ON THE VARIATIONAL PROPERTIES OF THE PRESCRIBED RICCI CURVATURE FUNCTIONAL

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ABSTRACT. We study the prescribed Ricci curvature problem for homogeneous metrics. Given a (0,2)-tensor field $T$, this problem asks for solutions to the equation $\text{Ric}(g) = cT$ for some constant $c$. Our approach is based on examining global properties of the scalar curvature functional whose critical points are solutions to this equation. We produce conditions under which it has a global maximum and find a large class of spaces where it admits saddle critical points, even though the Palais–Smale condition fails to hold. Finally, we consider several examples, including ones where the functional admits critical submanifolds of various dimensions.

The prescribed Ricci curvature problem consists in finding a Riemannian metric $g$ on a manifold $M$ such that $\text{Ric}(g) = T$ for a given (0,2)-tensor field $T$. As was suggested by DeTurck [16] and Hamilton [21], it is more natural to ask instead whether one can solve the equation

$$\text{Ric}(g) = cT$$

for some constant $c$. In fact, on a compact manifold, such a constant appears to be necessary. For example, if $M$ is a surface, this follows from the Gauss–Bonnet theorem.

The prescribed Ricci curvature problem has been studied by many authors since the 1980s; see, e.g., [11] for an overview of the literature. Considering the difficulty of the equation involved, it is natural to make symmetry assumptions. More precisely, suppose that the metric $g$ and the tensor $T$ are invariant under a Lie group $G$ acting on $M$. In the case where the quotient $M/G$ is one-dimensional, the problem was addressed by Hamilton [21], Cao–DeTurck [13], Pulemotov [29, 30] and Buttsworth–Krishnan [10]. The case where $M$ is a homogeneous space $G/H$ has been studied extensively; see the survey [11] and the more recent references [12, 25, 26, 4, 3]. In some situations, the equation can be solved explicitly, as shown, e.g., in [31, 9].

Assume that $T$ is positive-definite. General existence theorems in the homogeneous setting rely on the fact, proven in [31], that $G$-invariant metrics on $G/H$ with Ricci curvature $cT$ are precisely (up to scaling) the critical points of the scalar curvature functional $S$ on the set $\mathcal{M}_T = \mathcal{M}_T(G/H)$ of $G$-invariant metrics on $G/H$ subject to the constraint $\text{tr}_g T = 1$. We will assume that $G$ and $H$ are compact Lie groups. The goal of this paper is to study the global behavior of the scalar curvature functional. In [19, 32] one finds a general theorem for the existence of a global maximum, under certain assumptions on the isotropy representation of $G/H$. Our first result removes these assumptions, thereby expanding greatly the class of spaces to which the theorem applies. Moreover, we improve the conditions for the existence of the global maximum and interpret them geometrically.

If $H$ is maximal in $G$, the supremum of $S$ is always attained. Otherwise, the global behavior of $S$ depends on the set of intermediate subgroups, i.e., Lie groups $K$ with $H \subset K \subset G$. For every such $K$, one has the homogeneous fibration

$$K/H \to G/H \to G/K.$$
If we fix homogeneous metrics $g_F$ and $g_B$ on the fiber $K/H$ and the base $G/K$, it is natural to study the two-parameter variation

$$g_{s,t} = \frac{1}{s} g_F + \frac{1}{t} g_B$$

of $g = g_F + g_B$. Notice that we define this variation using the reciprocals of $s$ and $t$. In fact, we often deal with inverses of metrics rather than metrics themselves, which enables us to view $\mathcal{M}_T$ as a precompact space. Solving the constraint $\text{tr}_g T = 1$ for $s$, we obtain a one-parameter family $g_t \in S_{|\mathcal{M}_T}$, called a canonical variation, with scalar curvature given by

$$S(g_t) = \frac{S_F}{T_1^*} + T_2^* \left( \frac{S_B}{T_2^*} - \frac{S_F}{T_1^*} \right) t - \frac{t^2 T_1^*}{1 - t T_2^*} |A|_g.$$ 

