Oral Exam Practice Problems

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Here is an incomplete list outlining the things I did in preparation for the oral exam. I have not included the problems from the actual oral exam in this document. I would also like to thank Alon Benhaim for contributing some interesting problems.

1 Algebraic Number Theory and Algebraic Geometry

1.1 References with Good Problems

- John William Scott Cassels and Albrecht Fröhlich. *Algebraic Number Theory.*
- Daniel Marcus. *Number Fields.*
- Robin Hartshorne. *Algebraic Geometry.*

1.2 Other References

- Serge Lang. *Algebraic Number Theory.*
- Jürgen Neukirch. *Algebraic Number Theory.*
- Jean-Pierre Serre. *Local Fields.*
- Philip Griffiths and Joseph Harris. *Principles of Algebraic Geometry.*

1.3 Supplementary Problems

- Why is the product of two nonsquares a square in a finite field? Is this true for local fields or global fields?
- Find all quadratic extensions of the local and global rational fields.
- Do there exist local or global rational solutions to $x^2 - x - 3 = 0$?
- The local-global principle holds for quadratic forms, but does not hold in general. For example, show that there are no global rational solutions to $2y^2 - x^4 + 17z^4 = 0$, but there are local solutions at every place of $\mathbb{Q}$. Do the same for $3x^3 + 4y^3 + 5z^3 = 0$.
- Does Fermat's Last Theorem hold for $\mathbb{Q}_p$? How about $\mathbb{F}_q(t)$?
- What is the class group of $\mathbb{Q}(\sqrt{17})$? Exhibit an infinite family of number fields with nontrivial class group.
- Compute the class groups of $\mathbb{Q}(\sqrt{n})$ for $n \in \{-14, -5, 82\}$. Also do it for an extension of $\mathbb{Q}$ after adjoining one root of $x^3 - 11$.
- Find all integer solutions to $x^3 - y^2 = 2$.
- Explain how one can find all integral solutions to Pell's equation $x^2 - ny^2 = 1$ for any integer n.
- Find all extensions K of $\mathbb{Q}$ such that a prime p splits in K if and only if $p \equiv 1 \pmod 7$.
- Let K be a number field. Explain the relationship between the residue field degrees $f_1, \ldots, f_g$ corresponding to the ramification behavior of a prime p at K and the cycle type of any Frobenius associated to p in the Galois closure of K.
- Classify how primes can ramify in $\mathbb{Q}(\sqrt[3]{2})$, and give their corresponding densities. Do the same for an extension of $\mathbb{Q}$ by adjoining one root of $x^3 - x - 1$.
- Find a number field with Galois group $\mathbb{Z}/2 \times \mathbb{Z}/2$. Classify how primes ramify in such an extension, and give their corresponding densities. Can $p = 3$ ramify with ramification index 4? Can $p = 2$ ramify with ramification index 4? Do the same problem with $\mathbb{Z}/2 \times \mathbb{Z}/2$ replaced by $\mathbb{Z}/2 \times \mathbb{Z}/3$.
- What are the primes that splits in the maximal totally real subfield of $\mathbb{Q}(\zeta_n)$?
- Explicitly determine the ramification behavior of primes in the maximal totally real subfield of $\mathbb{Q}(\zeta_{11})$.
- Let K be a number field with an inert prime. Show that $\mathrm{Gal}(K/\mathbb{Q})$ is cyclic.

