Story begins with $N \in \mathbb{Z}_{\geq 1}$ and $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$ a Dirichlet character.

Attached to χ is $\rho_\chi : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL_1(\mathbb{C})$ a Galois representation

$$\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \twoheadrightarrow \mathrm{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{Q}) \cong (\mathbb{Z}/N\mathbb{Z})^\times \xrightarrow{\chi} GL_1(\mathbb{C}).$$

Next: GL_2 story. Say f is a cuspidal modular form and an Eigenform for T_ps. Say $T_p f = \lambda_p f$, $\lambda_p \in \mathbb{C}$. Turns out that the subfield of $\mathbb{C}$ generated by λ_p is a number field E_f. If ℓ is a prime number and $\lambda | \ell$ is a prime in E_f, then following a suggestion of Serre, Deligne constructured $\rho_f : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL_2(\overline{E_{f,\lambda}})$.

↳ ρ_f is attached to f in some way. If f is a modular form, f has

→ level $N \geq 1$ → weight $k \geq 1$ → character χ.

Turns out that ρ_f is unramified outside $N\ell$. If $p \nmid N\ell$ is prime, then $\rho_f(\mathrm{Frob}_p)$ has characteristic polynomial $x^2 - \lambda_p x + p^{k-1}\chi(p)$. (Cebotarev density theorem implies there is at most one such semisimple ρ_f).

- A word on constructing ρ_f: Deligne used étale cohomology, with nontrivial coefficients. (and then using trivial coefficients). This is for $k \geq 2$; $k=1$ in 1974 by Deligne-Serre.

Question from construction: If $p | N\ell$, what does ρ_f look like locally at p?

↳ Case 1: $p|N$ and $p \neq \ell$. The answer is given by local Langlands correspondence.

↳ Case 2: $p = \ell$. Then we should use p-adic Local Langlands correspondence.

Easier question: Instead of asking for ρ_f, could instead ask for

$$\overline{\rho}_f : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL_2(\overline{\mathbb{F}}_\ell) \quad \text{(← residue field of } E_\lambda \text{ at } \lambda\text{)} \quad (\overline{\rho}_f \subset \ell\text{-torsion of an appropriate abelian variety})$$

Are ρ_χ and ρ_f part of a general story?

Thm (Harris, Lan, Taylor, Thorne 2013 ; Scholze later):

Let E be a totally real or CM number field

π be a cuspidal automorphic representation of $GL_n(\mathbb{A}_E)$.

Assume that π_∞ is "cohomological" (an algebraicity assumption). Then there exists some $\rho_\pi : \mathrm{Gal}(\overline{E}/E) \to GL_n(\overline{\mathbb{Q}}_\ell)$ attached to π in some canonical way (analogue of giving a characteristic polynomial of $\rho_f(\mathrm{Frob}_p)$). More details later ...

Seen: $\left(\begin{array}{c}\text{algebraic or analytic gadget} \\ \chi, f, \pi, \dots\end{array}\right) \xrightarrow[\text{machinery}]{\text{technical}} \left(\text{representations of Galois group}\right)$

Interesting question: Can we classify the image? i.e. say ρ: Galois group $\to GL_n(\text{field})$.
Is it isomorphic to a representation from an algebraic/analytic gadget?

1-dimensional case. Say $K/\mathbb{Q}$ is a finite Galois extension and $\rho : \mathrm{Gal}(K/\mathbb{Q}) \to GL_1(\mathbb{C})$.
It is isomorphic to some ρ_χ ... ?

By replacing K with a subfield, we can assume ρ is injective, so $\mathrm{Gal}(K/\mathbb{Q})$ is abelian.

- For $\rho_\chi : \mathrm{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{Q}) \to \mathbb{C}$, it gives rise to $\mathrm{Gal}(L/\mathbb{Q}) \xhookrightarrow{\rho_\chi} \mathbb{C}^\times$ for some $L \subset \mathbb{Q}(\zeta_N)$.

So question is now: If $K/\mathbb{Q}$ is Galois with abelian Galois group, does there exist $N \geq 1$ with $K \subset \mathbb{Q}(\zeta_N)$? Yes, by Kronecker - Weber theorem.

$\hookrightarrow$ So: for all $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL_1(\mathbb{C})$ continuous (image finite), there exists some $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$ such that $\rho \cong \rho_\chi$.

2-dimensional case. If f is a cuspidal modular eigenform as before, then ρ_f has the following properties:

(1) ρ_f is absolutely irreducible

(2) ρ_f is odd, i.e. $\det \rho_f(\text{complex conjugation}) = -1$.

(3) ρ_f is unramified outside a finite set of primes, and

[condition in p-adic Hodge theory] ρ_f is potentially semistable at ℓ. $(\rho_f : Gal(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL_2(\overline{\mathbb{Q}}_\ell))$.

In early 1990s Fontaine-Mazur asked if the converse is true: if $\rho: Gal(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL_2(\overline{\mathbb{Q}}_\ell)$ satisfies (1) – (3), then is $\rho \cong \rho_f$ for some f?

↳ The conjecture is basically known by work of

Kisin, "The Fontaine-Mazur conjecture for GL_2"

Emerton, "Local-Global compatibility in the p-adic Langlands program for GL_2".

GL_n case: Is $\rho: Gal(\overline{E}/E) \to GL_n(\overline{\mathbb{Q}}_p)$ and assumptions enough to imply $\rho \cong \rho_\pi$ as in HLTT?

↳ Barnet-Lamb, Gee, Geraghty, Taylor proved this in many cases.

↳ see also 10 author paper, persiflage blog, Galois representations.

In this summer school we will give an updated version of Richard Taylor's Caltech 1992 course on an introduction to the Langlands program

2017-07-24 (2) Actual start.

Part 1: The Local Langlands Correspondence.

This correspondence for $GL_n(K)$, K a finite extension of $\mathbb{Q}_p$, is a canonical bijection

vaguely speaking: $\left\{\begin{array}{c}\text{certain (typically } \infty\text{-dimensional)} \\ \text{irreducible } \mathbb{C}\text{-representations of } GL_n(K)\end{array}\right\} \leftrightarrow \left\{\begin{array}{c}\text{certain } n\text{-dimensional } \mathbb{C}\text{-representations} \\ \text{of a group related to } Gal(\bar{K}/K)\end{array}\right\}$

$\underline{n=1}$: This is local class field theory.

$\underline{n>1}$: theorem of Harris-Taylor (2000); proofs also are global. The orange book.

Infinite Galois groups

Reminder of finite case: Let L/K be a finite extension. It is Galois if it is separable and normal. Write $Gal(L/K) = \{K\text{-automorphisms of } L\}$, finite of size $\dim_K L$.

There is an inclusion-reversing correspondence

$$(\text{Subgroups } H \subset Gal(L/K)) \longleftrightarrow (\text{fields } M \text{ with } K \subset M \subset L)$$

$$Gal(L/M) \longleftrightarrow M$$

Infinite case: Let L/K be an algebraic extension. It is Galois if separable and normal.

If $\lambda \in L$ then there is M, $K \subset M \subset L$, with M/K finite Galois and containing λ.

Thus, for $\varphi \in Gal(L/K)$, $\varphi(\lambda)$ is determined by image of φ in $Gal(L/K) \to Gal(M/K)$.

↳ In particular φ is determined by $\varphi|_M$ for all $K \underset{\text{Galois}}{\overset{\text{finite}}{\subset}} M \subset L$ and

$$Gal(L/K) \hookrightarrow \prod_{\substack{K \overset{\text{finite}}{\subset} M \subset L \\ \text{Galois}}} Gal(M/K).$$

Thus $Gal(L/K) = \varprojlim_{\substack{K \overset{\text{finite}}{\subset} M \subset L \\ \text{Galois}}} Gal(M/K)$.

Put product topology on $\prod \mathrm{Gal}(M/K)$, each $\mathrm{Gal}(M/K)$ with discrete topology.

$\mathrm{Gal}(L/K)$ is a closed subspace of $\prod \mathrm{Gal}(M/K)$, so give it the subspace topology.

<u>Fundamental theorem</u>: If L/K is Galois, then there is a bijection

$$(\text{closed subgroups of } \mathrm{Gal}(L/K)) \longleftrightarrow (\text{fields } M \text{ with } K\subset M \subset L).$$

$$\mathrm{Gal}(L/M) \longleftarrow M$$

<u>Example</u>: (0) $K=\mathbb{Q}$, $L=\bigcup_{n\geq 1}\mathbb{Q}(\zeta_{p^n})$. Writing $L_n=\mathbb{Q}(\zeta_{p^n})$, we know $\mathrm{Gal}(L_n/\mathbb{Q})=(\mathbb{Z}/p^n\mathbb{Z})^\times$.

It is clear that $\mathrm{Gal}(L/K)=\varprojlim(\mathbb{Z}/p^n\mathbb{Z})^\times=\mathbb{Z}_p^\times$.

(1) $K=\mathbb{F}_q$ and $L=\overline{K}$. Writing $L_n=\mathbb{F}_{q^n}$, note that $L_n\subset L_m$ iff $n\mid m$. So L is the filtered colimit of $\mathbb{F}_{q^n}$, and $\mathrm{Gal}(L/K)=\varprojlim(\mathbb{Z}/n\mathbb{Z})=\prod_p \mathbb{Z}_p=\hat{\mathbb{Z}}$.

(2) $K/\mathbb{Q}_p$ finite extension and $L=\overline{K}$ an algebraic extension. We'll fail to understand $\mathrm{Gal}(L/K)$, we can get some scraps though. Recall the (additive) normalized valuation $v_K: K^\times \twoheadrightarrow \mathbb{Z}$. We can uniquely extend v_K to v_L ($v_L: L^\times\to\mathbb{Q}$) on L as L/K is algebraic.

$$L \supset \mathcal{O}_L=\{0\}\cup\{\ell\in L : v_L(\ell)\geq 0\}\supset \mathfrak{p}_L, \quad \text{residue field } \mathcal{O}_L/\mathfrak{p}_L=k_L.$$

Note that k_L/k_K is algebraic. If L/K is Galois,

$$\mathrm{Gal}(L/K)\longrightarrow \mathrm{Gal}(k_L/k_K) \text{ is } \underline{\text{surjective}}.$$

Say L/K is unramified if the above map is an isomorphism.

- TFAE: (1) $\mathfrak{p}_L=\pi_K\mathcal{O}_L$ (2) $v_K(L^\times)=\mathbb{Z}$ (3) L/K unramified.
- The compositum of two unramified extensions is unramified.
- For L/K algebraic, there is a maximal unramified M/K inside L, and $\mathrm{Gal}(M/K)=\mathrm{Gal}(k_M/k_K)$.

2017-07-25 (1)

<u>Recall</u>: For $K/\mathbb{Q}_p$ finite and L/K a Galois extension, we define the <u>inertia group</u> to be $I_{L/K} = \ker(\mathrm{Gal}(L/K) \twoheadrightarrow \mathrm{Gal}(k_L/k_K))$.

To understand $\mathrm{Gal}(L/K)$, need to focus on $I_{L/K}$. Note that $I_{L/K}$ is a closed subgroup, so by fundamental theorem $I_{L/K} = \mathrm{Gal}(L/M)$ for some $K \subset M \subset L$. Here M is the union of all subfields of L unramified over K.

Special interesting case: $L = \overline{K}$

$$\begin{array}{cl} \overline{K} & \\ | & \Big\} \, I_{\overline{K}/K} \\ K^{nr} & \\ | & \Big\} \, \hat{\mathbb{Z}} \\ K & \end{array}$$

$\mathrm{Gal}(K^{nr}/K) = \mathrm{Gal}(\overline{k}_K/k_K)$

maximal unramified

<u>Example</u>: If $K = \mathbb{Q}_p$, then $K^{nr} = \bigcup_{p \nmid m \geq 1} \mathbb{Q}_p(\zeta_m)$.

$\boxed{\text{Now}}$ Assume L/K Galois, and $I_{L/K}$ finite. Put a filtration on $I_{L/K}$ as follow. If $\sigma \in I_{L/K}$, then $\sigma(\mathcal{O}_L) \subset \mathcal{O}_L$ and $\sigma(\mathfrak{p}_L) \subset \mathfrak{p}_L$. Because $I_{L/K}$ is finite, $\mathfrak{p}_L = (\pi_L)$ is principal, and also $v_L = (\# I_{L/K}) \cdot v_K$.

The <u>filtration</u>. For $i \geq 1$, define $I_{L/K,i} := \{ \sigma \in I_{L/K} : \frac{\sigma(\pi_L)}{\pi_L} \in 1 + \mathfrak{p}_L^i \}$.

Set $I_{L/K,0} = I_{L/K}$. Notice $I_{L/K,i} \supset I_{L/K,i+1}$.

The $I_{L/K,i}$ are normal subgroups in $\mathrm{Gal}(L/K)$. Furthermore if $i \gg 0$ then $I_{L/K} = \{1\}$.

Note that $I_{L/K}/I_{L/K,1} \hookrightarrow k_L^\times$ by $\sigma \mapsto \frac{\sigma(\pi_L)}{\pi_L}$, so $I_{L/K}/I_{L/K,1}$ is cyclic of order prime to p. If $i \geq 1$, then $I_{L/K,i}/I_{L/K,i+1} \hookrightarrow \mathfrak{p}_L^i/\mathfrak{p}_L^{i+1}$ by $\sigma \mapsto \frac{\sigma(\pi_L)}{\pi_L} - 1$ (in fact $\mathfrak{p}_L^i/\mathfrak{p}_L^{i+1} \cong k_L$), so $I_{L/K,i}/I_{L/K,i+1} \cong (\mathbb{Z}/p\mathbb{Z})^{\text{some power}}$.

<u>Upshot</u>. $I_{L/K,1}$ is the unique Sylow p-subgroup, and $I_{L/K}$ is solvable.

We say L/K is

- tamely ramified if $I_{L/K,1} = \{1\}$ (example: unramified extensions)
- wildly ramified if $I_{L/K,1} \neq \{1\}$.

We are really interested in $L = \overline{K}$ case, where $I_{\overline{K}/K}$ is not finite. Unfortunately $I_{L/K,i}$ does not behave well with respect to extensions.

⌜If $L'/L/K$ with L'/K and L/K Galois, $I_{L'/K}$ finite, then $I_{L'/K} \twoheadrightarrow I_{L/K}$ but $I_{L'/K,i}$ is not identified with $I_{L/K,i}$.⌟

A fix: Relabelling. Again let L/K Galois with $I_{L/K}$ finite. Set $g_i = \# I_{L/K,i}$, and say $g_0 \geq g_1 \geq \cdots \geq g_M = 1$ for $M \gg 0$. Define $\varphi: [0,\infty) \to [0,\infty)$ a piecewise linear (linear on $(i,i+1)$) and continuous function by

- $\varphi(0) = 0$,
- on $(i,i+1)$, φ has slope $\frac{g_{i+1}}{g_0}$. (so eventually φ has slope $\frac{1}{g_0}$).

Clearly φ is a strictly increasing bijection. For $v \in \mathbb{R}_{\geq 0}$, let $I_{L/K,v} = I_{L/K,\lceil v \rceil}$.

Def: For $u \in \mathbb{R}_{\geq 0}$, let $I^u_{L/K} := I_{L/K,\varphi^{-1}(u)}$.

Prop: If $L'/L/K$ all Galois and $I_{L'/K}$ is finite, then $I^u_{L/K} = \mathrm{im}(I^u_{L'/K})$

⌜Hasse Arf Theorem: If L/K is abelian then the jumps for $I^u_{L/K}$ are integers!⌟

If L/K is any Galois extension, we can glue $I^u_{M/K}$, for M/K algebraic with $I_{M/K}$ finite, and use that to define $I^u_{L/K}$.