Here $T_1^* = \text{tr}_{g_F} T|_F$ and $T_2^* = \text{tr}_{g_B} T|_B$ are the traces of $T$ on the fiber and the base, $S_F$ and $S_B$ are the scalar curvatures of $g_F$ and $g_B$, and $A$ is the O'Neil tensor of the Riemannian submersion. As a consequence,

$$\lim_{t \to 0} S(g_t) = \frac{S_F}{T_1^*} \quad \text{and} \quad \lim_{t \to 0} \frac{dS(g_t)}{dt} = T_2^* \left( \frac{S_B}{T_2^*} - \frac{S_F}{T_1^*} \right),$$

which means that $S_F$ and $S_B - S_F$ control the behavior of $S$ at infinity. It is easy to see that $S$ is bounded from above if $T$ is positive-definite, and it is therefore natural to search for a global maximum of $S$. The above formulas motivate two invariants of the intermediate subgroup $K$ with Lie algebra $\mathfrak{k}$:

$$\alpha_t = \sup \left\{ \frac{S(h)}{\text{tr}_h T|_F} \middle| h \in \mathcal{M}_T(K/H) \right\} \quad \text{and} \quad \beta_t = \sup \left\{ \frac{S(h)}{\text{tr}_h T|_B} \middle| h \in \mathcal{M}_T(G/K) \right\}.$$ 

These quantities maximize the limits and the “derivatives” of $S$ at infinity. We furthermore introduce an invariant of the homogeneous space $G/H$:

$$\alpha_{G/H} = \sup_t \alpha_t,$$

where the supremum is taken over all Lie algebras of intermediate subgroups. We can now state our first result.

**Theorem A.** Let $G/H$ be a compact homogeneous space and $T$ a positive-definite $G$-invariant $(0,2)$-tensor field on $G/H$. If $H$ is not maximal in $G$, then the set of intermediate subgroups $K$ with $\alpha_t = \alpha_{G/H}$ is non-empty. If $K$ is such a subgroup of the lowest possible dimension and $\beta_t - \alpha_t > 0$, then $S_{|\mathcal{M}_T}$ achieves its supremum at some metric $g \in \mathcal{M}_T$ and hence $\text{Ric}(g) = c T$ for some $c > 0$.

In fact, we show that there exists $\epsilon > 0$ such that the set \( \{ g \in \mathcal{M}_T(G/H) \mid S(g) > \alpha_{G/H} + \epsilon \} \) is non-empty and compact, which implies the result. This requires careful estimates of the behaviour of $S_{|\mathcal{M}_T}$ at infinity.

Our next goal is to study the existence of critical points of other types, in particular, saddle points. For this one needs a compactness result. Recall that the functional $S$ satisfies the Palais–Smale condition if any sequence of metrics $g_i$ with

$$\lim_{i \to \infty} S(g_i) = \lambda \in \mathbb{R} \quad \text{and} \quad \lim_{i \to \infty} |\text{grad} S_{|\mathcal{M}_T}(g_i)|_{g_i} = 0$$

has a convergent subsequence. We will see, however, that this condition is never satisfied for $S_{|\mathcal{M}_T}$ unless $H$ is maximal in $G$. More precisely, one can always detect a divergent Palais–Smale sequence along one of the canonical variations; see Theorem 5.3. We establish several properties of general divergent Palais–Smale sequence in Theorem 5.7. This enables us to find saddle points in many interesting cases.

A large class of homogeneous spaces has been studied in the recent literature, the so-called generalized Wallach spaces. Some typical examples are $U(p + q + r)/U(p)U(q)U(r)$, $O(p + q + r)/O(p)O(q)O(r)$, $Sp(p + q + r)/Sp(p)Sp(q)Sp(r)$ and the Ledger–Obata space $H^4/\text{diag}(H)$. For
a classification of generalized Wallach spaces, see \[14, 27\]. Each such space admits precisely three intermediate subgroups \(K_i\). We have the following result.

**Theorem B.** Let \(G/H\) be a generalized Wallach space with inequivalent isotropy summands and intermediate subgroups \(K_i, i = 1, 2, 3\). If two of the quantities \(\beta_i - \alpha_i\) are negative, then the functional \(S_{|\mathcal{M}_T}\) has a critical point with co-index 0 or 1.

Theorem B also holds in the case of equivalent summands, as long as one restricts to the set of diagonal metrics; see Section 6.

