Dessin d'Enfants and Superpotentials

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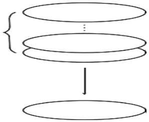
Branched covers

Our main object of study are *covers between compact Riemann surfaces branched over three points with base space the Riemann sphere*. Here are visual representations of...

a (compact Riemann) surface
Σ_g of genus g

a covering map
$X \longrightarrow Y$



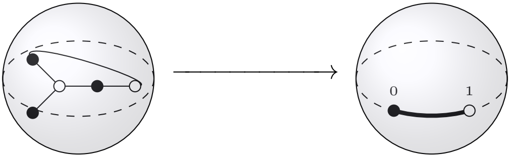
- $f : \Sigma_g \longrightarrow \Sigma_0$ is a ***branched cover*** if it behaves like a covering map over all but a finite subset $\{y_1, \ldots, y_n\}$ of Σ_0. These points are called ***branch points***.
- Every branched cover considered from this point forward will be branched over at most three points $\{y_1, y_2, y_3\}$.
- After a Möbius transformation, assume $\{y_1, y_2, y_3\} = \{0, 1, \infty\}$.

Dessin d'enfants

A *dessin d'enfant* $\mathcal{D}$ associated to the isomorphism class of a branched cover $f : \Sigma_g \longrightarrow \Sigma_0$ branched over $0, 1, \infty$ is the preimage of the interval $[0, 1]$. This is a connected bipartite graph embedded on $\mathcal{D}$, where:

- the preimages of 0 (resp. 1) are colored black (resp. white),
- the preimages of ∞ corresponds to the faces of $\mathcal{D}$, i.e. connected components of $\Sigma_g \setminus \mathcal{D}$.

In this talk we will focus on the case $\Sigma_g = \Sigma_0$. Below is such an example for $\Sigma_g = \Sigma_0$.



Branched covers of trees

A branched cover $f : \Sigma_0 \longrightarrow \Sigma_0$, with dessin d'enfant $\mathcal{T}$ a tree, can be expressed generally as follow.

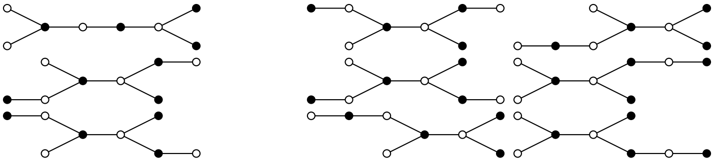
- Say $\mathcal{T}$ has black (resp. white) vertices of degrees $e_1, \ldots, e_n$ (resp. $f_1, \ldots, f_m$) with coordinates $b_1, \ldots, b_n$ (resp. $w_1, \ldots, w_m$).
- After a transformation to make sure $b_i, w_j \neq \infty$, the map f is a rational function satisfying

$$f(z) = K_1 \prod_{i=1}^{n} (z - b_i)^{e_i},$$

$$f(z) - 1 = K_2 \prod_{j=1}^{m} (z - w_j)^{f_j},$$

  where $K_1, K_2 \in \mathbb{C}$ are nonzero constants.
- The problem to find the branched cover of a tree (or general dessin) given just its graph is highly nontrivial. A complete classification of this exists only for trees of degrees at most ten, and for general dessins of degrees at most four.

An example



- The branch covers of the left hand side trees are of the form

$$f(z) = z^3(z-a)^2\left(z^2 + \left(-\frac{7}{2} + 2a\right)z + \frac{8}{5}a^2 - \frac{28}{5}a + \frac{21}{5}\right)$$

where $24a^3 - 84a^2 + 98a - 35 = 0$.

