Adelic class groups via two quick examples

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In this brief expository note we explain what the class groups of the linear algebraic groups GL_n and O_f are. (Here O_f is the orthogonal group of a classically integral quadratic form.) We will assume some familiarity with adeles and ideles, and strong approximation on algebraic groups.

Recurring symbols in adelic formulation

Symbol	Meaning
K	Global field (i.e. number field or function field)
$\mathcal{O}_K$	Ring of integers of K
v	Place (also called valuation or prime) of K
$v \nmid \infty$	v is a finite place (also called nonarchimedean valuation/prime)
$v \mid \infty$	v is an infinite place (also called archimedean valuation/prime)
K_v	Completion of K with respect to v
$\mathcal{O}_v$	Elements $x \in K_v$ with $\lvert x\rvert_v \leq 1$
$\mathbb{A}_K$	Ring of adeles of K
S	Finite set of places of K (usually just the archimedean ones)
$\mathbb{A}_{K,S}$	Subring of $\mathbb{A}_K$ avoiding the places in S
$\mathbb{A}_K^S$	Ring of S-adeles of K
$\mathbb{A}_{K,f}$	Ring of finite adeles of K
$\mathbb{A}_{K,\infty}$	Infinite part of $\mathbb{A}_K$
$\mathbb{A}_K^\infty$	Ring of ∞-adeles of K
$\mathbb{A}_K^\times$	Ring of ideles of K

1 The general linear group

Let K be a number field. Then one can define its *class group* to be the group I_K of fractional ideals modulo the group P_K of principal ideals of K. In adelic formulation, this can be written as

$$Cl(K) := \frac{\mathbb{A}_{K,f}^\times}{K^\times \prod_{v\nmid\infty} \mathcal{O}_v^\times} = (\mathbb{A}_K^\infty)^\times \setminus \mathbb{A}_K^\times / K^\times,$$

which is precisely the double coset for GL_1 in the previous section. A way to see that the two definitions agree is to consider the surjective homomorphism

$$\begin{aligned} A_{K,f}^\times &\longrightarrow I_K/P_K \\ (\alpha_v) &\longmapsto \prod_{v\nmid\infty} v^{\mathrm{ord}_v(\alpha_v)} \end{aligned}$$

which has kernel $K^\times \prod_{v\nmid\infty} \mathcal{O}_v^\times$. It is well-known that the class group of K is finite. More generally, we will define class groups for linear algebraic groups in the next section.

Let us concentrate on the example GL_n for now. Then its class group is defined to be the set of double cosets

$$Cl(\mathrm{GL}_n(K)) := \mathrm{GL}_n(\mathbb{A}_K^\infty) \setminus \mathrm{GL}_n(\mathbb{A}_K) / \mathrm{GL}_n(K).$$

Example 1. Here is a trivial example. Let $K = \mathbb{Q}$. Then, as

$$\mathbb{A}_{\mathbb{Q}}^{\times} = \mathbb{Q}^{\times}(\hat{\mathbb{Z}}^{\times} \times \mathbb{R}_{>0}^{\times}),$$

one easily sees that $Cl(\mathrm{GL}_n(K)) = 1$. There is a similar decomposition of $\mathbb{A}_{\mathbb{K}}^{\times}$ by modding out units, but this does not help in computing class groups; see the theorem directly below instead.

Notice that $Cl(\mathrm{GL}_1(K)) = Cl(K)$ by definition.

Theorem 2. $Cl(\mathrm{GL}_n(K)) = Cl(K)$.

