work with groups modulo an insignificant "Serre class" of groups. For example, we'll see that $\pi_i(S^n)$ looks like $\pi_i(K(\mathbb{Z}, n))$ modulo their torsion. Since $\pi_i(K(\mathbb{Z}, n)) = 0$ for $i \neq n$, we deduce that $\pi_i(S^n)$ is a torsion group whenever $i \neq n$.

Definition 14. A **Serre class of abelian groups** is a nonempty collection $\mathcal{C}$ of abelian groups satisfying:

1. If $0 \to A \to B \to C \to 0$ is a short exact sequence, then $B \in \mathcal{C}$ if and only if $A, C \in \mathcal{C}$.

The Serre class may also satisfy the following optional axioms:

2A. If $A, B \in \mathcal{C}$, then $A \otimes B \in \mathcal{C}$ and $\mathrm{Tor}_i(A, B) \in \mathcal{C}$ for all $i \geq 1$.

2B. If $A \in \mathcal{C}$, then $A \otimes B \in \mathcal{C}$ for any abelian group B.[2]

3. If $A \in \mathcal{C}$, then $H_i(A; \mathbb{Z}) := H_i(K(A, 1); \mathbb{Z}) \in \mathcal{C}$ for all $i \geq 1$.

Example 15.
The following Serre classes satisfy axioms 1, 2A, 2B, and 3:

(a) The trivial class $\mathcal{C}_{triv}$ of abelian groups.

(b) The class $\mathcal{C}_{tors}$ of torsion abelian groups. Axiom 3 holds by either by arguing as in p271 of [4], or by a group cohomology argument: write the torsion group A as a direct limit of its n-torsion subgroups, and further write each n-torsion subgroup A_n as a direct limit of its finite subgroups A_{nm}. Since homology commutes with direct limits,

$$H_i(A; \mathbb{Z}) \cong \varinjlim_{n,m} H_i(A_{nm}; \mathbb{Z}),$$

and since the homology of a finite group is torsion whenever $i \geq 1$, we deduce that $H_i(A; \mathbb{Z})$ is also torsion.

The following Serre classes only satisfy axioms 1, 2A, and 3:[3]

(a) The class $\mathcal{C}_{finite}$ of finite abelian groups. The second part of Axiom 2A follows from the additivity of $\mathrm{Tor}_i^{\mathbb{Z}}(A, B)$ in the first argument and the computation

$$\mathrm{Tor}_i^{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, B) = \begin{cases} B[n] & \text{if } i = 1 \\ 0 & \text{if } i > 1 \end{cases}$$

where $B[n]$ is the n-torsion part of B. Axiom 3 holds by:

[1] This section blatantly plagiarizes §10.1-2 of Davis and Kirk [4].

[2] Axioms 1 and 2B together imply Axiom 2A since $\mathrm{Tor}_i(A, B)$ is computed by taking a free resolution of A, tensoring by B, and taking the resulting homology.

[3] Axiom 2B fails in the next two examples since the tensor product of the finite group $\mathbb{Z}/2\mathbb{Z}$ by the direct sum $\oplus_{\mathbb{N}}\mathbb{Z}/2\mathbb{Z}$ is neither finite nor finitely generated.

- $K(A \times B, 1) \cong K(A, 1) \times K(B, 1)$,
- The Künneth theorem and axiom 2A,
- $K(\mathbb{Z}, 1) = S^1$,
- The spectral sequence computation ([4] theorem 9.30)

$$H_i(\mathbb{Z}/n\mathbb{Z}; \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } i = 0 \\ \mathbb{Z}/n\mathbb{Z} & \text{if } i = \text{odd} \\ 0 & \text{if } i > 0 \text{ is even} \end{cases}.$$

(b) The class $\mathcal{C}_{FG}$ of finitely generated abelian groups. The axioms are verified as in the previous example.

