

# Jacobi Polynomials via Pictures

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Notes for Pizza Seminar

## Abstract

Jacobi polynomials are an important class of orthogonal polynomials that includes examples like the Chebyshev and Legendre polynomials. The purpose of this expository note is to flesh out two pictorial appearances of these polynomials in physics and combinatorial number theory. We give a quick overview of Jacobi polynomials, followed by applications in electrostatics and dessins d'enfants.

## 1 Some Facets of Jacobi Polynomials

In this section we will introduce the bare minimum on Jacobi polynomials needed to understand this talk. A good reference on this is [4].

### 1.1 Chebyshev Polynomials

Let start with an example of Jacobi polynomials. Recall from high school calculus that one can expand  $\cos(n\theta)$  as a polynomial in  $\cos(\theta)$ . For example,

$$\begin{aligned}\cos(2\theta) &= 2\cos^2(\theta) - 1, \\ \cos(3\theta) &= 4\cos^3(\theta) - 3\cos(\theta),\end{aligned}$$

and more generally, using Euler's formula on complex numbers,

$$\cos(n\theta) = \sum_{j=0}^{\lfloor n/2 \rfloor} \binom{n}{2j} (\cos^2(\theta) - 1)^j \cos^{n-2j}(\theta).$$

By replacing  $\cos(\theta)$  with a variable, this gives us the *Chebyshev polynomials*:

$$T_n(x) = \sum_{j=0}^{\lfloor n/2 \rfloor} \binom{n}{2j} (x^2 - 1)^j x^{n-2j}.$$

(There is a completely analogous function by expanding  $\sin(n\theta)$ ; we will omit this here.)

The Chebyshev polynomials have a number of properties, and the first two will be essential later.

- The zeroes of  $T_n$  are simple and lie in  $(-1, 1)$ . In this case we can explicitly write them down:

$$\cos \frac{(1 + 2k)\pi}{2n}, \quad k = 0, \dots, n - 1.$$

Therefore, the non-endpoint critical points of  $T_n$  interlaces with the zeros and lies in  $(-1, 1)$ . In fact, they are

$$\cos \frac{k\pi}{n}, \quad k = 1, \dots, n - 1.$$

- The polynomial  $T_n$  solves the Sturm-Liouville equation

$$(1 - x^2)y'' - xy' + n^2y = 0,$$

and are orthogonal with respect to the weight  $(1 - x^2)^{-1/2}$ .

- The Chebyshev polynomial  $T_n$  is the best degree  $n$  polynomial to interpolate a given function  $f$ .

## 1.2 Legendre Polynomials

Let us give another example that appears in spherical harmonics. This is the *Legendre polynomial*, defined as

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n.$$

For example,

$$\begin{aligned} P_1(x) &= x, \\ P_2(x) &= \frac{1}{2}(3x^2 - 1), \\ P_5(x) &= \frac{1}{8}(63x^5 - 70x^3 + 15x). \end{aligned}$$

The Legendre polynomials satisfy two similar properties with the Chebyshev polynomials, which will be shared with all Jacobi polynomials.

- The zeros of  $P_n$  are simple and lies in  $(-1, 1)$ .
- The polynomials  $P_n$  solves the Sturm-Liouville equation

$$(1 - x^2)y'' - 2xy' + n(n + 1)y = 0,$$

and are orthogonal with respect to the weight 1.

- The Legendre polynomial  $P_n$  is the best degree  $n$  polynomial for numerical integration on an interval, in the sense of Gaussian quadrature.

## 1.3 Jacobi Polynomials

Now we define the polynomials we want to look at in this talk.

**Definition 1.** Let  $\alpha, \beta > -1$  be half-integers. The *Jacobi polynomials* are defined by

$$J_n^{\alpha, \beta}(x) = \frac{1}{2^n n!} (1 - x)^{-\alpha} (1 + x)^{-\beta} \frac{d^n}{dx^n} ((1 - x)^\alpha (1 + x)^\beta (1 - x^2)^n).$$

(This is also known as the *Rodrigues expansion* of the Jacobi polynomial.)