Proof. Let $G = \mathrm{GL}_n$, and consider the determinant map $\det : G \longrightarrow \mathrm{GL}_1$. Then one observes that

$$\det(G(\mathbb{A}_K)) = \mathbb{A}_K^{\times}, \qquad \det(G(\mathbb{A}_K^{\infty})) = (\mathbb{A}_K^{\infty})^{\times}, \qquad \det(G(K)) = K^{\times}.$$

Hence there is an induced map

$$\det : G(\mathbb{A}_K^{\infty}) \setminus G(\mathbb{A}_K)/G(K) \longrightarrow (\mathbb{A}_K^{\infty})^{\times} \setminus \mathbb{A}_K^{\times} / K^{\times}.$$

This map is surjective, so it remains to show injectivity. Suppose

$$(\mathbb{A}_K^{\infty})^{\times} \det(g) K^{\times} = (\mathbb{A}_K^{\infty})^{\times} \det(h) K^{\times}.$$

We need to show that $G(\mathbb{A}_K^{\infty}) g G(K) = G(\mathbb{A}_K^{\infty}) h G(K)$. By assumption

$$\det(g) = x \det(h) y$$

for some $x \in (\mathbb{A}_K^{\infty})^{\times}$ and $y \in K^{\times}$. Picking $a \in G(\mathbb{A}_K^{\infty})$ and $b \in G(K)$ such that $\det(a) = x$ and $\det(b) = y$, one gets

$$\det(g) = \det(ahb).$$

It suffices to show g and ahb define the same double coset in $G(\mathbb{A}_K^{\infty}) \setminus G(\mathbb{A}_K)/G(K)$. Writing $t = ahb$, observe that

$$s := t^{-1} g \in H(\mathbb{A}_K),$$

where H is the subgroup SL_n of G. Since $U := t^{-1} H(\mathbb{A}_K^{\infty}) t$ is an open subgroup of $H(\mathbb{A}_K)$, by strong approximation

$$Us \cap H(A_{K,\infty}) H(K) \neq \emptyset.$$

Since $H(A_{K,\infty}) \subset U$, this gives the existence of $u \in H(\mathbb{A}_K^{\infty})$ and $v \in H(K)$ such that

$$t^{-1} u t s = v.$$

Rewriting, one gets $g = u^{-1} t v$, as desired. □

Remark. In the proof above we made use of strong approximation for SL_n. In fact, strong approximation does not hold for GL_n! See [3] for two explanations of this.

Recall a *lattice* is a finitely-generated $\mathcal{O}_K$-module in K^n containing a K-basis of K^n. A lattice in K^n is always free over K, but it might not be free over $\mathcal{O}_K$. However, by the structure theory of finitely generated modules over a PID, a lattice in $K_v^{\times}$ is always free for any finite place v. We assume the following classical result about the local behavior of lattices.

Theorem 3. *Let L be a lattice in $V = K^n$. If v is a finite place of K, write $L_v := L \otimes_{\mathcal{O}_K} \mathcal{O}_v$.*

1. *A lattice is uniquely determined by its localizations, i.e $L = \bigcap_{v \nmid \infty} (V \cap L_v)$.*

2. *If M is another lattice, then $L_v = M_v$ for almost all finite v.*

3. *For every v, let $N_v \subset V \otimes_K K_v$ be local lattices. If $N_v = L_v$ for almost all finite v, then there exists a unique lattice $M \subset V$ such that $M_v = N_v$ for all finite v.*

Proof. See [2, Theorem 1.15]. ☐

We now show that the class group of $\mathrm{GL}_n(K)$ (which is the class group of K by above) parametrizes lattices in K^n. This gives a geometric interpretation of the class group of a number field.

Corollary 4. *$Cl(\mathrm{GL}_n(K))$ is in one to one correspondence with the set of isomorphism classes of lattices in K^n.*

Proof. Let $\mathcal{L}$ be the set of all lattices. Then the previous theorem defines an action of $\mathrm{GL}_n(\mathbb{A}_K)$ on $\mathcal{L}$ as follows. If $g = (g_v) \in \mathrm{GL}_n(\mathbb{A}_K)$ and $L \in \mathcal{L}$, then $g_v \in \mathrm{GL}_n(\mathcal{O}_v)$ and $L_v = \mathcal{O}_v^n$ for almost all finite places v, implying $g_v L_v = L_v$. One then defines gL to be the unique lattice M such that $M_v = g_v L_v$ for all finite places v.

