Branching Rules for Splint Root Systems

(with Logan Crew and Alexandre Kirillov)

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Finite Groups

Let G be a finite group.

Definition

A (complex) representation of G is a homomorphism $\rho\colon G \longrightarrow \mathrm{GL}_n(\mathbb{C})$. The character $\chi\colon G \longrightarrow \mathbb{C}$ of ρ is defined by $\chi(g) = \operatorname{tr}\rho(g)$.

If H is a finite subgroup of G, consider the restricted representation

$$\operatorname{Res}_H^G \rho\colon H \longrightarrow \mathrm{GL}_n(\mathbb{C}).$$

A way to understand this is to consider the (finite) character table of H: one defines a simple inner product on the space of class functions, and using Maschke's theorem the coefficients of $\operatorname{Res}_H^G \rho$ in its irreducible decomposition can be determined by taking inner products with respect to the irreducible characters of H.

The Symmetric Group

The irreducible representations of S_n can be enumerated by the partitions λ of n, and one has the following classical result.

Theorem (Classical)

Let π_λ be an irreducible representation of S_n. If we view the partition $\lambda = (\lambda_1, \ldots, \lambda_l)$ as a Ferrers board, then

$$\mathrm{Res}^{S_n}_{S_{n-1}} \pi_\lambda \cong \bigoplus_{\substack{\lambda' \vdash n-1 \\ \lambda' \subset \lambda}} \pi_{\lambda'},$$

where the λ' are sub-Ferrers boards of λ.

For example,

$$\mathrm{Res}^{S_{15}}_{S_{14}} \pi_{(5,4,4,2)} \cong \pi_{(4,4,4,2)} \oplus \pi_{(5,4,3,2)} \oplus \pi_{(5,4,4,1)}.$$

Lie Groups

Now let G be a connected and simply-connected compact Lie group, and let H be a Lie subgroup of G.

Question

Can we explicitly understand the functor Res_H^G?

Maschke's Theorem still holds here, so it suffices to understand Res_H^G applied to irreducible (finite-dimensional complex) representations. By the fundamental theorem of Lie theory

$$\{\text{representations of } G\} \longleftrightarrow \{\text{representations of } \mathfrak{g} = \mathrm{Lie}(G)\}.$$

If $\mathfrak{h}$ is the Lie algebra of H, the question above is equivalent to understanding the restriction functor

$$\mathrm{Res}_{\mathfrak{h}}^{\mathfrak{g}}$$

in the category of Lie algebras.

Gelfand-Tsetlin Patterns for $\mathfrak{sl}_r$

Irreducible representations of $\mathfrak{sl}_{r+1}$ (with root system A_r) are indexed by Young tableaux with at most r rows.

Theorem (Gelfand-Tsetlin 1950)

If $\pi^r_{\lambda_1,\dots,\lambda_r}$ is a highest weight (irreducible) representation of $\mathfrak{sl}_{r+1}$, then

$$\mathrm{Res}^{\mathfrak{sl}_{r+1}}_{\mathfrak{sl}_r} \pi^r_{\lambda_1,\dots,\lambda_r} = \bigoplus_{\lambda_i \geq \mu_i \geq \lambda_{i+1}} \pi^{r-1}_{\mu_1,\dots,\mu_{r-1}}.$$

A similar characterization exists for the restriction functor $\mathrm{Res}^{\mathfrak{so}_{r+1}}_{\mathfrak{so}_r}$.

Splint Root Systems

Let Δ be the root system of a simple Lie algebra $\mathfrak{g}$. We focus on splint root systems, i.e. $\Delta = \Delta_1 \sqcup \Delta_2$, where

- Δ_1 is a root system itself,
- Δ_2 is embedded in such a way that the length of roots are scaled uniformly (the embedding may not be angle-preserving).

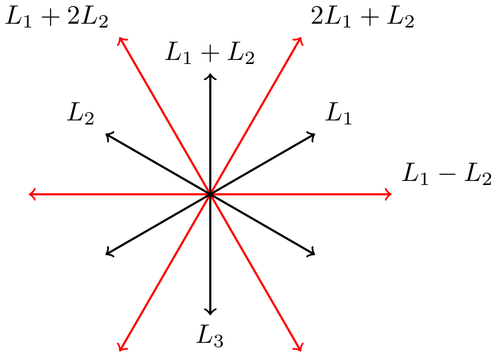
Richter (2012) has classified irreducible splint root systems.

Type	Δ	Δ_1	Δ_2
(I)	A_r $(r \geq 2)$	A_{r-1}	$(A_1)^{\oplus r}$
(II)	B_r $(r \geq 2)$	D_r	$(A_1)^{\oplus r}$
(III)	C_r $(r \geq 3)$	$(A_1)^{\oplus r}$	D_r
(IV)	G_2	A_2	A_2
(V)	F_4	D_4	D_4

We primarily discuss $\mathrm{Res}^{\mathfrak{g}}_{\mathfrak{a}}$ for Type IV, where $\mathfrak{a}$ is the Lie algebra of Δ_1. (Note that $\mathrm{Res}^{\mathfrak{g}}_{\mathfrak{a}}$ for Types I and II corresponds to Gelfand-Tsetlin patterns.)

Type IV Splint Root System

The root system A_2 embeds into G_2 via the long roots.