Definition 16. Given a Serre class $\mathcal{C}$, a homomorphism $\varphi : A \to B$ of abelian groups is a **$\mathcal{C}$-isomorphism** if both $\ker \varphi$ and $\operatorname{coker} \varphi$ are in $\mathcal{C}$. $\mathcal{C}$-isomorphism

Definition 17. Two abelian groups A and B are **$\mathcal{C}$-isomorphic** if there exists an abelian group C and two $\mathcal{C}$-isomorphisms $f : C \to A$ and $g : C \to B$,[4] and we denote this by $A \cong B \mod \mathcal{C}$.[5] $\mathcal{C}$-isomorphic

Example 18. For two abelian groups A and B,

- $A \cong B \mod \mathcal{C}_{triv}$ if and only if $A \cong B$;
- $A \cong B \mod \mathcal{C}_{tors}$ if and only if $A \otimes \mathbb{Q} \cong B \otimes \mathbb{Q}$.[6]

Theorem 19.

(1) (mod $\mathcal{C}$ Hurewicz theorem) Let X be 1-connected, and suppose $\mathcal{C}$ satisfies axioms 1, 2A, and 3.

(a) If $\pi_i X \in \mathcal{C}$ for all $i < n$, then $H_i X \in \mathcal{C}$ for all $0 < i < n$ and the Hurewicz map $\pi_n X \to H_n X$ is a $\mathcal{C}$-isomorphism.

(b) If $H_i X \in \mathcal{C}$ for all $0 < i < n$, then $\pi_i X \in \mathcal{C}$ for all $i < n$ and the Hurewicz map $\pi_n X \to H_n X$ is a $\mathcal{C}$-isomorphism.

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[4] Such diagram $\begin{tikzcd} & C \arrow[dl, "f"'] \arrow[dr, "g"] & \\ A & & B \end{tikzcd}$ is called a "roof". Looking at the category of abelian groups modulo these roofs is an example of localizing a category.

[5] This is indeed an equivalence relation.

[6] In other words, A and B are the same up to torsion (in the new sense) if and only if A and B are the same up to torsion (in the familiar sense). This example illustrates that roofs are needed for isomorphisms to be reflexive: $\mathbb{Z} \cong \mathbb{Q} \mod \mathcal{C}_{tors}$ since $\begin{tikzcd} & \mathbb{Z} \arrow[dl, "\mathrm{id}"'] \arrow[dr, hook] & \\ \mathbb{Z} & & \mathbb{Q} \end{tikzcd}$ forms a roof. However there is no $\mathcal{C}_{tors}$-isomorphism, let alone nonzero group homomorphism, $\mathbb{Q} \to \mathbb{Z}$.

(2) *(mod $\mathcal{C}$ relative Hurewicz theorem) Let $A \subset X$, A and X be 1-connected, and $\pi_2(X, A) = 0$. Suppose $\mathcal{C}$ satisfies axioms 1, 2B, and 3.*

   (a) *If $\pi_i(X, A) \in \mathcal{C}$ for all $i < n$, then $H_i(X, A) \in \mathcal{C}$ for all $i < n$ and the Hurewicz map $\pi_n(X, A) \to H_n(X, A)$ is a $\mathcal{C}$-isomorphism.*

   (b) *If $H_i(X, A) \in \mathcal{C}$ for all $i < n$, then $\pi_i(X, A) \in \mathcal{C}$ for all $i < n$ and the Hurewicz map $\pi_n(X, A) \to H_n(X, A)$ is a $\mathcal{C}$-isomorphism.*

(3) *(mod $\mathcal{C}$ Whitehead theorem) Let $f : A \to X$ be a map between 1-connected spaces, and $\pi_2(f)$ an epimorphism. Let $\mathcal{C}$ satisfy axioms 1, 2B, and 3. Then the following are equivalent:*

   (a) *$\pi_i(f)$ is a $\mathcal{C}$-isomorphism for $i < n$ and a $\mathcal{C}$-epimorphism for $i = n$.*

   (b) *$H_i(f)$ is a $\mathcal{C}$-isomorphism for $i < n$ and a $\mathcal{C}$-epimorphism for $i = n$.*

Application. The homotopy group $\pi_i(S^n)$ is finite if n is odd and $i \neq n$.