Note L/K tamely ramified $\Leftrightarrow I_{L/K,\varepsilon} = \{1\}$ for $\varepsilon > 0$

$\Leftrightarrow I^{\delta}_{L/K} = \{1\}$ for $\delta > 0$ ← good definition for any Galois L/K.

As compositum of tamely ramified is still, L/K contains a maximal tamely ramified extension.

So we now have

$$\begin{array}{cl} L & \\ | & \Big\} \text{ wild part} \\ K_2 & \\ | & \\ K_1 & \\ | & \Big\} \text{ maximal unramified} \\ K & \end{array} \qquad \text{maximal tamely ramified: } K_2 / K$$

What are the Galois groups?

Back to $\overline{K}/K$. If K_2/K^{nr} is Galois with group $\mathbb{Z}/m\mathbb{Z}$, by Kummer theory $K_2 = K^{nr}(\sqrt[m]{\alpha})$ for some $\alpha \in K^{nr}$. In fact it is not hard to check that $K_2 = K^{nr}(\sqrt[m]{\pi_K})$.

Thus the maximal tamely ramified extension is $K^t = \bigcup_{p \nmid m \geq 1} K^{nr}(\sqrt[m]{\pi_K})$.

Note that $\mathrm{Gal}(K^{nr}(\sqrt[m]{\pi_K})/K^{nr}) = \mu_m$ via map $\sigma \mapsto \dfrac{\sigma(\sqrt[m]{\pi_K})}{\sqrt[m]{\pi_K}}$. Thus

$$\mathrm{Gal}(K^t/K^{nr}) = \varprojlim_{p \nmid m} \mu_m \cong \varprojlim_{p \nmid m} \mathbb{Z}/m\mathbb{Z} = \prod_{\ell \neq p} \mathbb{Z}_\ell .$$

And so...

$$\begin{array}{cl} \overline{K} & \\ | & \Big\} \text{ some pro-}p\text{ group.} \\ K^t & \\ | & \Big\} \cong \prod_{\ell \neq p} \mathbb{Z}_\ell \\ K^{nr} & \\ | & \Big\} = \hat{\mathbb{Z}} = \prod_{\ell} \mathbb{Z}_\ell \\ K & \end{array}$$

Recall $\mathrm{Gal}(K^{nr}/K) = \mathrm{Gal}(k_K^{nr}/k_K)$ and contains the Frobenius $\mathrm{Frob}: x \mapsto x^q$, $q = \# k_K$.

If we lift Frob to $\mathrm{Gal}(K^t/K)$ then it acts by conjugation on the normal subgroup $\mathrm{Gal}(K^t/K^{nr})$: $\sigma \mapsto \mathrm{Frob} \circ \sigma \circ \mathrm{Frob}^{-1}$. Check that $\mathrm{Gal}(K^t/K^{nr}) = \varprojlim_m \mu_m(\overline{K})$ and the map induced by Frob is $\zeta \mapsto \zeta^q$ (the glue telling us what $\mathrm{Gal}(K^t/K)$ is).

2017-07-25 (2)

We have just seen an attempt to analyze $\mathrm{Gal}(\bar{K}/K)$, K a p-adic field, via an explicit attack on the inertia group. An obstacle: Sylow p-subgroup is hard to understand.

Another approach: try to understand the abelianization $\mathrm{Gal}(\bar{K}/K)^{ab}$ via

Local Class Field Theory

Def: Let $K/\mathbb{Q}_p$ finite. Recall $1 \to I_{\bar{K}/K} \to \mathrm{Gal}(\bar{K}/K) \to \hat{\mathbb{Z}} \to 1$. ← generated by "Frob", and equal $\mathrm{Gal}(K^{nr}/K)$

For $\mathrm{Frob} \in \hat{\mathbb{Z}}$, consider $(\mathrm{Frob})^{\mathbb{Z}} = \mathbb{Z} \subset \hat{\mathbb{Z}}$. Define the Weil group W_K of K by

$$\begin{array}{ccccccccc} 1 & \to & I_{\bar{K}/K} & \to & W_K & \to & (\mathrm{Frob})^{\mathbb{Z}} = \mathbb{Z} & \to & 1 \\ & & \| & & \downarrow & & \downarrow & & \\ 1 & \to & I_{\bar{K}/K} & \to & \mathrm{Gal}(\bar{K}/K) & \to & \hat{\mathbb{Z}} & \to & 1 \end{array}$$

Formally, $W_K := \{ g \in \mathrm{Gal}(\bar{K}/K) : \mathrm{im}(g) \text{ in } \hat{\mathbb{Z}} = \mathrm{Gal}(K^{nr}/K) \text{ is in } \mathbb{Z} = (\mathrm{Frob})^{\mathbb{Z}} \}$.

Topologize W_K with $I_{\bar{K}/K}$ open in it, and the quotient $W_K / I_{\bar{K}/K}$ with discrete topology.

(In particular, W_K does not have subspace topology.)

For G a topological group, let G^c be the topological closure of the commutator group.

Then G/G^c is the maximal abelian Hausdorff quotient of G, denoted G^{ab}.

Main theorem: Let $K/\mathbb{Q}_p$ be finite. Then there is a canonical isomorphism $r_K : K^\times \longrightarrow W_K^{ab}$, and it satisfies the following properties.

- $$\begin{array}{rccc} r_K : & K^\times & \xrightarrow{\sim} & W_K^{ab} \\ & \cup & & \cup \\ & \mathcal{O}_K^\times & \xrightarrow{\cong} & \mathrm{image}(I_{\bar{K}/K}) \\ & \cup & & \cup \\ (i \geq 1) & 1+\mathfrak{p}_K^i & \xrightarrow{\cong} & \mathrm{image}(I_{\bar{K}/K}^i) \end{array}$$

- $r_K(\pi_K) \in \mathrm{Frob}^{-1} \cdot \mathrm{Image}(I_{\bar{K}/K})$.

More properties to come ...

⌜ Note: r_K here is the one sending π_K to Frob^{-1}. ⌟

- If. L/K is finite, then $W_L \hookrightarrow W_K$ and $W_L^{ab} \to W_K^{ab}$ (not necessarily injective!). Also –

$$\begin{tikzcd} L^\times \arrow[r, "r_L"] \arrow[d, "Nm_{L/K}"'] & W_L^{ab} \arrow[d] \\ K^\times \arrow[r, "r_K"] & W_K^{ab} \end{tikzcd}$$

- There is a transfer map (verlagerung): for $H \subset G$ of finite index, there is this map $V: G^{ab} \to H^{ab}$ by $g \mapsto \prod_{i=1}^{n} \gamma_i g \gamma_i^{-1}$, the γ_i being coset representatives of G/H.

We have

$$\begin{tikzcd} L^\times \arrow[r, "r_L"] & W_L^{ab} \\ K^\times \arrow[u, hook] \arrow[r, "r_K"] & W_K^{ab} \arrow[u, "\text{transfer}"'] \end{tikzcd}$$

- If L/K is finite Galois, then W_K/W_L is finite with W_L normal in W_K, and $W_K/W_L = \mathrm{Gal}(L/K)$. Also $W_L^c \lhd W_K$, so we can define $W_{L/K} := W_K/W_L^c$, and so

$$1 \longrightarrow L^\times \overset{r_L}{=} W_L^{ab} \hookrightarrow W_{L/K} \longrightarrow \mathrm{Gal}(L/K) \longrightarrow 1.$$

This extension gives rise to an extension of $H^2(\mathrm{Gal}(L/K), L^\times)$. This element is called $\alpha_{L/K}$ and generates the group $H^2(\mathrm{Gal}(L/K), L^\times)$, which turns out to be cyclic.

($\alpha_{L/K}$ is called the fundamental class.)

<u>Upshot.</u> We now understand the Galois group of maximal abelian extension of $K/\mathbb{Q}_p$.

$$\begin{tikzcd} & \overline{K} \arrow[d, dash] \\ & K^{ab} \arrow[d, dash] \arrow[dl, dash] \\ K^{nr} \arrow[r, dash] & K \end{tikzcd}$$

Galois group is $\mathrm{Gal}(K^{ab}/K) = \mathrm{Gal}(\overline{K}/K)^{ab}$

$$\begin{tikzcd} \mathcal{O}_K^\times = I_{K^{ab}/K} \arrow[r] & \mathrm{Gal}(\overline{K}/K)^{ab} \arrow[r] & \mathrm{Gal}(K^{nr}/K) = \hat{\mathbb{Z}} \\ I_{K^{ab}/K} \arrow[u, equal] \arrow[r] & W_K^{ab} \arrow[u, hook] \arrow[r] & \mathbb{Z} \arrow[u, hook] \\ & K^\times \arrow[u, equal, "r_K"'] & \end{tikzcd}$$

$\therefore \mathrm{Gal}(\overline{K}/K)^{ab} \cong \mathcal{O}_K^\times \times \hat{\mathbb{Z}}$.

2017-07-26 (1)

Towards statements of Local Langlands -- Let $K/\mathbb{Q}_p$ finite.

Vaguely speaking: $\left(\begin{array}{c}\text{n-dimensional representations} \\ \text{of "Galois groups"}\end{array}\right) \leftrightarrow \left(\begin{array}{c}\text{some representations} \\ \text{of } GL_n(K)\end{array}\right)$.

Recall.

$$\begin{tikzcd} 1 \arrow[r] & I_{\bar{K}/K} \arrow[r] \arrow[d, equal] & Gal(\bar{K}/K) \arrow[r] & \hat{\mathbb{Z}} \arrow[r] & 1 \\ 1 \arrow[r] & I_{\bar{K}/K} \arrow[r] & W_K \arrow[r] \arrow[u, hook] & \mathbb{Z} \arrow[r] \arrow[u, hook] & 1 \end{tikzcd}$$

Let E be a field, and put discrete topology on E and $GL_n(E)$ ($n \in \mathbb{Z}_{>0}$ fixed). Let us consider a continuous homomorphism $\rho : W_K \to GL_n(E)$, so that $\ker\rho$ is open. As $I_{\bar{K}/K}$ is ~~open~~ compact, $\rho(I_{\bar{K}/K})$ is compact in $GL_n(E)$, so $\rho(I_{\bar{K}/K})$ is a finite set. Hence we can use the theory of lower numbering on $\rho(I_{\bar{K}/K})$.

$$\begin{tikzcd} & \bar{K} \arrow[dash, d] \arrow[dash, dr] & \\ I_{\bar{K}/K} & K^{nr} \arrow[dash, r, "\rho(I_{\bar{K}/K})"'] & L = L(\rho) \end{tikzcd}$$

$\rho(I_{\bar{K}/K}) = I_{L/K} \supset I_{L/K,1} \supset \cdots$

use fundamental theorem

Define the conductor of ρ to be

$$f(\rho) := \sum_{i=0}^{\infty} \frac{1}{[I_{L/K} : I_{L/K,i}]} \dim\left(V / V^{I_{L/K,i}}\right)$$

where V is defined by $GL_n(E) = Aut_E(V)$. (Note $f(\rho) = 0 \Leftrightarrow \rho$ is unramified).

Remark: $f(\rho)$ is an integer!

Example. Recall we have $v_K : K^\times \to \mathbb{Z}$, norm $|\lambda| = e^{-v_K(\lambda)}$ (many choices) and

$$\begin{tikzcd} r_K : K^\times \arrow[r, "\cong"] & W_K^{ab} \\ & {|\lambda|} \arrow[ul, two heads] \arrow[u] \end{tikzcd}$$

⌜ Interlude: If K is any field complete with a nontrivial nonarchimedean norm, we can set up a good theory of rigid geometry. ⌟

In our situation where $K/\mathbb{Q}_p$ is finite there is a canonical norm! As K is locally compact, it has an additive Haar measure μ, say normalized such that $\mu(\mathcal{O}_K) = 1$.

Thus $\mu(\mathfrak{p}_K) = q_K^{-1}$ as $\mathcal{O}_K = \coprod_{\lambda \in k_K} (\lambda + \mathfrak{p}_K)$.

Cute idea: If $a \in K^\times$, define $\|a\|$ to be the factor by which multiplication by a scales Haar measure, so $\|\lambda\| = \frac{\mu(\lambda X)}{\mu(X)}$. This gives some idea of "norm".

Back to $K/\mathbb{Q}_p$ finite, we scale our norm such that $|\pi_K| = q^{-1}$, the "natural norm"

Thus we have

$$\begin{tikzcd} W_K \arrow[r, two heads] \arrow[drr, dashed, "\text{induces}", "|\cdot|"'] & W_K^{ab} \arrow[r, "r_K", "\cong"'] & K^\times \arrow[d, "|\cdot|"] \\ & & \mathbb{Q}_{>0} \end{tikzcd}$$

which is an example of a representation $W_K \to GL_1(\mathbb{Q})$.

Exercise: $f(|\cdot|^m) = 0$ for all $m \in \mathbb{Z}_{>0}$.

What is $|\widetilde{Frob}|$? By looking at the definitions $|\widetilde{Frob}| = |\pi_K^{-1}| = q$.

Weil-Deligne representations

A Weil-Deligne representation (WD) is a pair (ρ_0, N), where

- $\rho_0 : W_K \to \mathrm{Aut}_E(V) \cong GL_n(E)$ is a continuous representation with $\mathrm{char}(E) = 0$
- $N : V \to V$ is an E-linear nilpotent endomorphism

such that, for all $\sigma \in W_K$, we require $\rho_0(\sigma) \circ N \circ \rho_0(\sigma)^{-1} = |\sigma| N$ ($|\cdot|$ as above).

Example: Every W_K-representation is a WD representation by setting $N = 0$.

Example: Let $E = \mathbb{Q}$ and $V = \mathbb{Q}\langle e_1, e_0\rangle \cong \mathbb{Q}^2$. Define ρ_0 by $\rho_0(\sigma) = \begin{bmatrix} |\sigma| & 0 \\ 0 & 1 \end{bmatrix}$

and N by $N(e_0) = e_1$, $N(e_1) = 0$ so $N = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

Example: $V = \mathbb{Q}^n = \mathbb{Q}\langle e_0, \ldots, e_{n-1}\rangle$ with $\rho(\sigma) e_i = |\sigma|^i e_i$ and $N(e_i) = e_{i+1}$, $N(e_{n-1}) = 0$.

We say a Weil-Deligne representation (ρ_0, N) is F-semisimple if $\rho_0(\widetilde{Frob})$ is semisimple (i.e. diagonalizable over $\overline{E}$). This is independent of choice of $\widetilde{Frob}$.

One side of LLC (Local Langlands Conjecture) is

$$\left\{ \begin{array}{l} n\text{-dimensional F-semisimple} \\ \text{WD representations of } W_K \end{array} \right\} \Big/ \cong .$$

2017-07-26 (2)

Representations of $GL_n(K)$ ($K/\mathbb{Q}_p$ finite)

Let E be a field with discrete topology. Let V be an E-vector space, possibly infinite dimensional. We consider a group homomorphism $\pi : GL_n(K) \to \mathrm{Aut}_E(V)$. Say π is

- smooth if $\mathrm{Stab}_\pi(v)$ is open for all $v \in V$
- admissible if, for all open $U \subset GL_n(K)$, with U a subgroup, V^U is finite-dimensional.

Example: Consider $\pi(g) = 1$ for all $g \in GL_n(K)$. This is smooth. It is admissible in case $\dim V = 1$, but not when $\dim V = \infty$.