Let us now fix $L = \mathcal{O}^n$. If M is another lattice in K^n, then for all finite v we can write $M_v = g_v(L_v)$ for some $g_v \in \mathrm{GL}_n(\mathcal{O}_v)$. Since $M_v = L_v$ for almost all finite v, there exists $g \in GL(\mathbb{A}_K)$ such that $M = g(L)$. (Notice we are using the previous theorem here.) Therefore the action defined in the previous paragraph is transitive. As the stabilizer of L is $\mathrm{GL}_n(\mathbb{A}_K^\infty)$, there is a bijection

$$\mathrm{GL}_n(\mathbb{A}_K^\infty) \setminus \mathrm{GL}_n(A_K) \longleftrightarrow \mathcal{L},$$

implying $Cl(\mathrm{GL}_n(K))$ is in bijection with $\mathcal{L}/\mathrm{GL}_n(K)$, the isomorphism classes of lattices in K^n. ☐

One can look at [2, Section 8.1], or sieve it out from the arguments in this section, for various ways to determine if a lattice in K^n is free over $\mathcal{O}_K$.

Remark. Strong approximation gives us a similar relationship between special linear groups and unimodular lattices; in particular for SL_2 one has

$$\mathrm{SL}_2(\hat{\mathbb{Z}}) \setminus \mathrm{SL}_2(\mathbb{A}_\mathbb{Q}) / \mathrm{SL}_2(\mathbb{Q}) = \mathrm{SL}_2(\mathbb{R}) / \mathrm{SL}_2(\mathbb{Z});$$

this is quotient has finite Haar volume and parametrizes unimodular lattices.

2 Some general theorems

Recall our convention that a linear algebraic group is an affine algebraic group with a fixed embedding into GL_n for some n. In general the class group of a linear algebraic group is defined just as in the case of GL_n.

Definition 5. Let G be a linear algebraic group. Then its *class group* is defined to be

$$Cl(G) := G(\mathbb{A}_K^\infty) \setminus G(\mathbb{A}_K)/G(K).$$

Theorem 6. *The class group of a linear algebraic group is always finite.*

Proof. See [2, Theorem 5.1]. ☐

Remark. In general the class group of an arbitrary algebraic group is not always finite; see [1, Example 1.5].

One can ask if it is possible to bound class groups via smaller subgroups. There are various results of this form in [2], and we record two of them here. Recall that G satisfies absolute strong approximation if the embedding $G(K) \longrightarrow \mathbb{A}_{K,f}$ is dense (strong approximation is when $\mathbb{A}_{K,f}$ is replaced by some $\mathbb{A}_{K,S}$; see [2, Chapter 5] for details).

Proposition 7. *Let G be a semidirect product of H and N, where N is a normal subgroup of G (and everything is defined over K). If N satisfies absolute strong approximation, then $Cl(G) \leq Cl(H)$.*

Proof. See [2, Proposition 5.4]. ☐

Proposition 8. *Let G be a reductive group, and let P be a parabolic K-subgroup of G. Then $Cl(G) \leq Cl(P)$.*

Proof. See [2, Theorem 8.11]. ☐

The main purpose of this section is to understand the following statement, which is a special case of Proposition 7 above.

Proposition 9. *The class group of a linear algebraic group G with absolute strong approximation has cardinality 1.*

Proof. Since G satisfies absolute strong approximation, $G(\mathbb{A}_{K,\infty})G_K$ is dense in $G(\mathbb{A}_K)$. Therefore the open set $G(\mathbb{A}_K^\infty)x$ intersects $G(\mathbb{A}_{K,\infty})G_K$ nontrivially for any $y \in G(\mathbb{A}_K)$, and consequently

$$G(\mathbb{A}_K) = G(\mathbb{A}_K^\infty)G(\mathbb{A}_{K,\infty})G_K = G(\mathbb{A}_{K,\infty})G_K,$$

where the second equality is because $G(\mathbb{A}_{K,\infty}) \subset G(\mathbb{A}_K^\infty)$. This implies $G(\mathbb{A}_K)$ has exactly one double coset, so $Cl(G) = 1$. □

Corollary 10. $Cl(\mathrm{SL}_n) = 1$.