Proof. For $n = 1$, covering space theory tells us that $\pi_i(S^1) = \pi_i(\mathbb{R}) = 0$ for $i > 1$. Now let $n > 1$ be odd. The sphere S^n is a finite CW complex so $H_i S^n \in \mathcal{C}_{FG}$ for all $i > 0$. Since S^n is 1-connected, by (1)(b), $\pi_i(S^n) \in \mathcal{C}_{FG}$ for all $i \geq 0$.

We now show that $\pi_i S^n$ is a torsion group for $i \neq n$. Let $f : S^n \to K(\mathbb{Z}, n)$ be the map inducing an isomorphism on π_n. Then $\pi_i(f)$ is an isomorphism for $i \leq n$ and an epimorphism for $i = n + 1$. Applying (3) with the class $\mathcal{C}_{triv}$, we see that

$$H_i(f) : H_i(S^n; \mathbb{Z}) \to H_i(K(\mathbb{Z}, n); \mathbb{Z})$$

is an isomorphism for $i \leq n$ and an epimorphism for $i = n + 1$.The homologies of S^n and $K(\mathbb{Z}, n)$ with coefficients in $\mathbb{Q}$ vanish in degrees $> n$,[7] so

$$H_i(f) : H_i(S^n; \mathbb{Q}) \to H_i(K(\mathbb{Z}, n); \mathbb{Q})$$

is an isomorphism for all i. By example 18,

$$H_i(f) : H_i(S^n; \mathbb{Z}) \to H_i(K(\mathbb{Z}, n); \mathbb{Z})$$

is a $\mathcal{C}_{tors}$-isomorphism for all i. Applying (3) with the class $\mathcal{C}_{tors}$, we deduce that

$$\pi_i(f) : \pi_i(S^n) \to \pi_i(K(\mathbb{Z}, n))$$

is a $\mathcal{C}_{tors}$-isomorphism for all i. Since $\pi_i(K(\mathbb{Z}, n)) = 0$ for $i \neq n$, $\pi_i(S^n) \in \mathcal{C}_{tors}$ for $i \neq n$.

We conclude that $\pi_i S^n$ is a finite group for $i \neq n$. $\square$

[7] Take the Poincaré dual of our result in example 8 to find $H_i(K(\mathbb{Z}, n); \mathbb{Q})$.

5 Proof of theorem 19

In theorem 19, parts (1)(b) and (2)(b) follow from (1)(a) and (2)(a), respectively, and part (3) follows from part (2).

It then remains to prove things in the following order: theorem 19(1)(a) and a lemma 20 needed in the proof of 19(1)(a). We refer the reader to [4] p277 for how theorem 19(2)(a) follows from theorem 19(1)(a).

5.1 Proof of 19(1)(a)

Strategy. To show that the Hurewicz map is an isomorphism for a space X, we consider the path fibration $\Omega X \to \mathcal{P}X \xrightarrow{f} X$ and the diagram

$$\begin{tikzcd}
\pi_n(X,x_0) \arrow[d, "\rho"] & \pi_n(\mathcal{P}X,\Omega X) \arrow[l, "f_*"', "\cong"] \arrow[r, "\partial", "\cong"'] \arrow[d, "\rho"] & \pi_{n-1}(\Omega X) \arrow[d, "\rho"] \\
H_n(X,x_0) & H_n(\mathcal{P}X,\Omega X) \arrow[l, "f_*"'] \arrow[r, "\partial", "\cong"'] & H_{n-1}(\Omega X)
\end{tikzcd} . \tag{1}$$

To show the leftmost ρ is a $\mathcal{C}$-isomorphism, it suffices to show the lower f_* and the rightmost ρ are $\mathcal{C}$-isomorphisms, since the ∂ and the top f_* are already isomorphisms.[8]

To show the lower f_* is a $\mathcal{C}$-isomorphism, we need the following spectral sequence lemma, whose proof is postponed to the next subsection:

Lemma 20. *Let $F \to E \xrightarrow{f} B$ be a fibration, B be simply-connected, and $\mathcal{C}$ a Serre class satisfying axioms 1 and 2A. Suppose further that $H_pB \in \mathcal{C}$ for $0 < p < n$ and $H_pF \in \mathcal{C}$ for $0 < q < n-1$. Then*

$$f_* : H_i(E,F) \to H_i(B,b_0)$$

is a $\mathcal{C}$-isomorphism for $i < n$.