$\boxed{\text{Fact}}$: If π is irreducible and smooth, then π is admissible.

⌈Recall: A basis of open neighborhoods of 1 in $GL_n(K)$ is

$$\{ M \in GL_n(\mathcal{O}_K) : M \equiv I_n \bmod \mathfrak{p}_K^m \} \quad (m \in \mathbb{Z}_{\geq 1}).$$ ⌋

Local class field theory gave $K^\times \xrightarrow{\cong} W_K^{ab}$, and we really wanted to understand W_K. Langlands reinterpretation is $\left\{ \begin{array}{c} \text{irreducible 1-dimensional} \\ \text{representations of } K^\times \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \text{irreducible 1-dimensional} \\ \text{representation of } W_K \end{array} \right\}$.

Local Langlands for GL_n: There is a canonical bijection

$$\left(\begin{array}{c} \text{Irreducible admissible} \\ \text{representations of } GL_n(K) \end{array} \right) \longleftrightarrow \left(\begin{array}{c} \text{F-semisimple } n\text{-dimensional WD} \\ \text{representations of } W_K \end{array} \right)$$

Next time: $n=2$ (and $n=1$), and lots of examples on both sides.

Recall statement of LLC: For $K/\mathbb{Q}_p$ finite and $E = \mathbb{C}$ (for concreteness), there is a canonical bijection

$$\left(\begin{array}{c}\text{F-semisimple } n\text{-dimensional} \\ \text{WD representations } /\cong\end{array}\right) \leftrightarrow \left(\begin{array}{c}\text{Smooth irreducible admissible} \\ \text{representations } \pi \text{ of } GL_n(K)\end{array}\right)$$

"Canonical" here means "satisfies many nice properties", for example duality of both sides and their L-functions should match up.

↳ Big list of nice properties became sufficiently long that one can prove there is at most one such "canonical" bijection.

⌈ For function field case, it is a theorem of Laumon-Rapoport-Stuhler.
For p-adic field case, it is a theorem of Harris-Taylor. ⌋

Two obvious observations:

(1) Brilliant generalization of local class field theory, to be checked shortly.

(2) Completely pointless bijection unless we understand both sets in the bijection better.

Remark. If G is any connected reductive group over K, there is a local Langlands correspondence:

$$\left(\begin{array}{c}\text{Smooth irreducible admissible} \\ \text{representations of } G(K)\end{array}\right) \xrightarrow[\substack{\text{finite fibers} \\ \text{"L-packets"}}]{\text{surjection}} \left(\begin{array}{c}\text{certain WD representations} \\ (\rho, N): W_K \to {}^LG(\mathbb{C})\end{array}\right)$$

and the map satisfies a big list of properties (see Borel Corvallis). Characterization not well understood yet.

LLC for $n=1$: Left hand side is 1-dimensional WD representations $(\rho_0, N): W_K \to GL_1(\mathbb{C})$.

Certainly $N = 0$ and ρ_0 factors through W_K^{ab}.

$\boxed{\text{LHS}}$ 1-dimensional Continuous $\mathbb{C}$-representations of W_K^{ab}.

Right hand side is smooth irreducible admissible representations of $K^\times = GL_1(K)$. One can see that $\dim \pi$ will be finite, so $\dim \pi = 1$.

$\boxed{\text{RHS}}$ Continuous group homomorphisms $K^\times \to \mathbb{C}^\times = GL_1(\mathbb{C})$.

Done by local class field theory as $K^\times = W_K^{ab}$.

Source of WD representations

<u>ℓ-adic representations</u>. Let $K/\mathbb{Q}_p$ be finite, and say $\rho : Gal(\overline{K}/K) \to GL_n(\mathbb{Q}_\ell)$ is a continuous representation, with $\ell \neq p$.

↳ These show up in nature, for example: Tate module of elliptic curves; ℓ-adic étale cohomology of algebraic variety $H^i_{et}(X_{\overline{K}}, \mathbb{Q}_\ell)$, X/K an algebraic variety; ℓ-adic deformations of examples.

<u>Remark</u>. If E/K is an elliptic curve with split multiplicative reduction, then

$$E(\overline{K}) \cong \overline{K}^\times / q^{\mathbb{Z}} \text{ for some } q \in K \text{ with } |q| < 1.$$

If ρ is an ℓ-adic representation as above, then $\rho(I^\varepsilon_{\overline{K}/K})$ is finite if $\varepsilon > 0$. Recall that $Gal(K^t/K^{nr}) = \prod_{r \neq p} \mathbb{Z}_r$. We should worry about the $\mathbb{Z}_\ell$ part.

Fix $t : Gal(K^t/K^{nr}) \twoheadrightarrow \mathbb{Z}_\ell$ and fix a $\varphi \in Gal(\overline{K}/K)$ lifting $\mathrm{Frob} \in Gal(K^{nr}/K)$.

<u>Prop</u> (Grothendieck): If $\rho : Gal(\overline{K}/K) \xrightarrow{\text{cont.}} GL_n(E)$, $E = \mathbb{Q}_\ell$, then there is a unique (up to isomorphism) WD representation $(\rho_0, N) : W_K \to GL_n(E)$ ← discrete topology!!

such that $\rho(\varphi^m \sigma) = \rho_0(\varphi^m \sigma) \underbrace{\exp(N \cdot t(\sigma))}_{\text{matrix exponential}}$ for all $\sigma \in I_{\overline{K}/K}$ and $m \in \mathbb{Z}$

Example: For Tate curve, $N = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

Remarks. If (ρ_0, N) arises as in proposition, then eigenvalues of $\rho_0(\varphi)$ will be ℓ-adic units.

If (ρ_0, N) is given and eigenvalues of $\rho_0(\varphi)$ are ℓ-adic units, then it comes from a ρ.

The isomorphism class of (ρ_0, N) is independent of t and φ.

Smooth admissible representations of $GL_n(K)$

Last thing about $n=1$. If $\pi : K^\times \to \mathbb{C}^\times$ is smooth irreducible admissible, define its conductor

$$f(\pi) = \begin{cases} 0 & \text{if } \quad \pi|_{\mathcal{O}_K^\times} = 1 \\ r & \text{if } \quad r \text{ is the smallest positive integer with } \pi|_{1+\mathfrak{p}_K^r \mathcal{O}_K} = 1 \end{cases}$$

If $\rho_0 = (\rho_0, N=0)$ corresponds to π under LLC for $n=1$, then $f(\rho_0) = f(\pi)$ [not easy].

$n=2$. Here is a cool construction of π. Say $\chi_1, \chi_2 : K^\times \to \mathbb{C}^\times$ are continuous characters.

Define $$I(\chi_1, \chi_2) := \left\{ \varphi : GL_2(K) \to \mathbb{C} \;:\; \begin{array}{l} \varphi \text{ is locally constant, and} \\ \varphi\left(\begin{bmatrix} a & b \\ 0 & d \end{bmatrix} g\right) = \chi_1(a)\chi_2(d) \left|\frac{a}{d}\right|^{\frac{1}{2}} \varphi(g) \end{array} \right\}$$

and let $\pi : GL_2(K) \to \mathrm{Aut}_{\mathbb{C}}(I(\chi_1, \chi_2))$ by translation:

$$(\pi(g)\varphi)(h) = \varphi(hg).$$

Lemma. Let $B(K) = \left\{ \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \in GL_2(K) \right\}$. Then $GL_2(K) = B(K) \cdot GL_2(\mathcal{O}_K)$.

Remark. If $\varphi : GL_2(K) \to \mathbb{C}$ is locally constant (and continuous), then it is continuous.

Hen $\varphi(GL_2(\mathcal{O}_K))$ is finite, and $\varphi(B(K))$ is controlled by definition of $I(\chi_1, \chi_2)$.

Proof of Lemma. Say $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL_2(K)$. By left multiplying γ by $\begin{bmatrix} (\det\gamma)^{-1} & 0 \\ 0 & 1 \end{bmatrix}$, we can assume $\gamma \in SL_2(K)$. Choose $\alpha \in K^\times$ so that $\alpha c, \alpha d \in \mathcal{O}_K$ and at least one of them is a unit, and by left multiplying with $\begin{bmatrix} \alpha^{-1} & 0 \\ 0 & \alpha \end{bmatrix}$ we can assume $c, d \in \mathcal{O}_K$ with at least one of c, d a unit. If $d =$ unit, $c \neq$ unit, then right multiply by $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \in GL_2(\mathcal{O}_K)$ to assume c is a unit. Finally notice that

$$\begin{bmatrix} 1 & -\frac{a}{c} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{c} \\ c & d \end{bmatrix} \in GL_2(\mathcal{O}_K). \quad \square$$

Exercise. $I(\chi_1, \chi_2)$ is smooth and admissible.

Remark. Deleting the fudge factor $\left|\frac{a}{d}\right|^{\frac{1}{2}}$ to define $I^{\text{naive}}(\chi_1, \chi_2)$. If $\chi_1 = \chi_2$ then $I^{\text{naive}}(\chi_1, \chi_2) \supset \{\text{invariant 1-dimensional subspace}\}$.

There is a duality, i.e. a natural pairing $I(\chi_1, \chi_2) \times I(\chi_1^{-1}, \chi_2^{-1}) \to \mathbb{C}$ involving an integral on G and B. At some point, we need to change left Haar measure on B to a right Haar measure, with a fudge factor $\left|\frac{a}{d}\right|$. This is where the "$\left|\frac{a}{d}\right|^{\frac{1}{2}}$" in $I(\chi_1, \chi_2)$ comes from, to make $I(\chi_1^{-1}, \chi_2^{-1})$ the dual of $I(\chi_1, \chi_2)$.

↳ [Godement's IAS notes, chapter I-II].

$I(\chi_1,\chi_2)$ is irreducible if $\chi_1\chi_2^{-1} \neq |\,|^{\pm 1}$.

↳ If $\chi_1\chi_2^{-1} = |\,|^{-1}$ then $g \mapsto (\chi_1 \times |\,|_k^{\frac{1}{2}})(\det g)$ is a 1-dimensional subrepresentation of $Gl_2(K)$ with quotient $S(\chi_1,\chi_2)$, say. $S(\chi_1,\chi_2)$ is irreducible.

↳ If $\chi_1\chi_2^{-1} = |\,|$ then $0 \to S(\chi_2,\chi_1) \to I(\chi_1,\chi_2) \to (\chi_2 \times |\,|^{\frac{1}{2}})\circ \det \to 0$.

[A reference: Bernstein - Zelevinsky, Theorem 1.21]

A little bit about another construction (§1 of Jacquet - Langlands).

If K is any field, then Weil constructed a presentation of $Sl_2(K)$ using generators $\begin{bmatrix} t & 0 \\ 0 & t^{-1} \end{bmatrix}$, $\begin{bmatrix} 1 & u \\ 0 & 1 \end{bmatrix}$, $\underbrace{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}_{w}$ and explicit obvious relations.

Upshot. We can construct representations of $Sl_2(k)$ on generators and checking relations.

Weil observed that we can have $Sl_2(K)$ act on the L^2-functions on K with

$\begin{bmatrix} t & 0 \\ 0 & t^{-1} \end{bmatrix}$: $f(x) \mapsto f(tx)$, $\begin{bmatrix} 1 & u \\ 0 & 1 \end{bmatrix}$: $f(x) \mapsto f(u+x)$, w: Fourier transform.

This gives another source of representations of $Sl_2(k)$, and hence $Gl_2(K)$.

Fact. If L/K is a quadratic extension and $\chi : L^\times \to \mathbb{C}^\times$ is admissible, and if $\chi \neq \chi\circ\sigma$ $(\sigma \in Gal(L/K)\setminus\{1\})$, then Jacquet-Langlands constructed an irreducible infinite dimensional representation $BC_L^K(\chi)$ of $Gl_2(K)$. [Thm 4.6 of Jacquet - Langlands]

Fact. $I(\chi_1,\chi_2)$, $S(\chi_1,\chi_2)$, $BC_L^k(\chi)$ are all infinite dimensional smooth and admissible, and $S(\chi_1,\chi_2)$, $BC_L^k(\chi)$ are always irreducible.

The only isomorphisms between these are $I(\chi_1,\chi_2) \cong I(\chi_2,\chi_1)$ $\cancel{(\chi_1 \neq \chi_2)}$ $(\chi_1\chi_2^{-1} \neq |\,|^{\pm 1})$.

If char(K) > 2 then these are all such representations of $Gl_2(K)$.

Let $K/\mathbb{Q}_p$ be finite. Recalled we have the following smooth irreducible admissible representations of $GL_2(K)$: $I(\chi_1,\chi_2)$, $S(\chi_1,\chi_2)$, $\chi\circ\det$, $BC_L^K(\Upsilon)$.

Fact: If $\mathrm{char}(k_K) > 2$, then these are all such representations of $GL_2(K)$.

Conductors

Let us stick to admissible irreducible representations π of $GL_2(K)$ with $\dim(\pi) = \infty$.

⌈Remark: If G is any connected reductive group and π a smooth irreducible admissible representation of $G(K)$, there is a notion of a "generic" π. In case $G = GL_2$, this is equivalent to $\dim\pi = \infty$.⌋

Theorem of Casselman (Antwerp proceedings)

For $n \geq 0$, define $U_1(\mathfrak{p}_K^n) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL_2(\mathcal{O}_K) : \begin{bmatrix} a & b \\ c & d \end{bmatrix} \equiv \begin{bmatrix} * & * \\ 0 & 1 \end{bmatrix} \bmod \mathfrak{p}_K^n \right\}$.

These are all compact and open, so $d(\pi,n) := \dim\left(\pi^{U_1(\mathfrak{p}_K^n)}\right) < \infty$.

Casselman showed there exists $f(\pi) \in \mathbb{Z}_{\geq 0}$ such that $d(\pi,n) = \max(0, 1+n-f(\pi))$. □

Exercise:
- Check the theorem directly for $I(\chi_1,\chi_2)$ [maybe hard?], $\chi_1\chi_2^{-1} \neq |\cdot|^{\pm 1}$
- Assume Casselman's theorem. Then $f(I(\chi_1,\chi_2)) = f(\chi_1) + f(\chi_2)$ and

$$f(S(\chi_1,\chi_2)) = \begin{cases} 1 & f(\chi) = 0 \\ 2f(\chi) & f(\chi) > 0 \end{cases}.$$

- (Schur's Lemma) If π is an irreducible admissible representation of $GL_n(K)$, then there is a character $\chi_\pi : K^\times \to \mathbb{C}^\times$, called central character, with $K^\times = Z(GL_n(K))$ acting as χ_π.

<u>Example</u>. The central characters $\chi_{I(\chi_1,\chi_2)} = \chi_1\chi_2$, $\chi_{S(\chi_1,\chi_1)} = \chi_1\chi_2$, $\chi_{\varphi\circ\det} = \varphi^2$.

<u>Local Langlands Correspondence for $GL_2(K)$</u>:

For characters $\chi_i : K^\times \to \mathbb{C}^\times$, associate it to $\rho_i : W_K \to \mathbb{C}^\times$ via the $GL_1(K)$ case.

The correspondence for $GL_2(K)$ is as follow.