- The branch covers of the right hand side trees are of the form

$$f(z) = z^3(z-a)^2\Bigg(z^2 + \left(\frac{4}{3}a^5 - \frac{34}{15}a^4 - \frac{26}{15}a^3 + \frac{7}{5}a^2 + \frac{20}{3}a - \frac{28}{5}\right)z \\ - \frac{8}{15}a^5 - \frac{32}{75}a^4 + \frac{172}{75}a^3 + \frac{148}{75}a^2 - \frac{14}{5}a - \frac{287}{75}\Bigg)$$

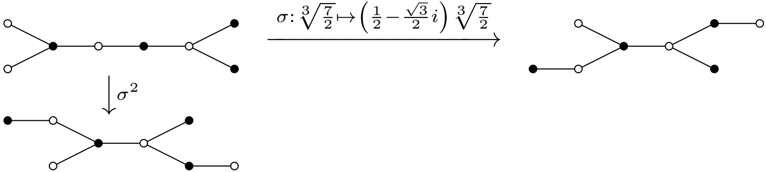
where $20a^6 - 84a^5 + 84a^4 + 56a^3 - 294a + 245 = 0$.

Galois action

Theorem (Belyi, 1980)

A compact Riemann surfaces S is defined over $\overline{\mathbb{Q}}$ if and only if there is a branched cover $f : S \longrightarrow \Sigma_0$ with at most three branched points.

- Belyi's theorem gives us an action of the absolute Galois group $\Gamma = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on branched covers branched at $\{0, 1, \infty\}$.
- For $S = \Sigma_0$, the action is on the coefficients on the rational function defining the branched cover.



Galois invariants

A property that is equal on two Galois-conjugate dessin d'enfants is called a *Galois invariant*. Below are some well-known Galois invariants of a dessin d'enfant $\mathcal{D}$:

- the number of edges, white vertices, black vertices, and faces of $\mathcal{D}$,
- the degree of white vertices, black vertices, and faces of $\mathcal{D}$,
- the genus of $\mathcal{D}$.

Nontrivial Galois invariants are important because they distinguish Galois orbits of dessin d'enfants, which actually corresponds to subsets of the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ via the following theorem.

Theorem

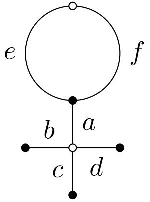
$\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ *acts faithfully on the set of dessin d'enfants.*

We only have enough time to study the superpotential algebra of a tree and state a conjecture on its Galois invariant property.

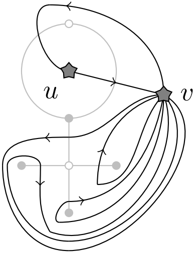
The superpotential algebra

We define the superpotential algebra via the following example.

$\mathcal{F} =$



$\mathcal{F}^\vee =$

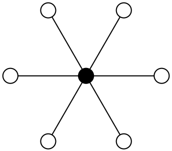


$$
\begin{aligned}
W_{\mathcal{F}} &= abcd + ef - aef - b - c - d \\
I_{W_{\mathcal{F}}} &= (\partial_x W_{\mathcal{F}})_{x \in \{a,b,c,d,e,f\}} \\
&= \begin{pmatrix} bcd - ef, cda - v, dab - v \\ abc - v, f - fa, e - ae \end{pmatrix}
\end{aligned}
$$

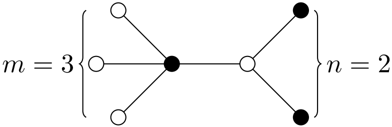
$$
\begin{aligned}
A_{\mathcal{F}} &:= \mathbb{Z}[a,b,c,d,e,f,u,v] \Big/ I_{W_{\mathcal{F}}} \cup \left\{ \begin{matrix} \text{path algebra} \\ \text{relations} \end{matrix} \right\} \\
&\cong \mathbb{Z}[e,f,u,v] \Big/ \left(\{efef - v\} \cup \left\{ \begin{matrix} \text{path algebra} \\ \text{relations} \end{matrix} \right\} \right).
\end{aligned}
$$

Superpotential algebra for stars and double stars

Below are two examples of stars and double stars.



$A_{\mathrm{Star}_6} \cong \mathbb{Z}[x]/(x^5 - 1)$



$A_{\mathrm{DStar}_{3,2}} \cong \mathbb{Z}[x]/(x^{3-2} - 1) \cong \mathbb{Z}$

In general we have the following computation.