Actual start of proof. To prove 19(1)(a), we induct on n. For $n = 1$, we have $\pi_1 X = 0 = H_1 X$. For $n = 2$, $\pi_1 X = 0 = H_1 X$ and so lemma 20 tells us the bottom f_* of diagram 1 is an isomorphism. Also $\pi_1(\Omega X) = \pi_2(X)$

[8] ∂ is an isomorphism because $\mathcal{P}X$ is contractible, and the top f_* is an isomorphism by applying the five lemma to the commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & \pi_n(\Omega X) \arrow[r] \arrow[d, equal] & \pi_n(\mathcal{P}X) \arrow[r] \arrow[d, equal] & \pi_n(\mathcal{P}X,\Omega X) \arrow[r] \arrow[d, "f_*"] & \cdots \\
\cdots \arrow[r] & \pi_n(\Omega X) \arrow[r] & \pi_n(\mathcal{P}X) \arrow[r] & \pi_n(X) \arrow[r] & \cdots
\end{tikzcd} .$$

is abelian, so by the π_1-version of the Hurewicz theorem[9], the rightmost ρ of diagram 1 is an isomorphism. Therefore the leftmost ρ is an isomorphism.

Now suppose $n > 1$ and inductively assume that whenever a simply-connected space Y satisfies $\pi_i Y \in \mathcal{C}$ for $i < n-1$, then $H_i Y \in \mathcal{C}$ for $0 < i < n-1$ and the Hurewicz map $\rho : \pi_{n-1} Y \to H_{n-1} Y$ is a $\mathcal{C}$-isomorphism.

Let X be a simply-connected space satisfying $\pi_i X \in \mathcal{C}$ for $i < n$.

Case 1: $\pi_2 X = 0$**.** Then $\pi_i(\Omega X) = \pi_{i+1} X$ so we apply the inductive hypothesis to ΩX and conclude the rightmost ρ of diagram 1 is a $\mathcal{C}$-isomorphism. Lemma 20 shows the bottom f_* is a $\mathcal{C}$-isomorphism.

Case 2: $\pi_2 X \neq 0$**.** We reduce to case 1 by replacing X by an X_2 constructed as follows: after replacing X by a CW complex, let $f : X \to K(\pi_2 X, 2)$ be the map[10] inducing identity on π_2. Let X_2 be its homotopy fiber, giving the fibration

$$K(\pi_2, 1) = \Omega K(\pi_2 X, 2) \to X_2 \to X.$$

Then X_2 satisfies:

(a) $H_i X_2 \cong H_i X \mod \mathcal{C}$ for all $0 < i \leq n$.[11]

(b) $\pi_i(X_2) \cong \pi_i(X)$ for all $i > 2$.[12]

(c) $\pi_i(X_2) = 0$ for $i = 1, 2$.[13]

So X_2 satisfies the same inductive hypotheses as X except now X_2 belongs to case 1, so the Hurewicz map for X_2 at level n is an isomorphism. Therefore, the same holds for X.[14]

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[9] Note: the only inputs needed for this theorem is the π_1-version of the Hurewicz theorem and the Serre spectral sequence

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[11] We assumed $\pi_2 X \in \mathcal{C}$, so by axiom 3, we deduce $H_i(K(\pi_2 X, 1)) \in \mathcal{C}$ for all $i > 0$. Therefore:

- $H_i(X_2) \to H_i(X_2, K(\pi_2 X, 1))$ is a $\mathcal{C}$-isomorphism for all $i > 0$.
- Together with lemma 20, $H_i(X_2, K(\pi_2 X, 1)) \to H_i(X, x_0)$ is a $\mathcal{C}$-isomorphism for $0 < i \leq n$. (Typo in [4] on p277 says $0 < i < n$.)