In order to prove Theorem B, we show that the scalar curvature functional satisfies the Palais–Smale condition on \(S^{-1}((a, b))\) if \(\alpha_i \notin (a, b)\) for \(i = 1, 2, 3\). We then show that there exists an \(\epsilon > 0\) such that \(\{g \in \mathcal{M}_T(G/H) \mid S(g) > \alpha_i - \epsilon\}\) has two components for one of the indices \(i\) with \(\beta_i - \alpha_i < 0\). The gradient flow of \(S\) applied to a curve connecting these two components gives rise to a family of curves that “remain hanging” at a critical point, which hence has co-index 0 or 1.

One may view Theorem B as an analogue, in this special case, of the graph theorem for the Einstein equation obtained in \[8\]. It is thus natural to ask whether Theorem B can be extended to a graph theorem for the prescribed Ricci curvature problem on general homogeneous spaces.

As the following observation illustrates, the functional \(S_{|\mathcal{M}_T}\) can have a large set of critical points. Let \(J\) be a complex structure on a compact generalized flag manifold \(G/H = G/C(\tau)\), where \(C(\tau)\) is the centralizer of a torus \(\tau\) in \(G\). It is well known that there exists a unique \(G\)-invariant Hermitian bilinear form \(h_J\) such that \(\text{Ric}(g) = -h_J\) for every Kähler metric \(g\) compatible with \(J\); see \[24, 2\]. Since the set of such metrics has the same dimension as the torus \(\tau\), we obtain a smooth critical submanifold in \(\mathcal{M}_T\) of dimension \(\dim \tau - 1\). Furthermore, this critical submanifold contains the unique Kähler–Einstein metric associated to \(J\). For instance, if \(G/H = SU(n + 1)/T^n\), there is an \((n - 1)\)-dimensional smooth submanifold of Kähler metrics consisting of critical points of \(S_{|\mathcal{M}_T}\) for \(T\) equal to \(-h_J\) (up to scaling). In general, there exist several inequivalent complex structures on \(G/H\) and hence non-isometric critical submanifolds.

Finally, we study the behavior of \(S\) in several examples.

- For the Wallach space \(SU(3)/T^2\) we draw the region where \(S_{|\mathcal{M}_T}\) has a global maximum or a saddle point in Figure 1. The graphs of \(S_{|\mathcal{M}_T}\) for some typical choices of \(T\) are shown in Figure 2, and the curve of critical Kähler metrics appears in Figure 3. A computer experiment with one million data points indicates that in this example Theorems A and B are optimal.
- On the space \((SU(2) \times SU(2))/S^1_{p,q}\) with \(p \neq q\), previously studied in \[33, 1, 8\] in the context of Einstein metrics, we are able to obtain a complete classification of critical points, showing that Theorem A is optimal.
- We examine the Stiefel manifold \(V_2(\mathbb{R}^4)\) of two planes in \(\mathbb{R}^4\). Here the metrics are determined by five parameters, and there are equivalent isotropy summands, which makes this example more interesting but also more difficult. We find:
  - A tensor \(T\) such that \(S_{|\mathcal{M}_T}\) admits a two-dimensional submanifold of critical points with boundary at infinity. This submanifold does not contain any Einstein metrics.
  - A large set of critical points that are global maxima among diagonal metrics but saddles in the space of all invariant metrics; see Figure 6.
  - A set of tensors \(T\) for which \(S_{|\mathcal{M}_T}\) has both a circle of saddles unrelated to Theorem B and a strict local maximum. Remarkably, for some of these \(T\), the functional does not achieve its global maximum.

There are also two circles of tensors \(T\) which are Einstein metrics isometric to the canonical product metric on \(V_2(\mathbb{R}^4) \simeq S^3 \times S^3\). Thus they lie on curves of critical points, which turn out to be local maxima.
Finally, we consider the Ledger–Obata space $H^3 / \text{diag}(H) \simeq H \times H$ with $H$ simple and $\text{Spin}(8)/G_2 \simeq S^7 \times S^7$. These are the only homogeneous spaces $G/H$ where the isotropy representation splits into two equivalent irreducible modules. We obtain sufficient conditions for the existence of a global maximum using Theorem A. However, we also find a large set of tensors $T$ for which $S_{|\mathcal{M}_T}$ has a local maximum even though the conditions of Theorem A are not satisfied; see Figure 7. The Ledger–Obata space (respectively $\text{Spin}(8)/G_2$) has three tensors $T$ which are Einstein metrics isometric to the canonical product metric on $H \times H$ (respectively $S^7 \times S^7$). Thus each of them lies on a curve of critical points.

