Music 101 for Science People

Pizza Seminar

September 27, 2019

What is music?

Music is noise organized in time. There are various aspects of music:

- pitch;
- dynamics;
- timbre;
- rhythm;
- texture.

The main principle govering music is the *wave equation*. Most instruments abuses the 1-D wave equation, and some percussion instruments make use of the 2-D and 3-D wave equations.

Today's Topics: Music 101; wave equation in math and music.

Musical instruments (from my perspective)

The instruments on the table today represent various broad categories.

- Cute Hamster: Electronic.
- Kalimba: Percussion.
- Melodica: Wind/Brass.
- Violin: String.

There are also keyboard instruments, like the piano. They are platypodes.

Here's a fun fact

The timpani is a pitched percussion instrument!

Let's listen to the timpani in Stravinsky's Firebird SCREAM.

Music eras (from my perspective)

Here are the classical music periods and some representative composers.

- Baroque (1600s–1750s): Bach, Handel, Scarlatti, Vivaldi.
- Classical (1750s–1820s): Beethoven, Clementi, Haydn, Mozart.
- Romantic (1820s–1920s): Berlioz, Brahms, Chopin, Liszt, Rachmaninoff, Schubert, Schumann, Tchaikovsky.
- Post-Romantic (1920s–??): Bartok, Britten, Copland, Gershwin, Prokofiev, Ravel, Shostakovich, Stravinsky.

For East Asian music, one probably divides them as follows:

- classical (pentatonic);
- contemporary (Westernization).

Aspects of music (from my perspective)

There seems to be three aspects of music.

- Performance: The noisy and artsy part of music.
- Theory & Composition: The intellectual and creative part of music.
- History: Good bedtime stories if taken casually. Extremely torturous if taken seriously.

We will be talking about the math behind music (the scientific one). One should add this to music theory to appease science people and frustrate music people.

We focus on non-percussion instruments, in particular the violin. After some derivation, the 1-D wave equation is

$$u_{tt} = c^2 u_{xx},$$

where:

- x is position and t is time;
- $u(x,t)$ is vertical displacement;
- $c^2 = T/\rho$, T is tension, and ρ is mass density.

For the purposes of music, the boundary conditions are

$$u(0,t) = u(L,t) = 0,$$

where L is the length.

Question: What are the initial conditions?

Math behind music: The 1-D wave equation

There are two different initial conditions. The first one also applies to all string/wind/brass/keyboard instruments.

- "Gently" Bowed (*arco*): For $0 < a < b < L$ with a and $b - a$ small,

$$u(x,0) = 0, \qquad \text{and} \qquad u_t(x,0) = \begin{cases} \text{constant} & \text{for } a \le x \le b, \\ 0 & \text{otherwise.} \end{cases}$$

- Plucked (*pizzicato*): For small c,

$$u(x,0) = \text{tent of height } c, \qquad \text{and} \qquad u_t(x,0) = 0.$$

Many other violin techniques also have the same initial conditions as *arco*: *staccato*, *tremolo*, *col legno*, *spiccato*, *ricochet*, *con sordino*, et cetera. However, they sound different. Here are some reasons:

- different constant values for $u_t(x,0)$;
- multiple wave equations in succession;
- varying energy dissipation, which the wave equation ignores;
- doctoring the body of the violin, which acts as a "boombox".

Two solutions to the 1-D wave equation

There are two ways to write the solution of

$$u_{tt} = c^2 u_{xx}$$

with conditions $u(0,t) = u(L,t) = 0$, $u(x,0) = f(x)$, $u_t(x,0) = g(x)$.

D'Alembert: If F and G are the odd extensions of f and g, then

$$u(x,t) = \frac{1}{2}F(x-ct) + \frac{1}{2}F(x+ct) + \frac{1}{2c}\int_{x-ct}^{x+ct} G(s)\,ds.$$

Fourier: The solution to the 1-D wave equation is

$$u(x,t) = \sum_{n=1}^{\infty}\left(\alpha_n \cos\frac{cn\pi t}{L} + \beta_n \sin\frac{cn\pi t}{L}\right)\sin\frac{n\pi x}{L},$$

where $\alpha_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$, and $\beta_n = \frac{2}{cn\pi}\int_0^L g(x)\sin\frac{n\pi x}{L}\,dx$.

D'Alembert: Shape of the 1-D wave equation

D'Alembert says: If F and G are the odd extensions of f and g, then

$$u(x,t) = \frac{1}{2}F(x-ct) + \frac{1}{2}F(x+ct) + \frac{1}{2c}\int_{x-ct}^{x+ct} G(s)\,ds.$$

This solution tells us how the wave will look like.

- For *arco*, it will look like a sharp wave traveling out and coming back in the opposite direction. Let's see a video of this, called Helmholtz motion.
- For *pizzicato*, if plucked very near to one end it will look very much like Helmholtz motion. If plucked in the middle it will look like a wave splitting apart. Here is a video of the latter.