π's		ρ's
$I(\chi_1, \chi_2)$	$\longleftrightarrow$	$\rho_0 = \rho_1 \oplus \rho_2$, $N=0$
$S(\chi_1, \chi_1 \lvert\,\rvert)$	$\longleftrightarrow$	$\rho_0 = \begin{bmatrix} \lvert\,\rvert \cdot \rho_1 & 0 \\ 0 & \rho_1 \end{bmatrix}$, $N = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.
$\chi_1 \circ \det$	$\longleftrightarrow$	$\rho_0 = \begin{bmatrix} \rho_1 \lvert\,\rvert^{\frac12} & 0 \\ 0 & \rho_1 \lvert\,\rvert^{-\frac12} \end{bmatrix}$, $N=0$
$BC_L^K(\psi)$	$\longleftrightarrow$	$\rho_0 = \mathrm{Ind}_{W_L}^{W_K}(\sigma)$, $N=0$
		($\psi : L^\times \to \mathbb{C}^\times \xleftrightarrow{CFT} \sigma : W_L \to \mathbb{C}^\times$)

(+ extra stuff if $\overset{\mathrm{char}(k_K)}{\overset{\shortparallel}{p}} = 2$)

For $GL_2(K)$ this is how LLC was proved (match things up explicitly).

For $GL_n(K)$, use representation theory techniques of Bernstein-Zelevinsky to reduce the problem to matching irreducible (ρ_0, N)'s to supercuspidal π's (for example $BC_L^K(\psi)$)

The matching is done via a global argument.

If (ρ_0, N) is a WD representation, define its <u>conductor</u>

$$f(\rho_0, N) = f(\rho_0) + \dim\left(V^{I_{K/K}} / (\ker N)^{I_{K/K}} \right)$$

Exercise. • Check for some examples that, under LLC for $GL_2(k)$,

$$f(\pi) = f(\rho_0, N) \quad \text{and} \quad \chi_\pi = \det(\rho_0).$$

• Check that if $p>2$ then the list in the previous page contains all the F-semisimple 2-dimensional WD representations. [2.2.5.2 in Tate's article in Corvallis will help.]

Let us talk about the $f(\pi)=0$ "unramified" case. Say $\pi \overset{LLC}{\leftrightarrow} (\rho_0, N)$. In this case necessarily $\pi = I(\chi_1, \chi_2)$ with χ_i unramified, and $(\chi =) \chi_1 \chi_2^{-1} \neq |\,|^{\pm 1}$, $\dim \pi^{GL_2(\mathcal{O}_k)} = 1$.

or $\pi = \chi \circ \det$ with χ unramified.

On the other side, $\rho_0 = \rho_1 \oplus \rho_2$ with $\rho_i|_{I_{\bar{k}/k}} \equiv 1$ and $N = 0$.

Say π is infinite-dimensional (so we are in the $I(\chi_1, \chi_2)$ case).

[General case: Let G/k be connected reductive, and assume G is unramified. We say an irreducible smooth admissible representation π of $G(k)$ is unramified if there is a hyperspecial maximal compact subgroup $H \subset G(k)$ such that $\pi^H \neq 0$. For example: $G = GL_n$, $H = GL_n(\mathcal{O}_k)$.]

We want to do calculations with π. We can start with $\pi^{GL_2(\mathcal{O}_k)}$; not $GL_2(k)$-invariant ... The trick is to use Hecke operators. Let $G = GL_2(k)$ (or actually any locally compact totally disconnected topological group), and let π be an admissible representation of G.

Let U, V be compact open subgroups (for example $U_1(\mathfrak{p}_k^n)$ or $GL_2(\mathcal{O}_k)$) and $g \in G$, then there is an Hecke operator $[UgV]: \pi^V \longrightarrow \pi^U$. This is a $\mathbb{C}$-linear map, defined thus: Write $UgV = \coprod_{i=1}^{r} g_i V$ (finite as UgV is compact and V is open).

Then, for $x \in \pi^V$, $[UgV]x := \sum_{i=1}^{r} g_i x$. (Clearly this is independent of choice.)

Let us concentrate on $GL_2(K)$ with $U = V = GL_2(\mathcal{O}_K)$ and $f(\pi) = 0$.

Def. $T := \left[U \begin{bmatrix} \pi_K & 0 \\ 0 & 1 \end{bmatrix} V \right]$ and $S := \left[U \begin{bmatrix} \pi_K & 0 \\ 0 & \pi_K \end{bmatrix} V \right]$.

Exercise. If $\pi = I(\chi_1, \chi_2)$ with $\chi_1 \chi_2^{-1} \neq |\,|^{\pm 1}$ (and $f(\pi) = 0$), then

$$T = \sqrt{q_K}\,(\chi_1(\pi_K) + \chi_2(\pi_K)) \quad \text{and} \quad S = \chi_1(\pi_K) \cdot \chi_2(\pi_K).$$

As a consequence, show the following. If π is an admissible irreducible representation of $GL_2(K)$ and $f(\pi) = 0$, then $\pi \xleftrightarrow{LLC} (\rho_0, N)$ where $N = 0$ and

$$\rho_0 : W_K \to W_K / I_{\bar{K}/K} \xrightarrow{\cong} \mathbb{Z} \longrightarrow GL_2(\mathbb{C}) \quad \text{with } \rho_0(\mathrm{Frob}) \text{ having characteristic}$$

polynomial $x^2 - \frac{t}{\sqrt{q_K}} x + S$.

[More ambitious: do for GL_n.]

If $G = G(K)$ as in general case, with G/K unramified, and if π is an unramified representation of G, then Langland's reinterpretation of the Satake isomorphism associates to π a semisimple conjugacy class in ${}^L G(\mathbb{C})$:

$$\pi \longrightarrow \rho_0 \text{ with } \rho_0(\mathrm{Frob}) \text{ this semisimple class}.$$

2017-07-28 (2)

Part 2: The global Langlands correspondence

In this part K will be a number field.

We start by talking about the structure of $\mathrm{Gal}(L/K)$, for L/K finite Galois, and in particular its relationship to local Galois groups. Then we take limits to understand $\mathrm{Gal}(\overline{K}/K)$.

Global analog of WD representations may be "representations of global Langlands group" (we don't know if this exist). However we still have ℓ-adic representations of $\mathrm{Gal}(\overline{K}/K)$, which is the working definition of the "ρ-side". The "π-side" is automorphic representations

Uncheckable conjecture: All automorphic representations of GL_n/K $\overset{??}{\longleftrightarrow}$ n-dimensional representations of "global Langlands group".

Checkable conjecture: Algebraic automorphic representations π of GL_n/K $\longleftrightarrow$ "nice" (unramified outside finite set, de Rham) $\rho: \mathrm{Gal}(\overline{K}/K) \to GL_n(\overline{\mathbb{Q}}_\ell)$

dream, does not exist

"motives"

p-adic automorphic representations of GL_n/K $\longleftrightarrow$ ρ's to $GL_n(\overline{\mathbb{Q}}_p)$, dropping de Rham?

For $n=1$, the uncheckable conjecture will be global class field theory.

Galois groups

Choose a nonzero prime (hence maximal) of $\mathcal{O}_K$, say $\mathfrak{p} \neq 0$, and write the residue field as $k_\mathfrak{p} = \mathcal{O}_K/\mathfrak{p}$. We can complete K at $\mathfrak{p}$, with $\mathcal{O}_{K_\mathfrak{p}} := \varprojlim_n \mathcal{O}_K/\mathfrak{p}^n$, $K = \mathrm{Frac}(\mathcal{O}_{K_\mathfrak{p}})$.

For $\lambda \in K^\times$ we can factorize $\lambda \mathcal{O}_K = \mathfrak{p}^{v_\mathfrak{p}(\lambda)} \cdot \left(\begin{array}{c}\text{other coprime}\\ \text{prime ideals}\end{array}\right)$, with $v_\mathfrak{p}: K^\times \to \mathbb{Z}$.

The $\mathfrak{p}$-norm on K is defined as $|0|_{\mathfrak{p}}=0$ and $|\lambda|_{\mathfrak{p}} = (q_{\mathfrak{p}})^{-v_{\mathfrak{p}}(\lambda)}$. This induces the obvious norm on K, and completing gives a local field $K_{\mathfrak{p}}$; a finite extension of $\mathbb{Q}_p$ with $\mathfrak{p} \cap \mathbb{Z} = (p)$.

Now let L/K be a finite extension of number fields. For $\mathfrak{p} \subset \mathcal{O}_K$ as above, we can factorize $\mathfrak{p}\mathcal{O}_L = P_1^{e_1} \cdots P_g^{e_g}$ into primes of $\mathcal{O}_L$. Since $Gal(L/K)$ fixes $\mathfrak{p}$ and acts on $\mathcal{O}_L$, it fixes the ideal $\mathfrak{p}\mathcal{O}_L \subset \mathcal{O}_L$ (but not pointwise). Since $\sigma(P_i)$ is still a prime ideal dividing $\sigma(\mathfrak{p}) = \mathfrak{p}$, so it permutes the P_i "transport de Structure". In particular $Gal(L/K)$ acts transitively on $\{P_1, \dots, P_g\}$, so all the e_i's are the same.

<u>Corollary</u>: $L_{P_1} \cong L_{P_2} \cong \dots \cong L_{P_g}$.

Define the <u>decomposition group</u> $D_P = D_{P_1} := \{\sigma \in Gal(L/K) : \sigma(P) = P\}$, so $Gal(L/K)/D_P \cong \{P_1, \dots, P_g\}$.
If $\sigma \in D_P$, by "transport de structure" σ extends to $L_P \xrightarrow{\sigma} L_P$ fixing $K_{\mathfrak{p}}$. Since $L_P/K_{\mathfrak{p}}$ is Galois, we have that $Gal(L_P/K_{\mathfrak{p}}) \cong D_P \hookrightarrow Gal(L/K)$.

<u>Logical flow</u>: For L/K number fields,

- choose $\mathfrak{p}$ in $\mathcal{O}_K$
- choose P dividing $\mathfrak{p}\mathcal{O}_L$
- then $Gal(L_P/K_{\mathfrak{p}}) = D_P \hookrightarrow Gal(L/K)$.

<u>Fact</u>: If $\mathfrak{p}$ does not divide $disc(L/K)$, then I_P is trivial, so $L_P/K_{\mathfrak{p}}$ is unramified and there exists $Frob_P \in D_P \hookrightarrow Gal(L/K)$. ($Frob_P$ depends on $\mathfrak{p}$ and $P \mid \mathfrak{p}\mathcal{O}_L$).

By transport de structure $\mathrm{Frob}_{\mathfrak{P}}$ is defined up to conjugation. Let us define (3)

$\mathrm{Frob}_{\mathfrak{p}}$ to be the conjugacy class of $\mathrm{Frob}_{\mathfrak{P}}$, which works for all $\mathfrak{p} \nmid \mathrm{disc}(L/K)$.

Will work more on this next week.

Let L/K be a finite Galois extension of number fields. Recall that if $\mathfrak{p}$ is a nonzero prime ideal of K, and if $P \mid \mathfrak{p}\mathcal{O}_L$, then we had the decomposition group

$$D_{P/\mathfrak{p}} := \{ \sigma \in \mathrm{Gal}(L/K) : \sigma(P) = P \}.$$

We also showed that $D_{P/\mathfrak{p}} = \mathrm{Gal}(L_P/K_\mathfrak{p})$, and mentioned that if $\mathfrak{p} \nmid \mathrm{disc}(L/K)$ then the inertia group $I_{P/\mathfrak{p}} = \{1\}$ for all $P/\mathfrak{p}$. Thus, for all primes $\mathfrak{p}$ not dividing $\mathrm{disc}(L/K)$, we have (where $P/\mathfrak{p}$) $D_{P/\mathfrak{p}} = \mathrm{Gal}(L_P/K_\mathfrak{p}) = \mathrm{Gal}(k_P/k_\mathfrak{p}) = \langle \mathrm{Frob}_P \rangle$.

↳ Also: For $\mathfrak{p} \in \mathrm{Spec}\, \mathcal{O}_L$, we get a conjugacy class $\{ \mathrm{Frob}_P : P \mid \mathfrak{p} \} =: \mathrm{Frob}_\mathfrak{p}$.

<u>Fact</u>: Given L/K as above, every conjugacy class in $\mathrm{Gal}(L/K)$ equals $\mathrm{Frob}_\mathfrak{p}$ for infinitely many $\mathfrak{p}$. In fact the density of such $\mathfrak{p}$ equals $\frac{\# C}{\# G}$, where C is a conjugacy class that equals $\mathrm{Frob}_\mathfrak{p}$ and $G = \mathrm{Gal}(L/K)$. [Čebotarev density theorem]

↳ There is a variant for infinite extensions. Let K be a number field and S be a finite set of maximal ideals of $\mathcal{O}_K$. Recall if L_1, L_2 with $K \subset L_i \subset \overline{K}$ is unramified outside S, then so is $L_1 L_2$. Define $K^S := \bigcup_{\substack{L/K \text{ finite Galois} \\ \text{unramified outside } S}} L$.

(may not be infinite extension: every number field other than $\mathbb{Q}$ is unramified at some prime.)

⌈Example. Let $K = \mathbb{Q}$ and $S = \{p\}$. Then $K^S \supset \mathbb{Q}(\zeta_{p^n})$ for all $n \geq 1$. In fact, if we let $S = \{\text{primes } p \mid N\}$ then $\mathbb{Q}(\zeta_N)$ is unramified outside S with Galois group $(\mathbb{Z}/N\mathbb{Z})^\times$. If $p \notin S$ then Frob_p is the conjugacy class p in $(\mathbb{Z}/N\mathbb{Z})^\times$. ⌋

If $K = \mathbb{Q}$ and $S = \{p\}$, then $K^S \supset \bigcup_{n \geq 1} \mathbb{Q}(\zeta_{p^n}) =: \mathbb{Q}(\zeta_{p^\infty})$ so $\mathrm{Gal}(K^S/K) \twoheadrightarrow \mathbb{Z}_p^\times$. In this case, if $r \neq p$ is a prime number, then $r = \mathrm{Frob}_r \in (\mathbb{Z}/p^n\mathbb{Z})^\times = \mathrm{Gal}(\mathbb{Q}(\zeta_{p^n})/\mathbb{Q})$. Take projective limit to get $\mathrm{Frob}_r \in \mathbb{Z}_p^\times = \mathrm{Gal}(\mathbb{Q}(\zeta_{p^\infty})/\mathbb{Q})$.

General story. If S is a finite set of places of a number field K, then

$$\mathrm{Gal}(K^S/K) = \varprojlim_{\substack{L/K \text{ finite Galois,} \\ \text{unramified outside } S}} \mathrm{Gal}(L/K).$$

For $\mathfrak{p} \notin S$ we get $\mathrm{Frob}_{\mathfrak{p}, L/K}$ and these glue together to get a conjugacy class $\mathrm{Frob}_{\mathfrak{p}} = \mathrm{Frob}_{\mathfrak{p}, K^S/K} \subseteq \mathrm{Gal}(K^S/K)$. The Cebotarev density theorem for finite extensions then implies that $\{\text{primes } \mathfrak{p} \notin S\} \longrightarrow \{\text{conjugacy class of } \mathrm{Gal}(L/K)\}$ by $\mathfrak{p} \mapsto \mathrm{Frob}_{\mathfrak{p}}$ is surjective. Thus we have the following.

Corollary. If L/K is infinite Galois and unramified outside S, then the union of the conjugacy classes $\{\mathrm{Frob}_{\mathfrak{p}} : \mathfrak{p} \notin S\}$ is dense in $\mathrm{Gal}(L/K)$. (Thus, if $F : \mathrm{Gal}(L/K) \to X$ is continuous and constant on conjugacy classes, then we get the data of F from $F(\mathrm{Frob}_{\mathfrak{p}})$ with $\mathfrak{p} \notin S$.)