In all these examples the continuous families of critical points form non-degenerate critical submanifolds. We note, however, that generic critical points are isolated. More precisely, as shown in [25], there exists an open and dense subset in the space of $G$-invariant metrics on which the Ledger map is a local diffeomorphism (up to scaling). It is also natural to conjecture that, for generic $T$, the restriction of $S$ to $\mathcal{M}_T$ is a Morse function.

Let us compare the behavior of $S_{|\mathcal{M}_T}$ to that of $S_{|\mathcal{M}_1}$, where $\mathcal{M}_1$ is the set of homogeneous metrics of volume 1. The critical points of the latter functional are homogeneous Einstein metrics, and they have been studied extensively. There are certain similarities between $S_{|\mathcal{M}_T}$ and $S_{|\mathcal{M}_1}$ but also important differences. Unlike $S_{|\mathcal{M}_1}$, the functional $S_{|\mathcal{M}_T}$ is bounded from above only if $H$ is maximal in $G$. If not, its behavior at infinity is again controlled by the collection of intermediate subgroups. A graph theorem, as well as the topology of the simplicial complex of intermediate subgroups, guarantee the existence of critical points of large co-index plus nullity in many examples; see [8, 6, 7]. Another difference is that $S_{|\mathcal{M}_1}$ satisfies the Palais–Smale condition on the set of metrics of positive scalar curvature, and hence the set of critical points has only finitely many components, each of them compact. As mentioned above, we construct examples where $S_{|\mathcal{M}_T}$ has non-compact connected sets of critical points.

Throughout the paper we assume that $T$ is positive-definite. Without this assumption, the behavior of $S_{|\mathcal{M}_T}$ is very different since this functional may not be bounded from above. Note that in Figures 1 and 7 we indicate the regions (found with Maple) consisting of indefinite bi-linear forms $T$ for which the prescribed Ricci curvature problem has a solution.

We now outline the strategy of our proofs. To describe the set of homogeneous metrics on $G/H$, one decomposes the tangent space $m \simeq g/\mathfrak{h}$ into a sum of irreducible modules $m_1 \oplus \cdots \oplus m_r$ and considers so-called diagonal metrics

$$g = x_1 Q_{|m_1} + x_2 Q_{|m_2} + \cdots + x_r Q_{|m_r},$$

where $Q$ is a fixed bi-invariant metric on the Lie algebra of $G$. Not every homogeneous metric is of this form when some of the summands $m_i$ are equivalent. However, it can always be reduced to this form by choosing another decomposition of $m$ as above. With the substitution $y_i = \frac{1}{x_i}$, the constraint $\text{tr}_g T = 1$ becomes simply $\sum d_i T_i y_i = 1$, where $d_i = \dim m_i$ and $T_{|m_i} = T_i Q_{|m_i}$. Thus the set of diagonal metrics in $\mathcal{M}_T$ is parametrized by a simplex $\Delta$ with stratified boundary $\partial \Delta$. In this construction, the strata are indexed by the sets of those variables $y_i$ that vanish. Some of them are marked by subalgebras $\mathfrak{k}$ with $\mathfrak{h} \subset \mathfrak{k} \subset \mathfrak{g}$ and denoted $\Delta_t$. If $(g_i) \subset \mathcal{M}_T$ is a divergent sequence of metrics of bounded scalar curvature, then there exist such an intermediate subalgebra and a subsequence of $(g_i)$ that converges to a point in $\Delta_t$. At the remaining strata, denoted $\Delta_{\infty}$, the scalar curvature goes to $-\infty$. We will in fact show that $\alpha_T$ is the largest possible limit of the scalar curvature achieved as one approaches $\Delta_T$. Since $S_{|\mathcal{M}_T}$ is bounded from above, it is natural to look at subalgebras $\mathfrak{k}$ with $\alpha_\mathfrak{k} = \alpha_{G/H}$ and determine conditions under which there are metrics with scalar curvature larger than $\alpha_{G/H}$. This is achieved by using the “derivatives” $\beta_\mathfrak{k} - \alpha_\mathfrak{k}$, and if such metrics exists, we show that $\{ g \in \mathcal{M}_T(G/H) \mid S(g) > \alpha_{G/H} + \epsilon \}$ is compact for some $\epsilon > 0$. One of the main difficulties is that, generally speaking, the scalar curvature does not extend continuously
to $\partial\Delta$ and $\alpha_t$ may not be achieved by a metric on $K/H$. In addition, $\beta_t - \alpha_t$ may not be an actual derivative. This also explains, in part, why we cannot arbitrarily choose a subalgebra $\mathfrak{k}$ with $\alpha_t = \alpha_{G/H}$ in Theorem A. In order to prove our theorems, we need to produce careful estimates for the scalar curvature near $\partial\Delta$. We also need to conduct a detailed analysis of the behavior of Palais–Smale sequences. The final difficulty is to extend our theorems to the set of all metrics in $\mathcal{M}_T$ when some of the isotropy summands are equivalent and $m$ can be decomposed into irreducible modules in many substantially different ways. This requires further careful estimates.