It is not hard to check that:

- the Chebyshev polynomials  $T_n$  correspond to the case  $\alpha = \beta = -1/2$ ;
- the Legendre polynomials  $P_n$  correspond to the case  $\alpha = \beta = 0$ .

As mentioned before, the Jacobi polynomials share two properties similar to the Chebyshev and Legendre polynomials. We will simply take them on faith here; see [4] for a good treatment on Jacobi, and more generally, orthogonal, polynomials.

**Theorem 2** ([4, Theorem 3.3.1]). *The zeroes of  $J_n^{\alpha, \beta}$  are real and distinct, and lies in  $(-1, 1)$ . Therefore, the non-endpoint critical points of  $J_n^{\alpha, \beta}$  interlaces with the zeros and lie in  $(-1, 1)$ .*  $\square$

**Theorem 3** ([4, Theorems 4.2.1; 4.23.2]). *The Jacobi polynomials satisfies the Sturm-Liouville equation*

$$(1 - x^2)y'' + (\beta - \alpha - (\alpha + \beta + 2)x)y' + n(n + \alpha + \beta + 1)y = 0,$$

and are orthogonal with respect to the weight  $(1 - x)^\alpha (1 + x)^\beta$ . (A second linearly independent solution to the differential equation above can be explicitly written as a hypergeometric function.)  $\square$

Of course, once we have the definition of Jacobi polynomials as above, we can extend it to the complex plane, and all half integers  $\alpha$  and  $\beta$ . The downside is that  $J_n^{\alpha, \beta}$  may not be a polynomial anymore, and the theorems above do not work unless  $\alpha, \beta > -1$ . Nevertheless, we will still call them “Jacobi polynomials”. This extension will be needed in the last section when we discuss dessin d’enfants.

## 2 Electrostatics

This section aims to answer the following problem.

**Problem.** Consider “flatland physics”, i.e. physics on the 2-dimensional space. Arrange  $n + 2$  equally charged particles on a line, with the two end particles fixed. How can we arrange the particles in between so that they are in electrostatic equilibrium?

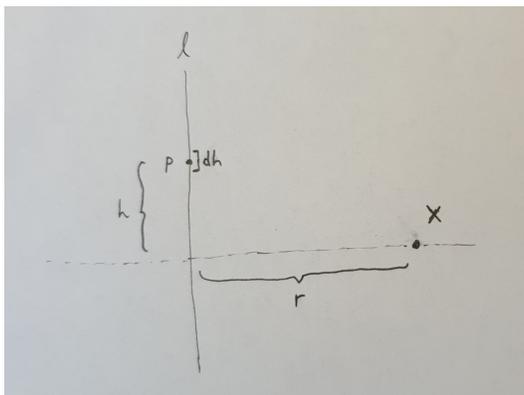
### 2.1 Electrostatic Forces

The “flatland” Coulomb’s Law is not an inverse-square law, but rather just an inverse law:

$$F \sim \frac{q_1 q_2}{r}.$$

Consequently, the “flatland” electrostatic energy obeys a logarithmic law.

The physical laws in “flatland” is actually related to the usual one we are familiar with. In our setting, it is related as follow. Consider an infinite charged line  $l$  in 3-dimensional space, such that  $l$  has uniform charge  $q$  and is perpendicular to the  $xy$ -plane. Without loss of generality, say  $l$  is the  $z$ -axis. Now consider any point  $x$  not on the line. If we just consider these two things interacting with each other, we can reduce this to a 2-dimensional problem. A picture of this is as shown below.