The action of the Weyl group $W_{A_2} \cong S_3$ on L_1, L_2, L_3 is simply by permuting the indices. The fundamental weights ω_1, ω_2 for G_2, and Ω_1, Ω_2 for A_2, are

$$\omega_1 = L_1 + L_2, \qquad \omega_2 = 2L_1 + L_2, \qquad \Omega_1 = L_1 + L_2, \qquad \Omega_2 = L_1.$$

Type IV Splint Root System

Let $\Pi_{k,l}$ be the highest weight representation of G_2 with weight $k\omega_1 + l\omega_2$. By the Weyl dimension formula, this representation has dimension

$$\frac{(k+1)(k+l+2)(2k+3l+5)(k+2l+3)(k+3l+4)(l+1)}{120}.$$

Similarly, if we let $\pi_{\alpha,\beta}$ be the highest weight representation of A_2 with weight $\alpha\Omega_1 + \beta\Omega_2$, then it has dimension

$$\frac{(\alpha+1)(\beta+1)(\alpha+\beta+2)}{2}.$$

Question

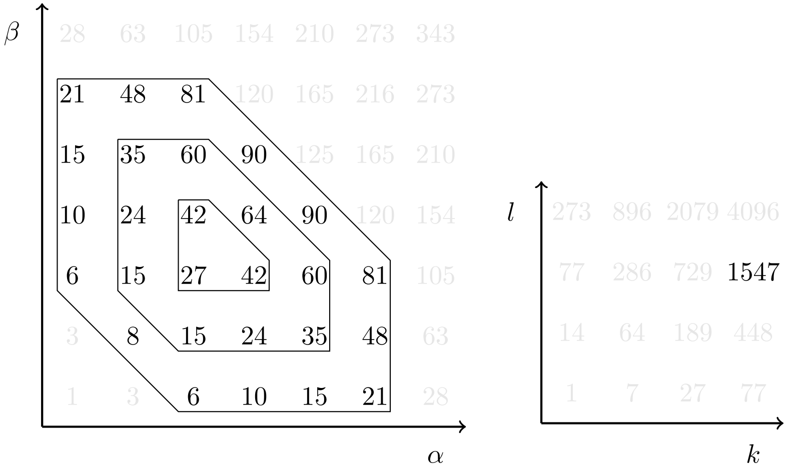
What is $\mathrm{Res}^{\mathfrak{g}_2}_{\mathfrak{sl}_3}$?

Type IV Splint Root System

β							
	28	63	105	154	210	273	343
	21	48	81	120	165	216	273
	15	35	60	90	125	165	210
	10	24	42	64	90	120	154
	6	15	27	42	60	81	105
	3	8	15	24	35	48	63
	1	3	6	10	15	21	28
							α

l				
	273	896	2079	4096
	77	286	729	1547
	14	64	189	448
	1	7	27	77
				k

Type IV Splint Root System

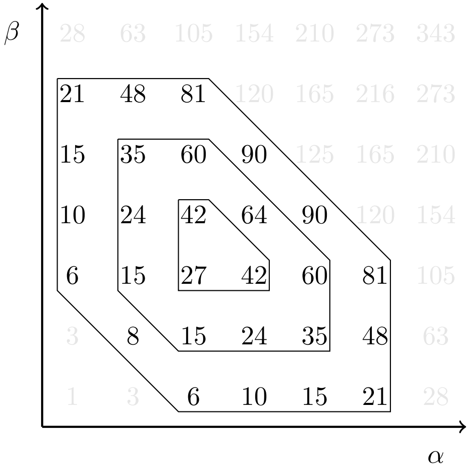


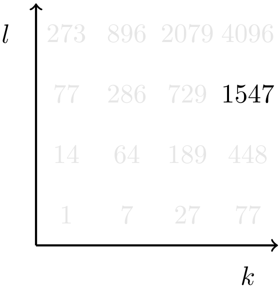
Observe that

$$1547 = \text{ sum of the three hexagons.}$$

(Think: tower of hexagons.)

Type IV Splint Root System





Thus one can conjecture that

$$\mathrm{Res}^{\mathfrak{g}_2}_{\mathfrak{sl}_3} \Pi_{3,2} = \bigoplus_{\substack{\text{sum over the}\\ \text{three hexagons}}} \pi_{\alpha,\beta}.$$

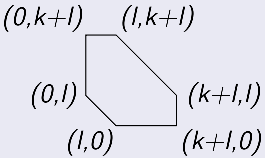
Type IV Splint Root System

Theorem

One has

$$\mathrm{Res}^{\mathfrak{g}_2}_{\mathfrak{sl}_3} \Pi_{k,l} = \bigoplus_{\alpha,\beta} n_{\alpha,\beta}\pi_{\alpha,\beta},$$

where (α, β) *are integral points on and inside of the hexagon*



and $n_{\alpha,\beta}$ *are positive integers determined as follows.*

- *If* (α, β) *lies on a* h^{th} *hexagon layer, then* $n_{\alpha,\beta} = h + 1$.
- *The hexagon* H *degenerates at the* $m^{th} = min(k, l)^{th}$ *layer to a triangle. Set* $n_{\alpha,\beta} = m + 1$ *for all points* (α, β) *on this triangle.*

Proof: Weyl character formula; Pieri's rule.

Type V Splint Root System

We can understand

$$\mathrm{Res}^{F_4}_{D_4}$$

via the following steps.

- There is an embedding $D_4 \hookrightarrow B_4 \hookrightarrow F_4$.
- One has a formula for $\mathrm{Res}^{B_4}_{D_4}$ via the Gelfand-Tsetlin patterns.
- Understand $\mathrm{Res}^{F_4}_{B_4}$.

Here are some conjectural formulas derived by comparing dimensions.

$$\mathrm{Res}^{F_4}_{D_4} \Pi_{k,0,0,0} = \bigoplus_{0 \le s'+s''+t' \le k} \pi_{s',s'',t',k-s'-s''-t'},$$

$$\mathrm{Res}^{F_4}_{D_4} \Pi_{0,0,0,k} = \bigoplus_{0 \le s'+t'+t'' \le k} (k+1-s'-t'-t'')\pi_{s',0,t',t''}.$$