Brauer-Nesbitt Theorem

Recall $\rho : G \to GL_n(E)$ is semisimple if it is a direct sum of irreducible representations. (Here G is a group and E is a field.) If $\rho_1, \rho_2 : G \to GL_n(E)$ are two semisimple representations with the characteristic polynomial agreeing with each other for all $\rho_i(g)$, then in fact $\rho_1 \cong \rho_2$. (Proof of this is algebra.)

Remark. If $\mathrm{char}(E) = 0$ then $\mathrm{tr}\,\rho_1 = \mathrm{tr}\,\rho_2$ implies $\rho_1 \cong \rho_2$. Not true if $\mathrm{char}(E) \neq 0$: there exists an example with $G = \mathbb{Z}/2\mathbb{Z}$ and $E = \mathbb{F}_2$.

<u>Upshot</u>. Let $E/\mathbb{Q}_\ell$ be a finite extension. If $\rho: \mathrm{Gal}(K^S/K) \to GL_n(E)$ is a continuous semisimple representation and if $\rho(\mathrm{Frob}_\mathfrak{p})$ is given for all $\mathfrak{p} \notin S$, then ρ is <u>uniquely</u> determined.

<u>Example</u>. Let $K = \mathbb{Q}$ and $S = \{p\}$ and $L = \mathbb{Q}(\zeta_{p^\infty})$. Then $\mathrm{Gal}(L/K) = \mathbb{Z}_p^\times = GL_1(\mathbb{Z}_p)$.

and $\rho: \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \twoheadrightarrow \mathrm{Gal}(\mathbb{Q}(\zeta_{p^\infty})/\mathbb{Q}) = GL_1(\mathbb{Z}_p) \hookrightarrow GL_1(\mathbb{Q}_p)$.

$$\begin{tikzcd}
\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \arrow[d] & \mathrm{Gal}(\mathbb{Q}(\zeta_{p^\infty})/\mathbb{Q}) & \\
\rho: \mathrm{Gal}(\mathbb{Q}^S/\mathbb{Q}) \arrow[ur] & & r \\
\rotatebox{90}{$\in$} & & \\
\mathrm{Frob}_r,\ r \neq p \arrow[uurr, mapsto] & &
\end{tikzcd}$$

ρ is the <u>cyclotomic character</u> and is determined by the data $\rho(\mathrm{Frob}_r) = r$ for $r \neq p$.

Call this p-adic cyclotomic character ω_p.

Let p and ℓ be two different primes, and let $S = \{p, \ell\}$. Then we have two representations ω_p and ω_ℓ of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Note that Frob_r, $r \notin S$ are dense subsets of $\mathrm{Gal}(\mathbb{Q}^S/\mathbb{Q})$ with $\omega_p(\mathrm{Frob}_r) = \omega_\ell(\mathrm{Frob}_r) = r$. <u>But</u>: Brauer-Nesbitt Theorem does not apply as $\mathbb{Z}_p^\times \neq \mathbb{Z}_\ell^\times$, so $\omega_p \neq \omega_\ell$. In fact these two representations are very different; $\mathbb{Q}(\zeta_{p^\infty})$ and $\mathbb{Q}(\zeta_{\ell^\infty})$ are totally disjoint.

2017-07-31 (2)

⌈Let K be a number field ($K=\mathbb{Q}$ is fine). Let E/K be an elliptic curve and S_0 a finite set of places of K where E has bad reduction. For ℓ a prime number, $\mathrm{Gal}(\overline{K}/K)$ acts on $E[\ell^n](\overline{K})$. This gives a Galois representation $\rho_{E,\ell}: \mathrm{Gal}(\overline{K}/K) \to GL_2(\mathbb{Z}_\ell)$ which factors through $\mathrm{Gal}(K^{S_0 \cup \{p \mid \ell\}}/K)$ and the characteristic polynomial of

$\rho_{E,\ell}(\mathrm{Frob}_p) = X^2 - a_p X + \overset{\text{abs. norm}}{N(p)}$, $a_p = 1 + N(p) - \# \overline{E}(k_p)$. (Diamond-Shurman has details.) ⌋

ℓ-adic representations

Now let K be a number field and E a finite extension of $\mathbb{Q}_\ell$. Also let S be a finite set of maximal ideals of $\mathcal{O}_K$. If $\rho: \mathrm{Gal}(K^S/K) \to GL_n(E)$ is continuous (with respect to the ℓ-adic topology), we call ρ an <u>ℓ-adic representation</u> of $\mathrm{Gal}(\overline{K}/K)$.

Say ρ is <u>rational over E_0</u>, where E_0 is a subfield of E, if for all $\mathfrak{p} \notin S$ the characteristic polynomial of $\rho(\mathrm{Frob}_\mathfrak{p})$ lies in $E_0[x]$.

↳ <u>Example</u>: cyclotomic character, $\rho = H^i_{et}(X_{\overline{K}}, \mathbb{Q}_\ell)$ ℓ-adic étale cohomology of smooth proper algebraic variety rational over $\mathbb{Q}$

Say ρ is <u>pure of weight w</u> if ρ is rational over some number field E_0 and, for all $i: E_0 \hookrightarrow \mathbb{C}$ and all eigenvalues α of $\rho(\mathrm{Frob}_\mathfrak{p})$ we have $|i(\alpha)| = q_\mathfrak{p}^{-\frac{w}{2}}$. (Some people may use different sign conventions.)

↳ Deligne proved that $H^i_{et}(X_{\overline{K}}, \mathbb{Q}_\ell)$ is pure of weight i, X as above.

Example: • cyclotomic character is pure of weight -2. ($H^2(\mathbb{P}^1_{\bar{K}}, \mathbb{Q}_\ell) = \omega_\ell^{-1}$).

• T_ℓ(elliptic curve) is pure of weight -1. The roots of $X^2 - a_p X + N(p)$ are complex conjugates with $|a_p| \le 2\sqrt{N(p)}$ [Hasse-Weil bound.]

Now ℓ will vary! Let S_0 be a finite set of finite places, E_0 a number field. Say we are given the following data: for all $p \notin S_0$, a polynomial $F_p(x) \in E_0[x]$. Say also that for all maximal ideals $\lambda \in \mathrm{Spec}(\mathcal{O}_{E_0})$ we have an ℓ-adic representation $\rho_\lambda : \mathrm{Gal}(K^{S_0 \cup \{p|\ell\}}/K) \to GL_n(\overline{E_{0,\lambda}})$. We say ρ_λ is a compatible system of λ-adic representations if, for all λ and for all $p \notin S_0$ with $p \nmid \ell$, the $\rho_\lambda(\mathrm{Frob}_p)$ all have the same characteristic polynomial independent of λ, say $F_p(x)$.

↳ Examples: • cyclotomic characters have $F_p(x) = x - p$.

• T_ℓ(elliptic curve) has $F_p(x) = x^2 - a_p x + N(p)$.

• $H^i_{et}(X_{\bar{K}}, \mathbb{Q}_\ell)$ is known to be a compatible system..

Cool generalization: use local Langlands. Say ρ_λ as above are strongly compatible if, for all $p \in S_0$, $\mathfrak{p} | p$, all λ of E_0 with $\lambda \nmid \mathfrak{p}$, the

$$\rho_\lambda|_{\mathrm{Gal}(\bar{K}_p/K_p)} \underset{\substack{\text{Langlands} \\ \text{(via WD representation)}}}{\overset{\text{Local}}{\rightsquigarrow}} \pi : GL_n(K_p) \to (\text{something})$$

and this is independent of λ.

[Related to weight monodromy conjecture.]

↳ This is unknown for étale cohomology of smooth projective varieties.

Global class field theory

The goal is to understand $\mathrm{Gal}(\overline{K}/K)^{ab}$ for K a number field.

Say $K/\mathbb{Q}$ is of degree d. Then there are d embeddings into $\mathbb{C}$. Let r_1 (respectively $2r_2$) be the number of real (respectively, complex) embeddings, so $r_1 + 2r_2 = d$. Write $K_\infty := \prod_{\substack{v \\ \text{infinite}}} K_v$, so $K_v = \mathbb{R}$ if v is real and $K_v \cong \mathbb{C}$ if v is complex. Thus $K_\infty \cong \mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$. Notice $K_\infty^\times \cong (\mathbb{R}^\times)^{r_1} \times (\mathbb{C}^\times)^{r_2}$ is not connected in general, and $(K_\infty^\times)^\circ = (\mathbb{R}_{>0})^{r_1} \times (\mathbb{C}^\times)^{r_2}$.

Next time we will continue by using the language of adeles ...

$$\mathbb{A}_K = \prod_{\mathfrak{p}}{}' K_{\mathfrak{p}} \times K_\infty .$$

Again K is a number field. Recall we defined $K_\infty = \prod_{v|\infty} K_v \overset{!!}{=} K \otimes_{\mathbb{Q}} \mathbb{R}$.

Example: For $K = \mathbb{Q}(\sqrt[3]{2})$, $K \otimes_{\mathbb{Q}} \mathbb{R} = \mathbb{R}[x]/(x^3-2) = \mathbb{R}[x]/(x-\sqrt[3]{2})(x+\sqrt[3]{2}x+\sqrt[3]{4}) \cong \mathbb{R} \times \mathbb{C}$.

The adeles is defined to be $\mathbb{A}_K := \mathbb{A}_{K,f} \times K_\infty$, where $\mathbb{A}_{K,f}$ are the finite adeles:

$\mathbb{A}_{K,f} := \prod'_{v \nmid \infty} K_v$, where the restricted product is with respect to $\mathcal{O}_v$. This is a topological ring with the obvious topology. Notice $\mathbb{A}_{K,f} = \mathbb{A}_{\mathbb{Q},f} \otimes_{\mathbb{Q}} K$ and $\mathbb{A}_K = K \otimes_{\mathbb{Q}} \mathbb{A}_{\mathbb{Q}}$.

Lemma: $\mathbb{A}_{\mathbb{Q},f} = \mathbb{Q} + \prod_p \mathbb{Z}_p$.

↳ clearly RHS ⊂ LHS. Now let $x = (x_p) \in \mathbb{A}_{\mathbb{Q},f}$. Then there is a finite S such that $x_p \in \mathbb{Z}_p$ for $x_p \notin S$. Induct on S. Clear if $\#S = 0$. Inductively if $x_p \in S$ then $x_p = a_{-n}p^{-n} + a_{-n+1}p^{-n+1} + \dots + a_{-1}p^{-1} + a_0 + a_1 p + \dots$, and then add $a_{-n}p^{-n} + \dots + a_{-1}p^{-1}$ to λ.

Exercise. $\mathbb{A}_{K,f} = K + \prod_{\mathfrak{p}} \mathcal{O}_{K_\mathfrak{p}}$.

We are actually interested in the ideles $\mathbb{A}_K^\times = \prod'_v K_v$ where the restricted product is with respect to $\mathcal{O}_v^\times$ (this is not the subspace topology from $\mathbb{A}_K$ as multiplication will not be continuous!).

Global class field theory

let $\mathrm{Gal}(\bar{K}/K)^c$ be the closure of the commutator subgroup, and let $\mathrm{Gal}(\bar{K}/K)^{ab}$ be the quotient $\mathrm{Gal}(\bar{K}/K)/\mathrm{Gal}(\bar{K}/K)^c$. By Galois theory

$$\begin{tikzcd}[column sep=small] \bar{k} \arrow[dash, dd] \arrow[dash, dr] & \\ & k^{ab} \arrow[dash, dl, "\text{Galois group } \mathrm{Gal}(\bar{k}/k)^{ab}"] \\ k & \end{tikzcd}$$

where k^{ab} is the maximal abelian Galois extension of k. Clearly $k^{ab} \supset \bigcup_{N \geq 1} \mathbb{Q}(\zeta_N)$, and in fact $k^{ab} = \bigcup_{N \geq 1} \mathbb{Q}(\zeta_N)$ by the Kronecker-Weber theorem if $k = \mathbb{Q}$.

Theorem. There is a continuous group homomorphism $k^\times \backslash \mathbb{A}_k^\times \xrightarrow{r_k} \mathrm{Gal}(k^{ab}/k)$, called the global Artin map. Its kernel $\ker r_k$ is the topological closure of the image of $(k_\infty^\times)^\circ$ in $k^\times \backslash \mathbb{A}_k^\times$. [More properties to come ...]

Remark: If k is $\mathbb{Q}$ or an imaginary quadratic field, then the image of $(k_\infty^\times)^\circ$ in $k^\times \backslash \mathbb{A}_k^\times$ is already closed. This is not true in general.

- For a finite place $\mathfrak{p}$, we have

$$\begin{tikzcd} k^\times \backslash \mathbb{A}_k^\times \arrow[r, "r_k"] & \mathrm{Gal}(k^{ab}/k) \\ k_\mathfrak{p}^\times \arrow[u] \arrow[r, "r_{k_\mathfrak{p}}"] & \mathrm{Gal}(\overline{k_\mathfrak{p}}/k_\mathfrak{p})^{ab} \arrow[u] \end{tikzcd}$$

If L/k is a finite extension,

$$\begin{tikzcd} L^\times \backslash \mathbb{A}_L^\times \arrow[r, "r_L"] \arrow[d, "\mathrm{Nm}"'] & \mathrm{Gal}(L^{ab}/L) \arrow[d] \\ k^\times \backslash \mathbb{A}_k^\times \arrow[r, "r_k"] & \mathrm{Gal}(k^{ab}/k) \end{tikzcd}$$

Remark. Global class field theory tells us what $Gal(K^{ab}/K)$ is, but we don't really know what K^{ab} is in general.

Let us now look at $\mathbb{Q}^\times \backslash \mathbb{A}_\mathbb{Q}^\times$.

Lemma: $\mathbb{A}_\mathbb{Q}^\times = \mathbb{Q}^\times \cdot \left(\prod_p \mathbb{Z}_p^\times \times \mathbb{R}_{>0} \right)$, and $\mathbb{Q}^\times \backslash \mathbb{A}_\mathbb{Q}^\times = \prod_p \mathbb{Z}_p^\times \times \mathbb{R}_{>0}$.

↳ Let $(x_v) \in \mathbb{A}_\mathbb{Q}^\times$. Let $S = \{ p \neq \infty : x_p \notin \mathbb{Z}_p^\times \}$. Do induction on $\#S$. If $\#S = \emptyset$ then simply pick $\lambda = \pm 1$ for $\lambda(x_v) \in \prod_p \mathbb{Z}_p^\times \times \mathbb{R}_{>0}$. Inductively if $p \in S$ then $x_p = p^n u$ for $u \in \mathbb{Z}_p^\times$, so multiply λ by p^n.

Exercise. $K^\times \backslash \mathbb{A}_K^\times / \left(\prod_p \mathcal{O}_{K_p}^\times \times K_\infty^\times \right) \cong Cl(K)$. Replacing $K_\infty^\times$ by $(K_\infty^\times)^\circ$ gives the narrow class group!

In fact $\mathbb{A}_\mathbb{Q}^\times = \mathbb{Q}^\times \times \prod_p \mathbb{Z}_p^\times \times \mathbb{R}_{>0}$ as $\mathbb{Q}^\times \cap \left(\prod_p \mathbb{Z}_p^\times \times \mathbb{R}_{>0} \right) = \{1\}$, so we get our claim for $\mathbb{Q}^\times \backslash \mathbb{A}_\mathbb{Q}^\times$ to equal $\prod_p \mathbb{Z}_p^\times \times \mathbb{R}_{>0}$. Hence $\ker r_\mathbb{Q} = \mathbb{R}_{>0}$ and

$$Gal(\overline{\mathbb{Q}}/\mathbb{Q})^{ab} = Gal(\mathbb{Q}^{ab}/\mathbb{Q}) = \prod_p \mathbb{Z}_p^\times = \hat{\mathbb{Z}}^\times = \varprojlim_N (\mathbb{Z}/N\mathbb{Z})^\times .$$

Def: Let K be a number field. A Größencharacter (GC), or Hecke character, is a continuous group homomorphism $K^\times \backslash \mathbb{A}_K^\times \longrightarrow \mathbb{C}^\times$.