- $A_{\mathrm{Star}_n} \cong \mathbb{Z}[x]/(x^{n-1} - 1)$, where n is the number of leaves.
- $A_{\mathrm{DStar}_{n,m}} \cong \begin{cases} \mathbb{Z}[x, x^{-1}] & \text{if } n = m \\ \mathbb{Z}[x]/(x^{m-n} - 1) & \text{if } n < m \end{cases}$

The superpotential rank

Theorem

The superpotential algebra $A_{\mathcal{T}}$ *for a tree* $\mathcal{T}$ *is isomorphic to* $\mathbb{Z}[x, x^{-1}]$ *or* $\mathbb{Z}[x]/(x^l - 1)$ *for some non-negative integer* l*. Furthermore,*

- *every edge* z *is identified with* $x^{m(z)}$ *in* $A_{\mathcal{T}}$ *for some integer* $m(z)$*,*
- *leaves are identified with* x *in* $A_{\mathcal{T}}$*.*

If we define the *superpotential rank* $l(\mathcal{T})$ for a tree $\mathcal{T}$ to be

$$l(\mathcal{T}) = \begin{cases} l & \text{if } A_{\mathcal{T}} \cong \mathbb{Z}[x]/(x^l - 1), \\ \infty & \text{if } A_{\mathcal{T}} \cong \mathbb{Z}[x] \text{ or if } A_{\mathcal{T}} \cong \mathbb{Z}[x, x^{-1}], \end{cases}$$

then we have the following conjecture.

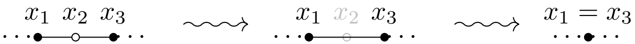
Conjecture

$l(\mathcal{T})$ is a Galois invariant on trees.

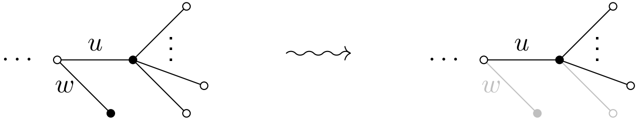
This conjecture is provable in special cases, and checked for trees of degrees up to ten.

The proof of the previous theorem relies on the following reductions for a tree $\mathcal{T}$. Here $\mathcal{T}', \mathcal{T}'', \mathcal{T}'''$ are the trees after reduction.

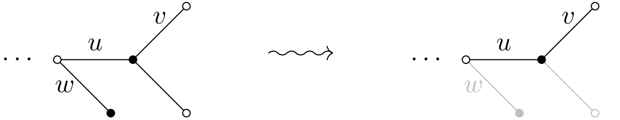
- $A_{\mathcal{T}} \cong A_{\mathcal{T}'}$.



- $A_{\mathcal{T}} \cong A_{\mathcal{T}''}$.

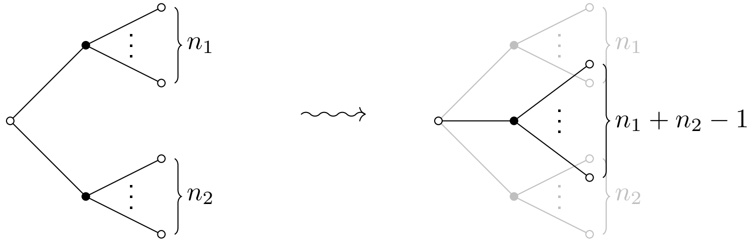


- $A_{\mathcal{T}} \cong A_{\mathcal{T}'''}[v^{-1}]$.



Proof idea of theorem

After doing the previous three reductions till it is not impossible, a technical lemma implies that we can apply the following reduction to get a tree $\mathcal{T}''''$.



The reduction gives us a superpotential algebra $A_{\mathcal{T}''''}$ for which $A_{\mathcal{T}}$ is a certain quotient of. Iterating the four reductions above reduces us to the star or double star, which gives us the theorem two slides ago, and defines the superpotential rank.

- Assuming the conjecture that the superpotential rank is a Galois invariant, the reductions imply that, although there is no canonical way to track the reductions of Galois conjugate trees, they are actually reduced to the same star or double star.