Proof. SL_n satisfies absolute strong approximation. □

3 The orthogonal group

Let $G \subset \mathrm{GL}_n$ be a linear algebraic group acting on an affine m-dimensional variety X. If x and y lie in the same $G(\mathcal{O}_K)$-orbit of $X(\mathcal{O}_K)$, then they clearly lie in the same $G(K)$-orbit of $X(K)$, and $G(\mathcal{O}_v)$-orbit of $G(K_v)$, for all finite place v. A naive local-global problem we can ask if the following: does the converse always hold? One will expect that it usually does not hold, and consequently ask for a measurement of the failure of this local-global problem. We make all these ideas concrete via the following example/motivation.

Example/Motivation 11 (Quadratic forms)**.** Let f be a classically integral quadratic form over $\mathbb{Q}$, so

$$f = \sum_i a_{ii}X_i^2 + \sum_{j\neq k} 2a_{jk}X_jX_k, \qquad a_{ii}, a_{jk} \in \mathbb{Z}.$$

Given such a quadratic form one can associate to it the symmetric matrix $A_f = (a_{ij})$. Define the *class* $\mathrm{cl}(f)$ of f to be the collection of all classically integral quadratic forms f' that are equivalent over $\mathbb{Z}$, i.e. such that $g^tfg = f'$ for some $g \in \mathrm{GL}_2(\mathbb{Z})$, and define the *genus* $\mathrm{gen}(f)$ to be the collection of all classically integral quadratic forms f' such that they are equivalent over $\mathbb{Q}$ and $\mathbb{Z}_p$ for all primes p (but not necessarily over $\mathbb{Z}$). Clearly

$$\mathrm{gen}(f) = \bigsqcup_{i\in I_f} \mathrm{cl}(f_i),$$

where f_i is a set of representatives in the genus of f. We define the *number of classes $c(f)$ of f* to be the cardinality of I_f.

In general $c(f) \neq 1$ by considering the quadratic form $f = 5x^2 + 11y^2$. This is because the quadratic form

$$f' = x^2 + 55y^2$$

lies in the same genus and in a different class of f. To see this, consider

$$g_1 = \begin{bmatrix} 1/4 & -11/4 \\ 1/4 & 5/4 \end{bmatrix}, \qquad g_2 = \begin{bmatrix} 1/7 & -22/7 \\ 2/7 & 5/7 \end{bmatrix}.$$

Then $g_1^tfg_1 = f'$ and $g_2^tfg_2 = f'$. Since $g_1 \in \mathrm{GL}_2(\mathbb{Z}_p)$ for all $p \neq 2$ and $g_2 \in \mathrm{GL}_2(\mathbb{Z}_p)$, we see that f and f' are in the same genus. However, a direct computation shows that there does not exist $g \in \mathrm{GL}_2(\mathbb{Z})$ such that $g^tfg = f'$, so they cannot be in the same class.

We recall that there is a brute-force way to determine the number of classes of a binary quadratic form f over $\mathbb{Q}$. Namely, write down all the classes forms equivalent to f under $\mathrm{SL}_2(\mathbb{Z})$ (which is bounded by the class number of $\mathbb{Q}[\sqrt{\mathrm{disc}(f)}]$), identify those equivalent under $\mathrm{GL}_2(\mathbb{Z})$, and check pairwise if they are equivalent under $\mathbb{Q}$ and $\mathbb{Z}_p$. Using this method, one can show that $c(f) = 2$ for the form $f = 5x^2 + 11y^2$ in the previous paragraph. For a general quadratic form f, we will compute $c(f)$ below as the class group of the orthogonal group of f.