What is the net force acting on  $x$ ? By symmetry, there is no net vertical force. To calculate the horizontal force, pick a point  $p$  on the line and consider a small segment  $dh$  around it. If  $p$  is of height  $h$  relative to  $x$ , and  $x$  is of distance  $r$  away from  $l$ , then the usual Coulomb’s Law tells us that the horizontal force  $p dh$  exerts on  $x$  is

$$F_{hor,p} \sim \frac{p dh}{r^2 + h^2} \frac{r}{\sqrt{r^2 + h^2}}.$$

Therefore the net horizontal force exerted by the line is

$$F_{hor} \sim \int_{-\infty}^{\infty} \frac{q}{r^2 + h^2} \frac{r}{\sqrt{r^2 + h^2}} dh = \frac{2q}{r},$$

which is precisely the inverse-law in “flatland” physics.

### 2.2 Arrangement of Charged Particles

We now seek to answer the Problem posed at the beginning of this section. It is perhaps surprising that the solution to involves zeros of certain Jacobi polynomials! The exposition here loosely follows [2, 4].

**Theorem 4.** *There exists a unique equilibrium position in the Problem stated above. In fact, all local minimums of the energy functional are the same, and equal to the global minimum.*

*Proof.* Let  $p_1, p_2, \dots, p_{n-1}, p_n$  be the positions of the non-endpoint charged particles in increasing order. Then, by “flatland” physics, we would like to show that, for each  $k \neq 1, n$ , there is a unique solution to the equation

$$\frac{1}{p_k - 1} + \frac{1}{p_k + 1} + \sum_{j \neq k} \frac{1}{p_k - p_j} = 0.$$

It suffices to show something stronger: that there is a unique solution set  $p_2, \dots, p_{n-1}$  minimizing the energy

$$E_k(p_2, \dots, p_{n-1}) = -\log |p_k - 1| - \log |p_k + 1| - \sum_{j \neq k} \log |p_k - p_j|.$$

Once we have this, Lemma 5 below (with  $\alpha = \beta = 1$ ) will show that the solution set to each  $E_k$  is actually the same one, thus proving our theorem.

Let us now show the uniqueness of solution to  $E_k$ . Observe that a global minimum to  $E_k$  must exist by continuity. Let  $p_1, \dots, p_n$  be such a solution set guaranteeing global minimum, and suppose  $p'_1, \dots, p'_n$  is a set of solutions giving a local minimum. Write

$$s_i = \frac{p_i + p'_i}{2}.$$

Then, by the AM-GM inequality,

$$|s_j - s_k| \geq |p_j - p_k|^{\frac{1}{2}} |p'_j - p'_k|^{\frac{1}{2}} \quad \text{and} \quad |s_j \pm 1| \geq |1 \pm p_j|^{\frac{1}{2}} |1 \pm p'_j|^{\frac{1}{2}}$$

with equality if and only if  $p_j = p'_j$  for all  $j$ . If equality does not hold, then we have produced a solution set with lower minimum, a contradiction.  $\square$

**Lemma 5.** *Let  $x_1, \dots, x_n$  denote the  $n$  zeros of the Jacobi polynomial  $J_n^{\alpha, \beta}(x)$  in increasing order. Then, for each  $x_k$ , one has the relation*

$$\frac{1}{2} \left( \frac{\alpha + 1}{x_k - 1} + \frac{\beta + 1}{x_k + 1} \right) + \sum_{j \neq k} \frac{1}{x_k - x_j} = 0.$$

*Proof.* Let us write the Jacobi polynomial  $J_n^{\alpha, \beta}(x)$  as  $f(x)$ , and factorize it:

$$f(x) = c \prod_j (x - x_j).$$

Then one immediately observes that

$$f'(x) = \sum_i \prod_{j \neq i} (x - x_j), \quad f''(x) = \sum_{\substack{i, l \\ i \neq l}} \prod_{j \neq i, l} (x - x_j) = 2 \sum_{\substack{i < l \\ i \neq l}} \prod_{j \neq i, l} (x - x_j).$$