The composite $H_i(X_2) \to H_i(X, x_0)$ is then a $\mathcal{C}$-isomorphism for $0 < i \leq n$.

[12] This follows from the long exact sequence for the fibration.

[13] As in (b), note that f induces an isomorphism on π_2.

[14] Formally, we would draw the diagram and observe the middle ρ is an isomorphism for $i > 2$:

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & \pi_i(X_2) & \xrightarrow{\cong} & \pi_i(X) & \longrightarrow & \pi_i(K(\pi_2 X, 2)) & \longrightarrow & \cdots \\
 & & \cong\downarrow\rho & & \downarrow\rho & & \downarrow\rho & & \\
\cdots & \longrightarrow & H_i(X_2) & \xrightarrow{\cong} & H_i(X) & \longrightarrow & H_i(K(\pi_2 X, 2)) & \longrightarrow & \cdots
\end{array}.$$

5.2 Proof of lemma 20

The spectral sequence for a relative fibration $F \to (E, F) \to (B, b_0)$ states that

$$E^2_{p,q} \cong H_p(B, b_0; H_q F) \Rightarrow H_\bullet(E, F).$$

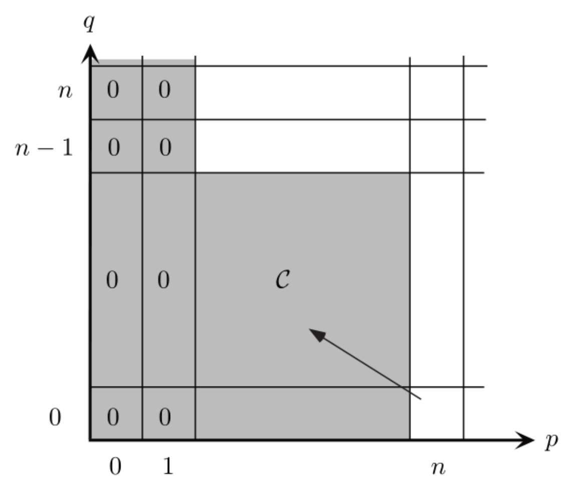
In the range $p = 0, 1$ or $(p, q) \in (0, n) \times (0, n-1)$, $E^2_{p,q} \in \mathcal{C}$[15] so $E^\infty_{p,q} \in \mathcal{C}$ as well.[16]

Now $H_i(E)$ is filtered by the E^∞ terms on the $p + q = i$ diagonal. For $i \le n$, entries of this diagonal lie in $\mathcal{C}$ except possibly $E^\infty_{i,0}$. This gives a map $H_i(E, F) \twoheadrightarrow E^\infty_{i,0}$ that is a $\mathcal{C}$-isomorphism.

Also, $E^\infty_{i,0} \subset E^2_{i,0}$ is a $\mathcal{C}$-isomorphism: $E^{k+1}_{i,0}$ is the homology at entry $E^k_{i,0}$ where the arrows are slanted so that (for $i \le n$) the kernel and cokernel either lie in the shaded region in the first quadrant (hence take values in $\mathcal{C}$), or lie in the other quadrants (so zero).

By Theorem 9.12 of [4], we can identify the composite of these two maps with f_* so f_* is a $\mathcal{C}$-isomorphism as well.

Figure 1: From [4]



[15] Since in the latter range for (p, q), $E^2_{p,q} = (H_p(B, b_0) \otimes H_q F) \oplus \mathrm{Tor}(H_p(B, b_0), H_q F)$ has all parts lying in $\mathcal{C}$ by hypothesis. In the range $p = 0, 1$, the $E^2_{p,q}$ terms are zero since B is 1-connected.

[16] Suppose that $E^i_{p,q} \in \mathcal{C}$. We get $E^{i+1}_{p,q}$ by taking homology at entry $E^i_{p,q}$, so by axiom 1, $E^{i+1}_{p,q} \in \mathcal{C}$.

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