We now generalize all the definitions in the above example/motivation.

Definition 12. Let Let $G \subset \mathrm{GL}_n$ be a linear algebraic group acting on an affine m-dimensional variety X, and let $x \in X(\mathcal{O}_K)$.

- The *genus* $\mathrm{gen}(x)$ of x is the collection of all $y \in X(\mathcal{O}_K)$ such that $y = g_K x$ for some $g_K \in G(K)$, and $y = g_v x$ for some $g_v \in G(\mathcal{O}_v)$ for all finite places v.
- The *class* $\mathrm{cl}(x)$ of x is the $G(\mathcal{O}_K)$-orbit of x.
- If one writes
$$\mathrm{gen}(x) = \bigsqcup_{i \in I_x} \mathrm{cl}(f_x)$$
for some set of representatives f_x in the genus of x, then $f_G(x)$ is defined to be the cardinality of I_x.

Theorem 13. *Let $G_x = \{g \in G : gx = x\}$. Then $f_G(x)$ is the number of double cosets $G_x(\mathbb{A}_K^\infty) g G_x(K)$ of $G_x(\mathbb{A}_K)$ which are contained in $G(\mathbb{A}_K^\infty)G(K)$. In particular, $f_G(x)$ is finite.*

Proof sketch. Let $\eth$ be the quotient set obtained from $\mathrm{gen}(x)$ by identifying elements belonging to the same class. We will construct the bijection between $\eth$ and the set M of double cosets $G_x(\mathbb{A}_K^\infty) g G_x(K)$ of $G_x(\mathbb{A}_K)$ contained in $G(\mathbb{A}_K^\infty)G(K)$, and leave the verification to the reader (see [2, Theorem 8.2]). Let $\overline{g} = G_x(\mathbb{A}_K^\infty) g G_x(K) \in M$, and write $g = g_\infty g_K$ with $g_\infty \in G(\mathbb{A}_K^\infty)$ and $g_K \in G(K)$. Defining $y_g := g_K x$, the bijection $\theta : M \longrightarrow \eth$ is given by $\theta(\overline{g}) = y_g$. □

Corollary 14. *If f is a classically integral quadratic form over $\mathcal{O}_K$, then $c(f) = Cl(O_f)$, where*

$$O_f = \{g \in \mathrm{GL}_n : g^t A_f g = A_f\}.$$

Proof. Let $X \subset \mathbb{A}^{n^2}$ be the variety of $n \times n$ symmetric matrices, and consider the action of $G = \mathrm{GL}_n$ by $g(x) = g^t x g$. Clearly $G_f = O_f$. If we can show that $O_f(\mathbb{A}_K) \subset G(\mathbb{A}_K^\infty)G(K)$, then we are done by the theorem above.

For each finite place v, clearly $G(\mathcal{O}_v)$ contains a matrix with determinant -1, so any element $t \in O_f(\mathbb{A}_K)$ has $st \in \mathrm{SL}_n(\mathbb{A}_K)$ for a suitable element $s \in G(\mathbb{A}_K^\infty)$. But we know that $Cl(\mathrm{SL}_n(K)) = 1$ as SL_n satisfies absolute strong approximation, so $st = s_\infty s_K$ for some $s_\infty \in \mathrm{SL}_n(\mathbb{A}_K^\infty)$ and $s_K \in \mathrm{SL}_n(K)$. In particular,

$$t = s^{-1} s_\infty s_K \in G(\mathbb{A}_K^\infty)G(K),$$

as desired. □

Remark. The above corollary agrees with the philosophy that local-global classification problems are related to the class group, since they are the analog of first cohomology in geometry.

References

[1] Brian Conrad, Notes on finiteness of class numbers for algebraic groups.

[2] Vladimir Platonov and Andrei Rapinchuk, Algebraic groups and number theory. *Academic Press*, 1993.

[3] Andrei Rapinchuk, Strong approximation for algebraic groups. *MSRI Publications* (2013), **61**: 269–298.