Thus

$$f(x_k) = 0 \quad \text{and} \quad \frac{f''(x)}{f'(x)} = \sum_{j \neq k} \frac{2}{x_k - x_j}.$$

Now, recall that  $f(x)$  satisfies the differential equation

$$(1 - x^2)f'' + (\beta - \alpha - (\alpha + \beta + 2)x)f' + n(n + \alpha + \beta + 1)f = 0.$$

Substituting  $x = x_k$  and rearranging, we get

$$\frac{\beta - \alpha - (\alpha + \beta + 2)x}{1 - x^2} + \frac{f''(x)}{f'(x)} = 0.$$

We get the proposition by simplifying the above expression.  $\square$

By forcing  $\alpha + 1 = \beta + 1 = 2$ , we get the solution to our Problem.

**Corollary 6.** *Arrange  $n + 2$  equally charged particles in the interval  $[-1, 1]$ , with two fixed at  $-1$  and  $1$ . Then the solution to the Problem of this section is the zeros of the Jacobi polynomial  $J_n^{(1,1)}(x)$ .  $\square$*

**Remark.** By using similar arguments, the results above can be generalized to the case of varied weights on the end charges. In particular, the Chebyshev polynomial  $J_n^{(-\frac{1}{2}, -\frac{1}{2})}(x) = T_n(x)$  (resp. Legendre polynomial  $J_n^{(0,0)}(x) = P_n(x)$ ) will give the solution to the Problem if we had wanted the two end particles to have one-quarter (resp. one-half) the charge instead.

Here is a natural follow-up question: what if we want the charged particles to each have different charges, or lie on a different shape other than the line? This question has not been fully answered yet, and we don't have time to say much here; see the following three references for further reading.

- A generalization of the Chebyshev polynomial is given in [5], with limited application to varied charged particles on a line. The methods used there are similar to the ones presented above.
- A relationship between charged particles on the circle and the so-called “paraorthogonal polynomials” is discussed in [3].
- Finally, see [2] for a blanket overview of these kinds of problems.

### 3 Dessins d’Enfants

Let me now explain how I came across the electrostatics problem by way of combinatorial number theory.

#### 3.1 Belyi Functions

Let  $\mathbb{P}^1$  denote the complex projective line. In this section, we are mostly concerned with meromorphic functions  $\mathbb{P}^1 \rightarrow \mathbb{P}^1$  ramified in at most three points. If we apply a Möbius transformation, we can assume without loss of generality that the possible ramified points are  $0, 1, \infty$ .

**Definition 7.** A *Belyi function* is a meromorphic function  $\mathbb{P}^1 \rightarrow \mathbb{P}^1$  that is unramified outside  $\{0, 1, \infty\}$ .

**Theorem 8** (Belyi). *A Belyi function can be chosen in such a way that it is defined over  $\overline{\mathbb{Q}}$ .*

**Remark.** It should be noted that Belyi functions can be generalized by replacing the domain with a general Riemann surface or algebraic curve. Also, a consequence of Belyi’s theorem tells us that there is a natural Galois action on the set of Belyi functions, and in fact the action is transitive. One can find literature on studying these Galois orbits, or constructing invariants for these Galois actions. We will not concern ourselves with any of these here; see [1] for an exposition.

Belyi functions can be interpreted combinatorially as follow. Let  $f : \mathbb{P}^1 \rightarrow \mathbb{P}^1$  be a Belyi function, and consider the preimage of the segment  $[0, 1]$ . If we color the preimages of  $0$  black and the preimages of  $1$  white, we will set up a correspondence between Belyi maps and bipartite connected graphs on  $\mathbb{P}^1$ . If we write a Belyi function as

$$f(z) = \frac{p(z)}{r(z)},$$

we immediately notice a few things.

- If a black vertex  $b$  has degree  $d$ , then it contributes a factor of  $(x - b)^d$  to the polynomial  $p$ .
- Let us write

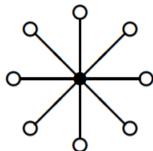
$$f(z) - 1 = \frac{q(z)}{r(z)}.$$

Then  $q$  records information about the white vertices just like how  $p$  records those for the black vertices.