Fourier says: The solution to the 1-D wave equation is

$$u(x,t) = \sum_{n=1}^{\infty} \left(\alpha_n \cos \frac{cn\pi t}{L} + \beta_n \sin \frac{cn\pi t}{L} \right) \sin \frac{n\pi x}{L},$$

where $\alpha_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L}\, dx$, and $\beta_n = \frac{2}{cn\pi} \int_0^L g(x) \sin \frac{n\pi x}{L}\, dx$. The essential elements of music can be extracted from this equation.

- Harmonics (Overtones): The numbers n, which comes from the eigenvalues of the 1-D Laplacian on the right hand side of the wave equation. The sound from a string is composed of much more than the fundamental frequency.
- Dynamics (Amplitude): The coefficients α_n and β_n. Notice the loudness depends on the initial conditions.
- Timbre (Quality): The summation over n, and the constant c. Recall that $c = T/\rho$, where T is tension and ρ is mass density.
- Pitch (Frequency): The constants c and L.

All of these aspects can be demonstrated on the violin.

Case study: Violin A-string

The A-string of a violin has:

$$L = 32.5\,\text{cm} = 0.325\,\text{m};$$
$$\rho = 0.60\,\text{g/m} = 0.00060\,\text{kg/m};$$
$$T = 49.08\,\text{N} = 49.08\,\text{kg m/s}^2.$$

Its fundamental frequency is

$$\frac{\text{speed}}{\text{wavelength}} = \frac{\sqrt{T/\rho}}{2L} \approx 440\,\text{Hz}.$$

However, if we substitute in the boundary conditions for *arco*, we see that the amplitudes for the higher harmonics is comparable to the one for the fundamental frequency. This does not agree with intuition.

The "Missing Fundamental" Phenomenon: The brain is wired to hear the pitch of the fundamental frequency (even if it is not present!). In fact, an example involves the G-string of a violin.

Vibrato: A key element in music

Vibrato is a musical effect on string instruments, and is the fancy term for a pulsating change of pitch.

- Mathematically, it is just changing the length L periodically.
- Musically, it gives more warmth to a note. If one plays most non-Baroque works on a violin without any vibrato, it will sound very unpleasant.

The vibrato can be imitated on wind/brass instruments by employing a pulsating change of air pressure.

Tangential Question: What do the pedals on a piano do?
[*The answer is different for uprights and grands*.]

Drums and the 2-D wave equation

The higher-dimensional wave equation is

$$u_{tt} = c^2 \Delta u,$$

where Δ is the Laplacian. For the ordinary circular drum, after observing u should not depend on θ, one can convert the 2-D wave equation into polar coordinates:

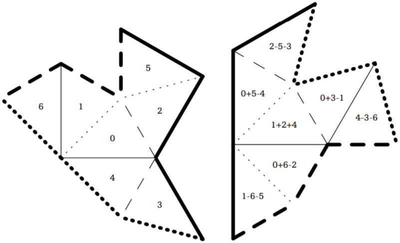
$$\frac{\partial^2 u}{\partial t^2} = \frac{c^2}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right).$$

We then solve it by separation of variables (analogous to the Fourier method in 1-D). The answer involves Bessel functions.

Question: Can one hear the shape of drums? More precisely, must drums of different shapes (but made with same materials) sound different?

Short Answer: No.

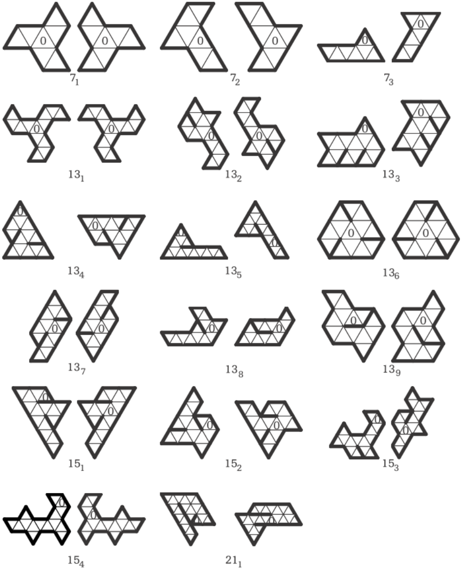
Long Answer: Using Sunada's method (1985), Gordon-Webb-Wolpert (1992) studied the Laplacian on Riemannian manifolds with boundary, and provided the first example of two irregular drums with the same eigenvalues and eigenfunctions.



We can verify this directly by symmetry considerations (BCDS 1994).

Drums and the 2-D wave equation

Here are more examples (BCDS 1994).



Higher-dimensional wave equations

Question: Can one hear the shape of a class of Riemannian manifolds?

Short Answer: Probably, probably not.

Slightly Longer Answer: Here are some stuff in the literature.

- Milnor (1964) exhibited two 16-dimensional tori that are distinct as Riemannian manifolds, but have the same eigenvalues.
- Zelditch (2000) showed that this is true for planar drums if one imposes some conditions (like convexity and analytic boundary).
- There are formulas that allows us to infer properties of the drum, such as area, by looking at the eigenvalues.

Some references

Music Theory

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Wave Equation

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