- Let K be a number field such that $\mathrm{Gal}(K/\mathbb{Q})$ is a p-group. Suppose there is exactly one prime q that ramifies in K. Show that K is abelian. Find all possible Galois groups for K in case $p = 5$ and $q = 101$.
- Classify the ramification behavior of primes in $\mathbb{Q}(\sqrt{p}, \sqrt{q})$ in terms of the Legendre symbol, where p and q are distinct prime numbers. Also give their corresponding densities.
- Compute the Frobenius associated to a prime that is unramified in $\mathbb{Q}(\sqrt{5}, \sqrt{-1})$.
- Show that $f(x) = x^5 - x + 1$ has discriminant 2869 and is irreducible over $\mathbb{Q}$. If α is a root of f, determine the number of real and complex embeddings of $\mathbb{Q}(\alpha)$. Show that $\mathcal{O}_F = \mathbb{Z}[\alpha]$, and that $\mathcal{O}_F$ is a PID.
- Let K be an extension of $\mathbb{Q}_p$. Let K^{nr} and K^t be the maximal unramified and maximal tame extensions of K in some fixed algebraic closure. Explicitly write down the action of $\mathrm{Gal}(K^{nr}/K)$ on $\mathrm{Gal}(K^t/K^{nr})$.
- For positive numbers $0 < a < b$, show that $\{x \in \mathbb{A}_K^\times : \|x\| \geq a\}$ and $\{x \in \mathbb{A}_K^\times : a \leq \|x\| \leq b\}$ are closed in $\mathbb{A}_K$, but $\{x \in \mathbb{A}_K^\times : \|x\| \leq b\}$ is not closed.
- Discuss the endomorphism rings of $\mathbb{A}_K$, and of $\mathbb{A}_K^\times$.
- What is Artin reciprocity for cyclotomic fields? Use Cebotarev's Density Theorem to prove the infinitude of primes in arithmetic progression. What is its density? Show that if natural density exists, so does its Dirichlet density, and they are both equal. Is the converse true?
- Let K be a number field and let v be a place of K. Let G be one of GL_2, SL_2, or O_f, where f is a classically integral quadratic form in two variables. Is $G(K_v)$ compact? Is $G(\mathbb{A}_K)$ compact? Is $G(\mathbb{A}_{K,f})$ compact? Write down a nonempty algebraic variety V over $\mathbb{A}_K$ such that $V(\mathbb{A}_K)$ is compact.
- Let K be a number field. What is the image of $\mathcal{O}_K^\times$ in $\mathbb{A}_{K,f}^\times$? Does the closure of $\mathcal{O}_K^\times$ in $\mathbb{A}_{K,f}^\times$ have finite or infinite index?
- Write down all the Dirichlet characters for $\mathbb{Q}$ and $\mathbb{F}_q(t)$.
- Define the Dirichlet, Artin, and Hecke L-functions associated to a Hecke character $\mathbb{A}_K^\times/K^\times \longrightarrow \mathbb{C}^\times$. Why is the Dirichlet L-function a special case of the latter two? Given an irreducible representation on the Galois group of an abelian extension L/K, show that its associated Artin L-function is a Hecke L-function.
- Reinterpret class field theory as a bijection between finite order Hecke characters and a certain other set of characters. Explicitly write down the Hecke character corresponding to a quadratic number field.
- Construct dihedral and quaternion extensions of a number field using class field theory. At some point you should compute the Verlagerung of order 8 groups, and make use of Hecke characters $\mathbb{A}_K^\times/K^\times \longrightarrow \mu_4$.
- Write down all the Hecke characters $\mathbb{A}_K^\times/K^\times \longrightarrow \mathbb{C}^\times$ when K is $\mathbb{Q}$ or a quadratic number field.
- Define what it means for a Hecke character $\chi : \mathbb{A}_K^\times/K^\times \longrightarrow \mathbb{C}^\times$ to be algebraic. Explain why χ has finite order if and only if it is trivial on the connected component of the identity. Use Dirichlet's Unit Theorem to show that the infinity component of χ is determined by its maximal CM subfield (or its maximal totally real subfield if a CM subfield does not exist).
- Define a CM elliptic curve. What are the possible endomorphism rings of such an elliptic curve? Give an example of a CM elliptic curve, and compute its endomorphism ring. Explain how Hecke L-functions can be used to determine the Hasse-Weil zeta function of a CM elliptic curve.
- Describe and prove the relationships between the following statements: $\mathbb{Q}(\sqrt{-163})$ has class number one; $e^{\pi\sqrt{163}}$ is almost an integer; the polynomial $x^2 + x + 41$ gives a prime number for $x = -1, 0, 1, \ldots, 40$.
- Compute the arithmetic and geometric genus of the curve $y^2 = x^3 + x^2$, which is possibly singular. Explain all the results that you use in the computation. Do the same for the curves $y^3 = x^6 + 1$, $y^4 = x^3 - x$, $y^4 = x^2(x-1)$, and $y^2 = (x-1)(x-2)^2 \cdots (x-100)^{100}$.
- Do there exist morphisms of nonsingular curves from genus 8 to 7? How about from genus 7 to 8? Discuss the existence of morphisms of nonsingular curves of genus g' to g with fixed degree. Given a prescribed list of ramification degrees satisfying the Riemann-Hurwitz formula, does there exist a morphism of nonsingular curves with such ramification degrees?
- Let C be a genus zero curve over a field K. Explain why C embeds as a conic in $\mathbb{P}_K^2$. If a K-rational point of C exists, explain why it must be isomorphic to $\mathbb{P}_K^1$. Explicitly classify genus zero curves over $\mathbb{R}$.
- Explain why the image of the the degree map on the Picard group of a genus zero curve is either $\mathbb{Z}$ or $2\mathbb{Z}$.
- Let P be a point on a Riemann surface of genus g. Consider the sequence of numbers $l(P), l(2P), l(3P), \ldots$. Use the Riemann-Roch Theorem to explain how this sequence of numbers changes at every step. Write down this sequence of numbers in genera zero and one.
- Consider a curve X over a finite field. Is $\mathrm{Pic}^0(X)$ finite? How about for curves over general fields? Relate $\mathrm{Pic}^0(X)$ to the Picard group for a function field.
- Relate the different ideal in number theory to the Kähler differential in algebraic geometry.