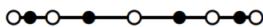
- The number of faces we get corresponds to the number of preimages for  $\infty$ . A closed face (i.e. a face excluding the outside face in a drawing of the graph) will always have  $2s$  sides, and such a face contributes  $s$  degrees to  $r(z)$ .

**Definition 9.** The graph constructed above is called a *dessin d’enfant*.

**Example 10.** Here is a simple example of a dessin d'enfant corresponding to the Belyi function  $f(z) = z^n$ .

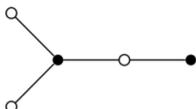


**Example 11.** The dessin d'enfant for the Chebyshev polynomial  $T_n$  is simply a straight line.



Notice that the vertices can be explicitly computed in this case; they are the critical points of  $T_n$ . If we want the critical points to map to  $0, 1$  instead of  $-1, 1$ , compose the map  $T_n$  by  $z \mapsto (z + 1)/2$  to get a Belyi map that suits the definition.

**Example 12.** In general, it is hard to represent a dessin d'enfant correctly in its “geometric form”, and we simply draw the lines in a convenient manner. For example, consider the dessin d'enfant below.



We can compute a Belyi function  $f(x)$  representing it (up to Möbius transformations) by letting the left-most black vertex be  $0$ , and the right-most white vertex be  $1$ . Then one has

$$f(x) = cx^3(x - \gamma) \quad \text{and} \quad f(x) - 1 = d(x - 1)^2(x - \alpha)(x - \beta),$$

giving

$$f(x) = -\frac{3}{4}x^3 \left( x - \frac{4}{3} \right).$$

This tells us that the vertices should not be spaced out equally.

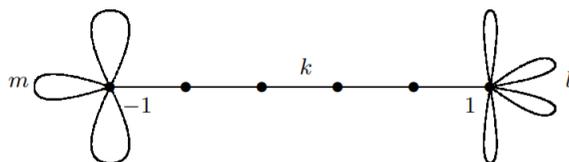
**Remark.** There are various combinatorial problems one can ask on dessin d'enfants.

- As mentioned earlier, a good understanding of Galois orbits and Galois invariants is desired.
- We can ask if Belyi functions with prescribed vertex and face degrees should exist. This is known as the *Hurwitz problem*.
- Belyi-extending maps are a good class of maps that generates more Galois invariants and Belyi functions from known ones. One can study such maps and hope to understand Belyi functions better.

We will not discuss any of them here; a literature search should give relevant articles on these things.

### 3.2 Example with Jacobi Polynomials

Let us now consider the following “double flower” (not drawn in its geometric form), where the white vertices are not shown and on the midpoints of each edge.



Without loss of generality, let's say that the leftmost black vertex is  $-1$ , and the rightmost one is  $1$ . Also assume there are  $k + l + m$  edges, with  $m$  petals on the left side, and  $l$  petals on the right side. We wish to find a Belyi function  $f$  that represents this. If we write

$$f(z) = \frac{p(z)}{r(z)}, \quad f(z) - 1 = \frac{q(z)}{r(z)}$$

we can immediately make a few observations.

- The degree of  $p$  is  $(2m + 1) + (2l + 1) + 2(k - 1)$ . In fact,

$$p(z) = (z + 1)^{2m+1}(z - 1)^{2l+1}P(z)^2,$$

where  $P$  is a degree  $k - 1$  polynomial.

- The degree of  $q$  is  $2(m + l + k)$ , and

$$q(z) = Q(z)^2.$$

- The degree of  $r$  is  $m + l$ .

All these implies that, in order to construct  $f$ , we need to find  $P, Q, r$  as above satisfying

$$(z + 1)^{2m+1}(z - 1)^{2l+1}P(z)^2 - Q(z)^2 = R(z).$$

We now see how Jacobi polynomials are related to double flowers via this equation. Recall that we can formally extend the definition of Jacobi polynomials to all half integers  $\alpha, \beta$ , though the extended functions may not be polynomials.