We will see that GCs are automorphic representations for GL_1/K.

Example: We know $\mathbb{Q}^\times \backslash \mathbb{A}_\mathbb{Q}^\times = \hat{\mathbb{Z}}^\times \times \mathbb{R}_{>0}$. Every continuous group homomorphism for $\mathbb{R}_{>0} \to \mathbb{C}^\times$ is of the form $x \mapsto x^s$, $s \in \mathbb{C}$. Clearly $\mathbb{C}^\times$ has no small subgroups, so a continuous $\hat{\mathbb{Z}}^\times \to \mathbb{C}^\times$ must factor through $(\mathbb{Z}/N\mathbb{Z})^\times$ for some N. Hence if we have a GC $\mathbb{Q}^\times \backslash \mathbb{A}_\mathbb{Q}^\times \to \mathbb{C}^\times$ then there is a pair (χ, s), where χ is a Dirichlet character and $s \in \mathbb{C}$, such that

$$\begin{array}{ccc} \mathbb{Q}^\times \backslash \mathbb{A}_\mathbb{Q}^\times & \longrightarrow & \mathbb{C}^\times \\ \| & & \uparrow (1,-s) \\ \hat{\mathbb{Z}}^\times \times \mathbb{R}_{>0} \longrightarrow (\mathbb{Z}/N\mathbb{Z})^\times \times \mathbb{R}_{>0} & \xrightarrow{(\chi,1)} & \mathbb{C}^\times \times \mathbb{R}_{>0} \end{array}$$

Corollary: The set of GCs for $\mathbb{Q}$ forms a Riemann surface: fixing a Dirichlet character χ, one has $\mathbb{C} \hookrightarrow \{\text{set of GCs}\}$ by $s \longmapsto (\chi, s)$.

Tate's thesis takes a GC ψ and gives $L(\psi) \in \mathbb{C} \cup \{\infty\}$. Hence L gives a function on the Riemann surface of all GCs (let's call it the $\mathbb{C}$-eigencurve for $GL_1/\mathbb{Q}$).

By the thesis L has a meromorphic extension to all of the $\mathbb{C}$-eigencurve, and checked that the restriction of L to the copy of $\mathbb{C}$ attached to χ is $L(\chi, s)$, i.e. that $L(\chi,s) = L(\psi)$ for ψ a GC attached to (χ, s).

Generalization to K: Recall for $K_\mathfrak{p}/\mathbb{Q}_p$ finite, there is a canonical norm such that $|\pi_\mathfrak{p}| = q^{-1}$ where $q = \#k_\mathfrak{p}$. This canonical norm extends to $\mathbb{A}_K$, so $\mathbb{A}_K$ has an additive Haar measure. We have a norm $\mathbb{A}_K^\times \xrightarrow{\|\ \|} \mathbb{R}_{>0}$ with kernel containing $K^\times$, hence there is a well-defined map $K^\times \backslash \mathbb{A}_K^\times \xrightarrow{\|\ \|} \mathbb{R}_{>0}$. The set of all GC for GL_1/K also becomes a Riemann surface $\coprod_{\text{infinite}} \mathbb{C}$, where $\psi_1, \psi_2 : K^\times \backslash \mathbb{A}_K^\times \to \mathbb{C}$ are in the same component if $\psi_1/\psi_2 = \|\ \|^s$ for some $s \in \mathbb{C}$.

$\hookrightarrow$ Tate's thesis defines one meromorphic function on this Riemann surface and proves functional equation.

Say $\psi = (\chi, s) : \mathbb{Q}^\times \backslash \mathbb{A}_\mathbb{Q}^\times \to \mathbb{C}^\times$ is a GC with χ trivial and $s = \sqrt{-c}$.

There is no Galois representations attached to ψ in general...

Idea: There should be a global Langlands group $L_\mathbb{Q}$ (or L_K for general K) with $(L_K)^{ab} = K^\times \backslash \mathbb{A}_K^\times$.

Theorem: There is a canonical bijection

$$\left(\begin{matrix} \text{automorphic representations} \\ \text{of } GL_1/K \end{matrix} \right) \longleftrightarrow \left(\begin{matrix} \text{1-dimensional representations} \\ \text{of } L_K \end{matrix} \right)$$

$$\text{"} (\psi : K^\times \backslash \mathbb{A}_K^\times \to \mathbb{C}^\times) \longleftrightarrow (L_K \to L_K^{ab} = K^\times \backslash \mathbb{A}_K^\times \xrightarrow{\psi} \mathbb{C}) \text{"}$$

For $\psi : K^\times \backslash \mathbb{A}_K^\times \to \mathbb{C}^\times$ and L/K finite, the norm map $Nm_{L/K}$ gives $BC_L^K(\psi) : L^\times \backslash \mathbb{A}_L^\times \to \mathbb{C}^\times$.

We will talk more about this, as well as generalizing it to GL_n and beyond.

Recall: If K, E_0 are number fields and S a finite set of finite places of $\mathcal{O}_K$, then a <u>Compatible system</u> of λ-adic Galois representations is, for all λ a finite place of E_0, a representation $\rho_\lambda : \mathrm{Gal}(\bar{K}/K) \to GL_n(\overline{E_{0,\lambda}})$, and for all $\mathfrak{p} \notin S$ a finite place of $\mathcal{O}_K$, a polynomial $F_\mathfrak{p}(x) \in E_0[x]$ of degree n, such that

(*) for all λ and $\mathfrak{p} \notin S$ with $\mathfrak{p} \nmid \ell$ (λ are ℓ-adic), ρ_λ is unramified at $\mathfrak{p}$ and $\rho_\lambda(\mathrm{Frob}_\mathfrak{p})$ has characteristic polynomial $F_\mathfrak{p}(x)$ independent of λ.

Example: $K = \mathbb{Q}$ and $S = \{\text{primes } p \mid N\}$. Then let

$$\begin{array}{ccccc} \rho_\lambda : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \twoheadrightarrow \mathrm{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{Q}) & & & & \\ \quad\quad\quad\quad\quad\quad\quad\quad \| & & & & \\ \quad\quad\quad\quad\quad\quad\quad\quad (\mathbb{Z}/N\mathbb{Z})^\times & \xrightarrow{\chi} & E_0^\times & \hookrightarrow & (E_{0,\lambda})^\times \end{array}$$

This is a compatible system with $F_p(x) = x - \chi(p)$.

Example: $K = E_0 = \mathbb{Q}$. We have the cyclotomic character

$$\begin{array}{ccc} \omega_\ell : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \mathrm{Gal}(\mathbb{Q}(\zeta_{\ell^\infty})/\mathbb{Q}) & & \\ \quad\quad\quad\quad\quad\quad\quad\quad \| & & \\ \quad\quad\quad\quad\quad\quad\quad\quad \mathbb{Z}_\ell^\times & \hookrightarrow & GL_1(\mathbb{Q}_\ell) \end{array}$$

We can take $S = \emptyset$, and $F_p(x) = x - p$.

Example: Tate module of elliptic curves (See Diamond for example).

Also recall a <u>Größencharacter</u> (GC) is a continuous group homomorphism $K^\times \backslash \mathbb{A}_K^\times \to \mathbb{C}^\times$, and we discussed the structure of GC for $K = \mathbb{Q}$ last lecture.

Example. $K=\mathbb{Q}(i)$. Let us understand the GC's for K. Here $\mathbb{A}_K^\times = \mathbb{A}_{K,f}^\times \times \mathbb{C}^\times$.

Let us look at $\mathbb{A}_{K,f}^\times$. Given $x=(x_p)\in \mathbb{A}_{K,f}^\times$, let $n_p = v_p(x_p)$ so that $n_p=0$ for almost all p. Define a fractional ideal $I(x) = \prod_p p^{n_p}$ of K. Thus we have a map $\mathbb{A}_{K,f}^\times \longrightarrow \{\text{fractional ideals of } K\}$, with $K^\times$ mapping to the principal ones.

Claim: $\mathbb{A}_K^\times = K^\times \cdot \left(\prod_p \mathcal{O}_p^\times \times K_\infty^\times\right)$, so K has class number 1. (Here $K=\mathbb{Q}(i)$).

↳ follows as $\mathcal{O}_K = \mathbb{Z}[i]$ is a PID. In fact it always works if a number field has class number 1.

[Notice, for any number field K of class number 1, to give a GC $\psi: K^\times\backslash\mathbb{A}_K^\times \to \mathbb{C}^\times$ is to give a continuous character ψ on $\prod_p \mathcal{O}_p^\times \times K_\infty^\times$ that is trivial on $K^\times \cap \left(\prod_p \mathcal{O}_p^\times \times K_\infty^\times\right)$]

Back to $K=\mathbb{Q}(i)$. Here a $\lambda \in K^\times \cap \left(\prod_p \mathcal{O}_p^\times \times K_\infty^\times\right)$ has $\lambda \in \mathcal{O}_K^\times = \{\pm 1, \pm i\}$. For the infinite part we require a continuous homomorphism $K_\infty^\times \cong \mathbb{C}^\times \to \mathbb{C}^\times$. Think of $\mathbb{C}^\times$ as $\mathbb{R}_{>0}\times S^1$. The only $\mathbb{R}_{>0} \to \mathbb{C}^\times$ are $x \mapsto x^s$ $(s\in\mathbb{C})$ and for $S^1 \to \mathbb{C}^\times$ it is $x\mapsto x^n$ $(n\in\mathbb{Z})$. Therefore we must have

$$K_\infty^\times \cong \mathbb{C}^\times = \mathbb{R}_{>0}\times S^1 \longrightarrow \mathbb{C}^\times \quad \text{by} \quad re^{i\theta} \longmapsto r^s e^{in\theta} \quad (s\in\mathbb{C}, n\in\mathbb{Z}).$$

Similar to $K=\mathbb{Q}$ case the finite part factors through some $n\neq 0$ with a character $\chi: (\mathbb{Z}[i]/n)^\times \to \mathbb{C}^\times$. Thus given such a χ and $(s,n)\in\mathbb{C}\times\mathbb{Z}$, we can get $\psi_0: \prod_p \mathcal{O}_p^\times \times K_\infty^\times \to \mathbb{C}^\times$. This may not be a GC as $\mathcal{O}_K^\times$ may not be in the kernel, so we have an easy fix by considering ψ_0^4.

Example: $K = \mathbb{Q}(\sqrt{2})$, so $\mathcal{O}_K = \mathbb{Z}[\sqrt{2}]$. The finite part $\prod_p K_p^\times$ is same as before. ③

The infinite part $K_\infty^\times = \mathbb{R}^\times \times \mathbb{R}^\times$, so any $\chi_\infty : K_\infty^\times \to \mathbb{C}^\times$ is parametrized by (up to sign) $(s_1, s_2) \in \mathbb{C} \times \mathbb{C}$. Hence we get $\psi_0 : \prod_p \mathcal{O}_p^\times \times K_\infty^\times \to \mathbb{C}^\times$, which is a GC iff $\psi_0|_{\mathcal{O}_K^\times} = 1$. Here $\mathcal{O}_K^\times$ is infinite, and $\mathcal{O}_K^\times = \{\pm 1\} \times \langle 1+\sqrt{2} \rangle$ (check by Dirichlet's unit theorem, say). Thus at $K_\infty^\times$ we need $|1+\sqrt{2}|^{s_1} |1-\sqrt{2}|^{s_2}$ to be a root of unity. A good way to make sure of this is to let $s_1 = s_2$. In any case we don't have two degrees of freedom for (s_1, s_2) anymore.

2017-08-02 (2)

Compatible system of 1-dimensional Galois representations.

We saw a few examples like the Dirichlet and cyclotomic characters. Also talked about GC. (Richard Taylor's notes: "There are too many of these".)

Definition: A GC ψ for $K^\times \backslash \mathbb{A}_K^\times$ is algebraic if, when restricted to $(K_\infty^\times)^\circ$, ψ looks like

$$(K_\infty^\times)^\circ \cong (\mathbb{R}_{>0})^{r_1} \times (\mathbb{C}^\times)^{r_2} \longrightarrow \mathbb{C}^\times$$

$$(x_1, \dots, x_{r_1}, z_1, \dots, z_{r_2}) \longmapsto x_1^{n_1} \cdots x_{r_1}^{n_{r_1}} z_1^{m_1} \bar{z}_1^{m_2} \cdots z_{r_2}^{m_{2r_2+1}} \bar{z}_{r_2}^{m_{2r_2+2}}$$

with all $n_i, m_j \in \mathbb{Z}$.

Example.. The norm $K^\times \backslash \mathbb{A}_K^\times \xrightarrow{\|\cdot\|} \mathbb{C}^\times$ is algebraic with all exponent 1.

Philosophy: If χ is a GC for K (which is an automorphic representation for GL_1/K) then it should correspond to a 1-dimensional representation of a global Langland group L_K (which is not defined).

Thm (Weil). If χ is an algebraic GC, then there is a compatible system of λ-adic Galois representations attached to χ. (The converse is true as well!)

$\hookrightarrow$ Idea: For $\chi : K^\times \backslash \mathbb{A}_K^\times \to \mathbb{C}^\times$ algebraic, want $\mathrm{Gal}(\bar{K}/K)^{ab} \simeq K^\times \backslash \mathbb{A}_K^\times / \overline{(K_\infty^\times)^\circ} \to GL_1(\bar{\mathbb{Q}}_\ell)$.

We need to somewhat manipulate χ to be trivial at $\overline{(K_\infty^\times)^\circ}$.

Proof. Given $\chi : K^\times \backslash \mathbb{A}_K^\times \to \mathbb{C}^\times$ algebraic, say

$$\chi|_{(K_\infty^\times)^\circ}(x_\infty) = \prod_{v \text{ real}} x_v^{n_v} \cdot \prod_{\substack{v = (\sigma, \bar{\sigma}) \\ \text{complex}}} (\sigma x_v)^{n_{v,1}} (\bar{\sigma} x_v)^{n_{v,2}}.$$

Define $\chi_0 : \mathbb{A}_K^\times \to \mathbb{C}^\times$ by $\chi_0(x) := \chi(x) \Big/ \left(\prod_{v \text{ real}} x_v^{n_v} \cdot \prod_{v \text{ complex}} (\sigma x_v)^{n_{v,1}} (\overline{\sigma x_v})^{n_{v,2}} \right)$.

χ_0 is trivial on $(K_\infty^\times)^\circ$, but <u>not</u> on $K^\times$. In fact $\chi_0(k) = \prod_{\sigma : K \hookrightarrow \mathbb{C}} \sigma(k)^{n_\sigma}$ for $k \in K^\times$.