- Let K be a function field. If $S = \{v\}$ is a valuation of K of degree d, and $\mathrm{Pic}^0(K)$ has cardinality h, what is the cardinality of $\mathcal{O}_{K,S}$, defined to be the elements of K with no poles outside v? Give a plausible explanation for why this is the analog of the ring of integers for number fields.

2 Algebraic Combinatorics and Representation Theory

2.1 References with Good Problems

- Richard Stanley. *Enumerative Combinatorics, Volume 1.*
- Richard Stanley. *Enumerative Combinatorics, Volume 2.*
- William Fulton and Joseph Harris. *Representation Theory.*

2.2 Other References

- Bruce Sagan. *The Symmetric Group.*
- Alexander Kirillov Jr. *An Introduction to Lie Groups and Lie Algebras.*

2.3 Supplementary Problems

Everything here will be over the complex numbers, unless otherwise stated. Also, most of the problems are in representation theory since the references on algebraic combinatorics listed above covers everything you need to know for the combinatorics oral exam.

- State Lie's theorems on representations of Lie groups and Lie algebras. How does one classify all the irreducible representations of a semisimple Lie algebra?
- Show that the representations $\mathbb{C}^n$, $S^k\mathbb{C}^n$, $\Lambda^k\mathbb{C}^n$ of $\mathfrak{sl}_n$ induced by the standard left action are irreducible. Compute the dimensions and highest weights of these representations.
- Write down a compact group. Explain why irreducible representations of compact groups are always finite dimensional.
- Use spherical harmonics to construct an irreducible representation of $\mathrm{SO}_3(\mathbb{R})$. What are the finite-dimensional representations of $\mathrm{SO}_3(\mathbb{R})$?
- Explicitly realize all finite dimensional irreducible representations of $\mathrm{SL}_2(\mathbb{C})$. Can you write down an infinite dimensional irreducible representation of $\mathrm{SL}_2(\mathbb{C})$?
- Every linear algebraic group over $\mathbb{R}$ (or $\mathbb{C}$) is also a Lie group, but the converse is not true. For example, show that the universal cover of $\mathrm{SL}_2(\mathbb{R})$ is a Lie group, but is not a linear algebraic group.
- Write down all the Dynkin diagrams, and write down a simple Lie algebra corresponding to each Dynkin diagram. Compute the roots and dominant integral weights for each Dynkin diagram.
- Explicitly show the Lie algebras isomorphisms $\mathfrak{sl}_2 \cong \mathfrak{so}_3$, $\mathfrak{so}_5 \cong \mathfrak{sp}_4$ and $\mathfrak{so}_4 \cong \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$.
- Discuss the similarities between the classification of irreducible representations of $\mathrm{SU}(n)$, $\mathfrak{sl}(n)$, and the symmetric group S_n.
- State the Weyl character and dimension formulas. Compute them for the simple Lie algebras of type A_2, B_2, D_2 and G_2. Embed D_2 into B_2 via the long roots, and embed A_2 into G_2 via the long roots. Explicitly describe the branching rules $\mathrm{Res}^{B_2}_{D_2}$ and $\mathrm{Res}^{G_2}_{A_2}$. The Littlewood-Richardson Rule may be helpful in the latter case.
- Why is the Weyl character formula for type A_n the same as the Schur function defined via the semistandard Young tableau?

3 Final Remark

It is also helpful to look at the Princeton Math Generals @ `https://web.math.princeton.edu/generals/`.