**Proposition 13** ([1, Proposition 2.5.9]). *The black vertices of the double flower corresponds to the zeros of  $J_{k-1}^{(l+\frac{1}{2}, m+\frac{1}{2})}(z)$ , and the white vertices corresponds to the zeros of  $J_{k+l+m}^{(-l-\frac{1}{2}, -m-\frac{1}{2})}(z)$ .*

*Proof.* Write

$$y_1(z) = J_{k+l+m}^{(-l-\frac{1}{2}, -m-\frac{1}{2})}(z) \quad \text{and} \quad y_2 = \left(\frac{z+1}{2}\right)^{m+\frac{1}{2}} \left(\frac{z-1}{2}\right)^{l+\frac{1}{2}} J_{k-1}^{(l+\frac{1}{2}, m+\frac{1}{2})}(z).$$

Using the notation above, we want

$$Q(z) = y_1(z) \quad \text{and} \quad (z + 1)^{m+\frac{1}{2}}(z - 1)^{l+\frac{1}{2}}P(z) = y_2(z).$$

A computation reveals the following:

- $y_1$  can be expanded into a formal Laurent series in  $z^{-1}$  using the generalized binomial theorem;
- $y_1^2$  is actually a polynomial in  $z$ ;
- $y_2$  is also a polynomial in  $z$ .

Also write  $L$  to be the formal differential operator for  $y_1$ :

$$L = (1 - z^2)\frac{d^2}{dz^2} + (-m + l - (-m - l + 1)z)\frac{d}{dz} + (k + l + m)k.$$

Another computation reveals that

$$L(y_1) = L(y_2) = 0.$$

Now, a third computation tells us that, if we have a Laurent series

$$y(z) = C_d z^d + C_{d-1} z^{d-1} + \dots$$

in  $z^{-1}$ , then

$$L(y) = (k + l + m - d)(d + k)C_d z^d + O(z^{d-1}).$$

Thus  $L(y) = 0$  implies that  $d = k + l + m$  or  $d = -k$ . By the definition of Jacobi polynomials, one observes that  $y_2 - y_1$  is of degree at most  $k + l + m - 1$ . In other words,

$$y_2 - y_1 = o(z^{k+l+m}),$$

and this forces  $y_2 - y_1 \sim z^{-k}$ . Therefore

$$y_2^2 - y_1^2 = (y_2 - y_1)(y_2 + y_1) \sim z^{l+m},$$

and since  $y_2^2$  and  $y_1^2$  are both polynomials, this implies

$$y_2^2 - y_1^2 = r(z)$$

for some degree  $l + m$  polynomial  $r$ . This is precisely what we wanted.  $\square$

As a final comment, let us notice that in the degenerate case  $l = m = 0$  this reduces to the Chebyshev polynomial case (check that the critical values and zeros of the relevant polynomials coincide with one another).

## References

- [1] Sergei Lando and Alexander Zvonkin. *Graphs on Surfaces and Their Applications*. Springer, 2004.
- [2] Francisco Marcellan, Andrei Martinez-Finkelshtein, and Pablo Martinez-Gonzalez. *Electrostatic models for zeros of polynomials: Old, new, and some open problems*. J. Comput. Appl. Math. **207** (2007), 258–272.
- [3] Brian Simanek. *An electrostatic interpretation of the zeros of paraorthogonal polynomials on the unit circle*. SIAM J. Math. Anal. **48**(3) (2016), 2250–2268.
- [4] Gabor Szego. *Orthogonal Polynomials*. American Mathematical Society, 1975.
- [5] Joris Van Deun. *Electrostatics and ghost poles in near best fixed pole rational interpolation*. Electron. Trans. Numer. Anal. **26** (2007), 439–452.