On the other hand χ_0 is trivial on the "continuous part" of $\mathbb{A}_K^\times$, and so $\mathrm{Im}(\chi_0) \subset E_0$ for some number field E_0 (strong approximation). Now say λ is a finite place of E_0. Notice $\chi_0|_{K^\times} : K^\times \to E_0^\times$ certainly extends to $E_{0,\lambda}^\times$ via injection $E_0 \hookrightarrow E_{0,\lambda}$, and $\chi_0|_{K^\times}$ extends to a continuous

$$\chi_\ell : (K \otimes_{\mathbb{Q}} \mathbb{Q}_\ell)^\times \longrightarrow (E_{0,\lambda})^\times.$$

We now define $\psi_\lambda(x) := \chi_0(x) / \chi_\ell(x_\ell)$, where $(x_\ell) = (x_p : p \mid \ell)$. Then ψ_λ is trivial on $K^\times$ and $(K_\infty^\times)^\circ$, so it extends to $\psi_\lambda : K^\times \backslash \mathbb{A}_K^\times / \overline{(K_\infty^\times)^\circ} \longrightarrow E_{0,\lambda}^\times$, with

$F_p(x) = X - \chi_0(\pi_p)$. $\square$

⌈For the <u>converse</u>: If ψ_λ is a compatible system we need to deal with the following question. If E_0 is a number field and $s \in \mathbb{C}$, then does $p^s \in E_0$ for all primes p imply $s \in \mathbb{Z}$? This is <u>true</u> by Waldschmidt transendence theory.⌋

<u>Big picture</u> $(n=1)$.

[?? means one or both sides are not defined.]

$$\begin{pmatrix}\text{Automorphic representations} \\ \text{for } GL_1/K\end{pmatrix} \xleftrightarrow{??} \begin{pmatrix}\text{1-dimensional representations} \\ \text{of } L_K\end{pmatrix}$$

$$\cup$$

$$\begin{pmatrix}\text{algebraic automorphic} \\ \text{representations for } GL_1/K\end{pmatrix} \underset{\text{Waldschmidt}}{\overset{\text{Weil}}{\rightleftarrows}} \begin{pmatrix}\text{compatible systems of 1-dimensional} \\ \ell\text{-adic representations of } \mathrm{Gal}(\bar{K}/K)\end{pmatrix}$$

$$\cap \qquad\qquad\qquad\qquad \cap \text{ [Brauer-Nesbitt]}$$

$$\begin{pmatrix}p\text{-adic automorphic} \\ \text{representations for } GL_1/K\end{pmatrix} \longleftrightarrow \begin{pmatrix}\text{continuous } p\text{-adic representations} \\ \mathrm{Gal}(\bar{K}/K) \to GL_1(\bar{\mathbb{Q}}_p)\end{pmatrix}$$

$\chi : K^\times \backslash \mathbb{A}_K^\times \to \bar{\mathbb{Q}}_p^\times$ or $\mathbb{C}_p^\times$

For general n :

$$\begin{array}{ccc}
\left(\begin{array}{c}\text{Automorphic representations}\\ \text{for } GL_n/K\end{array}\right) & \xleftrightarrow{??} & \left(\begin{array}{c}n\text{-dimensional representations}\\ \text{for } L_K\end{array}\right)\\
\cup & & \\
(\text{Algebraic ones}) & \xleftrightarrow{\circledast} & (\text{compatible } n\text{-dimensional system})\\
\cap & & \cap\\
(p\text{-adic ones}) & \xleftrightarrow{??} & (\text{continuous } p\text{-adic representation of } GL_n/K)
\end{array}$$

↳ overconvergent p-adic modular form an example?

$\circledast$: Conjectured by Clozel (1990)

<u>Fontaine-Mazur Conjecture</u>: If $\rho : Gal(\bar{K}/K) \to GL_n(E)$ ($E/\mathbb{Q}_p$ finite) is continuous semisimple unramified outside a finite set of places and potentially semistable (which implies Hodge-Tate), then does ρ come from a motive? If so, then ρ is part of a compatible system of ℓ-adic representations.

Rambling ~~Philosophy~~ of p-adic automorphic representation

Say $K = \mathbb{Q}$, $\ell = p$. Let $\chi : \mathbb{Q}^\times \backslash \mathbb{A}_{\mathbb{Q}}^\times \to \mathbb{C}_p^\times$

$\chi|_{\mathbb{Z}_p^\times}$ is a "weight". It is "<u>algebraic</u>" if $\frac{d}{dx}\chi(x)\big|_{x=1} \in \mathbb{C}_p$ is actually in $\mathbb{Z}$.

2017-08-03 C11

Last time the "web of modularity" was stated.

$$\begin{pmatrix} \text{Algebraic automorphic representations} \\ \text{of } G/K \end{pmatrix} \leftrightarrow \begin{pmatrix} \text{compatible system of semisimple} \\ \ell\text{-adic Galois representations} \\ \mathrm{Gal}(\overline{K}/K) \to {}^LG(\overline{\mathbb{Q}}_p) \end{pmatrix}$$

Here $G = GL_n$.

⌈For general connected reductive group G, there are some subtleties.

* More than one notion of algebraicity: C-algebraic and L-algebraic (see Toby-Gee paper).
* The correspondence above is not a bijection.
  - ↳ For left hand side there are local and global L-packets (not for GL_n). In particular π's in the same L-packet gives the same ρ.
  - ↳ Different global Langlands parameters on the right hand side may be isomorphic everywhere locally.
* One way of thinking about it: π should correspond to ρ_π defined up to some Tate-Shafarevich group (trivial for GL_n by Brauer-Nesbitt theorem.)⌋

Let us stick to $G = GL_n$.

- Langlands (Corvallis): dreaming of motives.
- Clozel (Ann Arbor): concrete conjecture, and statement of an actual theorem. If π is an automorphic representation of GL_n/K with K totally real or CM, a strong self duality condition on π, and strong algebraicity condition, then Clozel showed that ~~there is a compatible system ρ~~ there is a compatible system ρ

The idea to Clozel's proof

- Find an appropriate Shimura variety. (Eichler-Shimura relations comes in somewhere.)
- Relate the cohomology of this variety to automorphic forms.

Harris-Lan-Taylor-Thorne (2013) removed the self-duality condition.

↳ Idea: Given π, observed $\pi \oplus \pi^\vee$ is self-dual. Take limits of cohomology of Shimura varieties to get ρ.

↳ Scholze gave a second proof via perfectoid spaces.

Genesis of these ideas are Weil's construction from last lecture, and also

Eichler-Shimura $(k=2)$
Deligne $(k>2)$ ← etale cohomology magic.
Deligne-Serre $(k=1)$

: If f is a weight k modular eigenform, then there is a compatible system of 2-dimensional Galois representations $Gal(\bar{\mathbb{Q}}/\mathbb{Q})$.

For Deligne's theorem to fit into this we need to talk about automorphic representations

What isn't an automorphic representation ;

$\boxed{\text{Local Langlands conjecture}}$ Recall this said that, for $K/\mathbb{Q}_p$ finite, we have

$$\begin{pmatrix} \text{Smooth admissible irreducible} \\ \text{representations of } GL_n(K) \end{pmatrix} \longleftrightarrow \begin{pmatrix} n\text{-dimensional F-semisimple} \\ \text{WD representations} \end{pmatrix}.$$

$\boxed{\text{Global Langlands conjecture}}$ This is about automorphic representations of $GL_n(\mathbb{A}_K)$, K back to a number field. By definition this representation is irreducible and also

$GL_n(\mathbb{A}_K) = \prod_v' GL_n(K_v)$ restricted to $GL_n(\mathcal{O}_v)$.

⌈ If π is an irreducible representation of $G \times H$, with G and H finite groups, then $V = V_1 \otimes V_2$ with V_1 irreducible of G, V_2 irreducible of H. ⌋

If π is a nice well-behaved representation of $GL_n(\mathbb{A}_K)$, then it is true that $\pi = \bigotimes_v' \pi_v$ with π_v an irreducible admissible representation of $GL_n(K_v)$ [Strong multiplicity one and irreducibility stuff ; see Flath Corvallis.].

<u>Idea</u>: $\pi = \otimes \pi_v \xleftrightarrow{\text{Global}} \rho : \mathrm{Gal}(\overline{K}/K) \to GL_n(\overline{\mathbb{Q}}_\ell)$ a compatible system.

$GL_n(K_p) \circlearrowleft \pi_p \xleftrightarrow{\text{Local}}$ Local WD representations.

Program of Grothendieck:

$\left(\rho : \mathrm{Gal}(\overline{K}_p/K_p) \to GL_n(\overline{\mathbb{Q}}_\ell)\right)$

$\updownarrow$

(WD representation)

LLC

In these discussions an automorphic representation of $GL_n(K)$ <u>cannot be</u> just an arbitrary smooth admissible irreducible representation of $GL_n(\mathbb{A}_K)$.

$\hookrightarrow$ <u>Why not</u>? Say, in 1-dimensional case, we guess that the definition is just a representation of $\mathbb{A}_K^\times$. Let $K = \mathbb{Q}$. Say for all $\mathbb{Q}_p^\times$ with $p < 100$, send $\mathbb{Z}_p^\times \mapsto 1$ and $p \mapsto 7$, and for all other places it is trivial (Recall $\mathbb{Q}_p^\times \cong \mathbb{Z}_p^\times \times (p)$). If we believe in the Langlands philosophy, this π must correspond to some ρ_ℓ locally via LLC, and $\rho_\ell(\mathrm{Frob}_p) = 1$ for all $p \geq 100$, and ρ_ℓ trivial here by Cebotarev's density theorem. Hence $\rho_\ell(\mathrm{Frob}_2) = 1 \neq 7$, which does not agree to the Langlands correspondence.

Why was the π above not good? Previously we were looking at $\mathbb{Q}^\times \backslash \mathbb{A}_\mathbb{Q}^\times \to \mathbb{C}^\times$, but the π above is not trivial on $\mathbb{Q}^\times$: for example $\pi(2) = 7$.

$\ulcorner$ If G is a finite group, all the irreducible $\mathbb{C}$-representations of G is in the group ring: $\mathbb{C}[G] = \bigoplus \pi^{\dim}$ summed over all irreducibles.

If $H \subset G$ is a subgroup (maybe not normal) we can instead look at $\mathbb{C}[H \backslash G] \cong \bigoplus_{\pi \in S} \pi^{m(\pi)} \subset \mathbb{C}[G]$ where S is probably not all irreducibles, and $m(\pi) \leq \dim \pi$ (here $\mathbb{C}[H \backslash G]$ are functions $H \backslash G \to \mathbb{C}$.) $\lrcorner$

Idea for definition: let $G = GL_n(\mathbb{A}_K)$. Maybe we should focus on functions $\varphi : GL_n(K) \backslash GL_n(\mathbb{A}_K) \to \mathbb{C}$ with some "nice" property. Let $A_0(GL_n(K) \backslash GL_n(\mathbb{A}_K))$ be the set of all such functions, which is a $\mathbb{C}$-vector space with obvious action of $GL_n(\mathbb{A}_K)$ on the right. For $n=1$, a Größencharacter will be "nice", as is a finite sum of them. For general n, the $A_0(GL_n(K) \backslash GL_n(\mathbb{A}_K))$ will be a direct sum π of irreducibles, and maybe they are automorphic representations

2017-08-03 (2)

Let K be a number field and let S be a finite set of finite places. For $\mathfrak{p} \notin S$ we have conjugacy classes $\mathrm{Frob}_{\mathfrak{p}} \in \mathrm{Gal}(K^S/K)$. All the $\mathrm{Frob}_{\mathfrak{p}}$ are related in some vastly complex way that noone understands. In fact Cebotarev's density theorem says that the $\mathrm{Frob}_{\mathfrak{p}}$ are dense. (In particular it is very far from being a free group.)

Based on successes for GL_1, we will restrict to representations of $GL_n(\mathbb{A}_K)$ which shows up in $\mathcal{A}_0(GL_n(K)\backslash GL_n(\mathbb{A}_K)) =: \mathcal{A}_0(GL_n/K)$. What is a "nice" function?

$\boxed{n=1}$ GC's (Größencharacters) were nice. For a GC $\chi : GL_1(K)\backslash GL_1(\mathbb{A}_K) \to \mathbb{C}^*$ recall they are locally constant at finite places and smooth at infinite places. But there is more. The infinite places $x \overset{f}{\mapsto} x^s$ is not growing "too fast" and $xf'(x) = sf(x)$.

Interlude on differential equations:

Let G be a Lie group (for example $GL_n(K_\infty)$), and let $\mathfrak{g}$ be its Lie algebra. Write the exponential map $\exp : \mathfrak{g} \to G$. In case of $G = GL_n(\mathbb{R})$ and $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{R})$, $\exp(M)$ is the usual Taylor expansion.

For $X \in \mathfrak{g}$, if we think of it as a differential operator on C^∞-functions $G \to \mathbb{C}$,

$$Xf(g) = \frac{d}{dt}\Big|_{t=0} f(g\exp(tX)).$$

Example. If $G = GL_1(\mathbb{R})$ and $X = 1$, then $Xf(g) = gf'(g)$. In case $f(x) = x^s$, we have $Xf = s \cdot f$.

Example: If $G = GL_2(\mathbb{R})$ and $\mathfrak{g} = M_2(\mathbb{R})$, let $E = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $F = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Each of these gives a differential operator.

Let $V = \{ C^\infty\text{-functions } f: GL_2(\mathbb{R}) \to \mathbb{C} \}$. Certainly E, F, H, Z do not commute when viewed as actions on V, so we can't ask for a simultaneous eigenfunction (for example $EF - FE = 2H$). We need to use the universal enveloping algebra.

Let $\mathfrak{g}/\mathbb{R}$ be a Lie algebra, and $U(\mathfrak{g})$ its universal enveloping algebra. Recall $U(\mathfrak{g})$ is generated by a basis of $\mathfrak{g}$ by the Poincare-Birkhoff-Witt theorem.

↳ There is an adjoint functor (Associative algebra) $\to$ (Lie algebra) by $U(\mathfrak{g}) \leftarrow\!\shortmid \mathfrak{g}$.

We want to look at the center of $U(\mathfrak{g})$, which is a bunch of commutating operators, and we hope that they are simultaneously diagonalizable. Harish-Chandra figured out $Z(U(\mathfrak{g}\otimes\mathbb{C}))$. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ a reductive Lie algebra over $\mathbb{R}$.

Write $\quad U(\mathfrak{h}_{\mathbb{C}}) = \mathbb{C}\langle X_1, \dots, X_d\rangle / (X_iX_j - X_jX_i) = \mathbb{C}[X_1, \dots, X_d]$.

Thm: There is a canonical injection $Z(U(\mathfrak{g}\otimes\mathbb{C})) \hookrightarrow U(\mathfrak{h}_{\mathbb{C}})$, called the <u>Harish-Chandra homomorphism</u>, with image $U(\mathfrak{h}_{\mathbb{C}})^W$. $\leftarrow$ Weyl group

↳ Inside Kirillov's book. The p-adic version, called <u>Satake isomorphism</u>, is in Cartier's article in Corvallis.

<u>Example</u>: Let $G = GL_2(\mathbb{R})$, with $\mathfrak{h} = \left\{ \begin{bmatrix} * & 0 \\ 0 & * \end{bmatrix} \right\}$. Here $W = \mathbb{Z}/2\mathbb{Z}$ and

$U(\mathfrak{h}_{\mathbb{C}})^W = \mathbb{C}[x,y]^{\mathbb{Z}/2\mathbb{Z}}$ with $x \mapsto y$ and $y \mapsto x$ under the Weyl group action.

Hence $U(\mathfrak{h}_{\mathbb{C}})^W = \mathbb{C}[s,t]$ with $s = xy$ and $t = x+y$.

Let K be a number field and G/K a connected reductive group. Recall we were trying to figure out what a nice function is. We had a ~~nice function~~ temporary definiti

$$\mathcal{A}(G/K) = \{ \varphi : GL_n(K) \backslash GL_n(\mathbb{A}_K) \to \mathbb{C} \text{ with } \varphi \text{ "nice"} \}.$$

Remember $GL_1(\mathbb{A}_K) \supset K^\times \prod \mathcal{O}_p^\times \cdot K_\infty^\times$ and a "nice" φ is finite on the finite part in practice, so we were studying functions on $K_\infty^\times$.

<u>Fact</u>: $GL_n(K) \cdot \prod_p GL_n(\mathcal{O}_{K_p}) \cdot GL_n(K_\infty)$ is of finite index in $GL_n(\mathbb{A}_K)$.

↳ For GL_1 it was an exercise the quotient is the class group. We will look at proof of $GL_2/\mathbb{Q}$ later.

Recall that we wanted $\mathcal{A}(GL_1/\mathbb{Q})$ to contain the Grössencharacters, so at $K_\infty^\times = \mathbb{R}_{>0}$ the function $x \mapsto x^s$ must be "nice". (Last time, writing D to be 1 in $\mathfrak{g} = \mathbb{R}$, we had $Df(x) = xf'(x)$, so if $f(x) = x^s$ then $Df = s \cdot f$). The sum of two GC also needs to be "nice". (Abstractly $\mathbb{C}[D]$ acts on the C^∞-functions $\mathbb{R}_{>0} \to \mathbb{C}$, and for sums of them we can find some $D' \in \mathbb{C}[D]$ that annihilates them. If I is the ideal of annihilators of such a sum then it has finite codimension.)

Now, for general G as above, let $\mathfrak{g} = \mathrm{Lie}(G(K_\infty))$, with basis $e_1, \dots, e_d$. Certainly this is not commutative in general. The Harish-Chandra isomorphism allows us to pass to $Z(U(\mathfrak{g}_\mathbb{C})) \cong \mathbb{C}[T_1, \dots, T_d]^W$. This is a canonical source of higher-order differential operators.

↳ In case $G = GL_n(\mathbb{Q})$, one has $Z(U(\mathfrak{g}_\mathbb{C})) = \mathbb{C}[T_1, \dots, T_n]^W$, where W is the symmetric group S_n with permutations on T_i.

⌜Let $G = Gl_n(\mathbb{Q})$. What are the T_i's? Let T be the diagonals in Gl_n. Then $T(\mathbb{R})$ is a Lie subgroup of $G(\mathbb{R})$ with Lie algebra $\mathfrak{t} \subset \mathfrak{g}$. The Harish-Chandra isomorphism says $U(\mathfrak{t}) = \mathbb{C}[T_1, \dots, T_n]$, $Z(U(\mathfrak{g}_{\mathbb{C}})) = \mathbb{C}[\sigma_1, \dots, \sigma_n]$ with σ_i the i^{th} symmetric polynomial.⌟

Example: For $Gl_2/\mathbb{Q}$, we have $Z(U(\mathfrak{g}_{\mathbb{C}})) = \mathbb{C}[\Delta, Z]$ (Here the σ_i's are $t_1 + t_2$ and $t_1 t_2$.) Here $\mathfrak{g} = M_2(\mathbb{R})$ over $\mathbb{R}$, with standard basis

$$e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad f = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad z = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Certainly $[e,f] = h$, $[e,h] = -2e$, $[f,h] = 2f$, $[-,z] = 0$, so that

$$U(\mathfrak{g}_{\mathbb{C}}) = \mathbb{C}\langle E, F, H, Z\rangle / (EF - FE - H, \text{ etc.}).$$

One can check that, if $\Delta := H^2 + 2EF + 2FE$ (nothing to do with matrix multiplication), then Δ commutes with E, F, H, Z. Turns out Δ and Z generates $Z(U(\mathfrak{g}_{\mathbb{C}}))$. Here Δ is some differential operator on $\{f : GL_2^+(\mathbb{R}) \to \mathbb{C}\}$

↳ Recall $Gl_2^+(\mathbb{R})$ acts on the upper half plane $\mathfrak{h}$ transitively, so there is a surjection $Gl_2^+(\mathbb{R}) \twoheadrightarrow \mathfrak{h}$ by $\gamma \mapsto \gamma(i)$ with stabilizer $\mathbb{R}^\times \cdot SO_2(\mathbb{R})$. Hence

$$\mathfrak{h} = Gl_2^+(\mathbb{R}) / \mathbb{R}^\times \cdot SO_2(\mathbb{R}).$$

Now say $f : \mathfrak{h} \to \mathbb{C}$ and let F be the associated function on $Gl_2^+(\mathbb{R})$. Then ΔF descends to a function Δf on $\mathfrak{h}$. Up to a constant,

$$\Delta = -y^2 \left(\frac{d^2}{dx^2} + \frac{d^2}{dy^2} \right).$$

Upshot. Looks like we are interested in $f : \mathfrak{h} \to \mathbb{C}$ with $\Delta f = \lambda f$, $\lambda \in \mathbb{C}$, (and $Zf = \mu f$, μ central character)

Recall the following theorem: If f is an eigenform then it corresponds to some compatible system $\rho_f : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL_2(\overline{\mathbb{Q}_\ell})$ (with $\det \rho_f(\overset{\text{conjugation}}{c}) = -1$ from the Weil pairing).

Now let $f(x)$ be an irreducible polynomial with three real root α, β, γ, and let $K = \mathbb{Q}(\alpha,\beta,\gamma)$. Chances are that $\mathrm{Gal}(K/\mathbb{Q}) = S_3$ with $c = 1$. Fix

$$\rho_0 : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \mathrm{Gal}(K/\mathbb{Q}) \cong S_3 \xrightarrow{\text{irreducible}} GL_2(\mathbb{Q}),$$

with $\rho_0(c)$ having determinant 1. For all ℓ prime we get $\rho_\ell : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL_2(\mathbb{Q}) \hookrightarrow GL_2(\mathbb{Q}_\ell)$ all the same as ρ_0. Here we would still like ρ_ℓ to correspond to some π. Maaß wrote down a function $\mathfrak{h} \to \mathbb{C}$ that is not holomorphic and invariant under $\Gamma_1(N)$, $N = \mathrm{conductor}(\rho_0)$, with $\Delta f = \lambda f$, $\lambda \neq 0$. The discussions in previous few lectures leads to the following.

Def: Let G be a connected reductive group over a number field K. Let H_∞ be a maximal compact subgroup of $G(K_\infty)$. A function $\varphi : G(K) \backslash G(\mathbb{A}_K) \to \mathbb{C}$ is an <u>automorphic form</u> if the conditions below are satisfied.

(1) φ is smooth, i.e. if we write $G(\mathbb{A}_K) = G(\mathbb{A}_{K,f}) \times G(K_\infty)$ and $(x,y) \in G(\mathbb{A}_K)$, then for fixed x φ is C^∞ with respect to y, and for fixed y φ is locally constant.

(2) (a) There exists a compact open $U_f \subset G(\mathbb{A}_{K,f})$ with $\varphi(gu) = \varphi(g)$ for all $u \in U_f$.

(b) The $\mathbb{C}$-vector space spanned by $g \mapsto \varphi(g h_\infty)$ is finite-dimensional as $h_\infty \in H_\infty$.

(3) There exists $I \subset Z(U(\mathfrak{g}_\mathbb{C}))$ an ideal of finite codimension such that, for all $\delta \in I$, $\delta(y \mapsto \varphi(x,y)) = 0$ for all $x \in G(\mathbb{A}_{K,f})$.

(4) Growth conditions are satisfied: $|\varphi(x,y)| \leq (\text{constant}) \cdot \|y\|^N$ for some N.

Let G/K be connected reductive and $H_\infty \subset G(K_\infty)$ maximal compact open. An automorphic form $\varphi: G(K) \backslash G(\mathbb{A}_K) \to \mathbb{C}$ is one that is smooth, of moderate growth, H_∞-finite, $\mathfrak{z}$-finite. Let $\mathcal{A}(G)$ be the $\mathbb{C}$-vector space of automorphic forms of G.

If $g \in G(\mathbb{A}_{K,f})$ then we can define the obvious left action on $\mathcal{A}(G)$, by right translation. Unfortunately $G(K_\infty)$ does not act on this space; $gH_\infty g^{-1} \neq H_\infty$ generally. However H_∞ acts on it, and so does $\mathfrak{g}_{\mathbb{C}}$.

⌈Remark: there is a second way of defining $\mathcal{A}(G)$ as $L^2(G(K) \backslash G(\mathbb{A}_K), \omega)$ where $G(K_\infty)$ acts on. ⌋

There is something called a $(\mathfrak{g}, K)$-module ($(\mathfrak{g}, H_\infty)$-module). Then $\mathcal{A}(G)$ can also be defined as $(\mathfrak{g}_{\mathbb{C}}, H_\infty)$-module, and $\mathcal{A}(G)$ has a $G(\mathbb{A}_{K,f}) \times (\mathfrak{g}_{\mathbb{C}}, H_\infty)$ action. An automorphic representation π for G/K is an irreducible subquotient of $\mathcal{A}(G)$.

↳ Here it is not the algebraic vector space quotient.

We don't really know what it means, but here is a $\boxed{\text{fix.}}$

- A $\varphi \in \mathcal{A}(G)$ is cuspidal if $\int_{N(K) \backslash N(\mathbb{A}_K)} \varphi(xn)\, dn = 0$ for all x, where P is a maximal proper parabolic of G, and $P = MN$ with N the unipotent radical. (Check it agrees with the usual definition of modular form that are cuspidal in the classical case).

②

- Say Z is the center of G, and fix a character $\psi: Z(k)\backslash Z(\mathbb{A}_k) \to \mathbb{C}^\times$.

Define the space of cuspidal automorphic forms by

$$\mathcal{A}_0(G,\psi) := \left\{ \varphi \in \mathcal{A}(G) : \begin{array}{l} \varphi \text{ is cuspidal and} \\ \varphi(gz) = \psi(z)\varphi(g) \ \forall z \in Z(\mathbb{A}_k),\ g \in G(\mathbb{A}_k) \end{array} \right\}.$$

Def: A <u>cuspidal representation</u> π of $G(\mathbb{A}_k)$ is a representation π isomorphic to an irreducible subrepresentation of $\mathcal{A}_0(G,\psi)$ for some (central) character ψ.

<u>"Thm (Langlands)"</u>. If π is a noncuspidal representation, then $\pi \cong \mathrm{Ind}_P^G \pi_0$ where π_0 is cuspidal on some smaller group.

↳ "Upshot": For certain purposes it suffices to look at cuspidal representations. ⌋

<u>Example</u>. · For $G = GL_1 \times GL_1$ a cuspidal automorphic representation of G is a pair χ_1, χ_2 of Größencharacters.

- "$I(\chi_1,\chi_2)$" gives rise to noncuspidal automorphic representations of GL_2. Langlands showed every automorphic representation of GL_2 is either cuspidal or arises in this form.

<u>Global Langlands for GL_n/k (conjecture)</u>

$$\left(\begin{array}{l}\text{cuspidal automorphic} \\ \text{representations of } GL_n/k\end{array}\right) \overset{\text{LANGLANDS RECIPROCITY}}{\longleftrightarrow} \left(\begin{array}{l}\text{irreducible } n\text{-dimensional} \\ \text{representations of } L_k\end{array}\right)$$

$\leftarrow$ conjectured global Langlands group.

$\pi = \otimes' \pi_p$???

and the correspondence should be compatible with the Local Langlands correspondence.

Fact: Semisimple representations are a direct sum of irreducible representations

Philosophically fact corresponds to Langlands theorem above.

↳ Langlands theorem is an instance of functoriality.

In general reciprocity is a philosophy, and functoriality are concrete consequences that makes sense

Example: If π is an automorphic cuspidal representations of GL_2/K, philosophically it should correspond to so $\rho: L_K \to GL_2(\mathbb{C})$. So $Symm^2(\rho)$ should correspond to some $Symm^2(\pi)$ of $GL_3(K)$

↳ this does exist by a hard theorem in functional analysis.

Example (of an automorphic form for $GL_2/\mathbb{Q}$):

Recall that if $f: \mathfrak{h} \to \mathbb{C}$ is a function, $k \in \mathbb{Z}$, and $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL_2^+(\mathbb{R})$, then

$$(f|_k \gamma)(\tau) = (\det \gamma)^{k-1} (c\tau + d)^{-k} f(\gamma\tau).$$

Now say f is a modular form that is cuspidal, of level N, and weight k, so that $f|_k \gamma = f$ for all $\gamma \in \Gamma_1(N) = \left\{ \begin{bmatrix} * & * \\ 0 & 1 \end{bmatrix} \bmod N \text{ in } SL_2(\mathbb{Z}) \right\}$.

Let us try to define $\varphi: GL_2(\mathbb{A}_\mathbb{Q}) \to \mathbb{C}$ from f, with φ cuspidal. Recall that $GL_2(\mathbb{Q}_p) = B(\mathbb{Q}_p) GL_2(\mathbb{Z}_p)$, so that $GL_2(\mathbb{A}_f) = \underset{\uparrow \begin{bmatrix} \mathbb{A}_f^\times & \mathbb{A}_f \\ 0 & \mathbb{A}_f^\times \end{bmatrix}}{B(\mathbb{A}_f)} GL_2(\hat{\mathbb{Z}}) = B(\mathbb{Q}) GL_2(\hat{\mathbb{Z}})$.

If $U_f = U_1(N) = \left\{ m \in GL_2(\hat{\mathbb{Z}}) : m \equiv \begin{bmatrix} * & * \\ 0 & 1 \end{bmatrix} \bmod N \right\}$, then $GL_2(\hat{\mathbb{Z}}) = \coprod \tilde{\gamma}_i U_f$ where $\tilde{\gamma}_i$ is in $GL_2(\mathbb{Q})$ and lifts γ_i cosets for $\begin{bmatrix} * & * \\ 0 & 1 \end{bmatrix} \subset GL_2(\mathbb{Z}/N\mathbb{Z})$. Hence $GL_2(\mathbb{A}_f) = GL_2(\mathbb{Q}) U_1(N)$.

Thus $GL_2(\mathbb{A}_\mathbb{Q}) = GL_2(\mathbb{Q}) U_1(N) GL_2^+(\mathbb{R})$.

Given f, a modular form and before, and $s \in \mathbb{C}$, we define $\varphi: GL_2(\mathbb{Q}) \backslash GL_2(\mathbb{A}_\mathbb{Q}) \to \mathbb{C}$ by

$$\varphi(\underset{GL_2(\mathbb{Q})}{\gamma}\ \underset{U_1(N)}{u}\ \underset{GL_2^+(\mathbb{R})}{h}) = (f|_k h)(i) \cdot (\det(h))^s.$$

Then $\varphi \in \mathcal{A}(G)$ for $G = GL_2/\mathbb{Q}$. Observe that φ is well-defined: if $\gamma_1 u_1 h_1 = \gamma_2 u_2 h_2$, then $\gamma_2^{-1}\gamma_1 = u_2 h_2 h_1^{-1} u_1^{-1} \in U_1(N)\, GL_2^+(\mathbb{R}) \cap GL_2(\mathbb{Q}) = \Gamma_1(N)$, so $h_2 h_1^{-1} \in \Gamma_1(N)$. The other conditions are easy to check. In fact,

$$\Delta\varphi = (k^2 - 2k)\varphi \quad \text{and} \quad Z\varphi = (2s + k - 2)\varphi.$$

If f is an eigenform, then the representation spanned by π is irreducible cusipal, and is the automorphic representation attached to f.

↳ More details of this example in Gelbart's Automorphic Forms on Adele Groups.