The paper is organized as follows. In Section 1 we recall properties of homogeneous spaces and Riemannian submersion that we will need in our proofs. In Section 2 we describe the simplicial complex $\Delta$ and the stratification of $\partial\Delta$ by subalgebras, as well as the behavior of the scalar curvature near the strata in $\partial\Delta$. In Section 3 we use Riemannian submersions defined by intermediate subgroups to understand when there exist metrics $g$ with $S(g) > \alpha_t$ near a stratum $\Delta_k$. In Section 4 we vary the decompositions and prove Theorem A. In Section 5 we examine properties of divergent Palais–Smale sequences, which leads to the proof of Theorem B in Section 6. Finally, in Section 7 we study examples in detail and prove the existence of arcs and discs of critical points on the Wallach space and the Stiefel manifold.

1. Preliminaries

We first recall some basics of the geometry of a homogeneous space. Let $H \subset G$ be two compact Lie groups with Lie algebras $\mathfrak{h} \subset \mathfrak{g}$ such that $G/H$ is an almost effective homogeneous space. We fix a biinvariant metric $Q$ on $\mathfrak{g}$, which defines a $Q$-orthogonal $\text{Ad}_H$-invariant splitting $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, i.e., $\mathfrak{m} = \mathfrak{h}^\perp$. The tangent space $T_{eH}(G/H)$ is identified with $\mathfrak{m}$, and $H$ acts on $\mathfrak{m}$ via the adjoint representation $\text{Ad}_H$. A $G$-invariant metric on $G/H$ is determined by an $\text{Ad}_H$-invariant inner product on $\mathfrak{m}$. We denote by $\mathcal{M}(G/H)$, or sometimes simply by $\mathcal{M}$, the space of $G$-invariant metrics on $G/H$. We will assume throughout the paper that $G/H$ is not a torus since in this case all $G$-invariant metrics are flat, in particular, $\text{Ric}(g) = 0$ for all $g \in \mathcal{M}$.

We describe metrics in $\mathcal{M}$ in terms of $\text{Ad}_H$-invariant splittings. Under the action of $\text{Ad}_H$ on $\mathfrak{m}$, we decompose

$$\mathfrak{m} = \mathfrak{m}_1 \oplus \ldots \oplus \mathfrak{m}_r,$$

where $\text{Ad}_H$ acts irreducibly on $\mathfrak{m}_i$. Some of these summands may need to be one-dimensional if there exists a subspace of $\mathfrak{m}$ on which $\text{Ad}_H$ acts as the identity. We denote by $\mathcal{D}$ the space of all such decompositions and use the letter $D \in \mathcal{D}$ for a particular choice of decomposition. The space $\mathcal{D}$ has a natural topology induced from the embedding into the products of Grassmannians $G_k(\mathfrak{g})$ of $k$-planes in $\mathfrak{g}$. Clearly, $\mathcal{D}$ is compact.

If $T$ is a $G$-invariant symmetric bi-linear form field on $G/H$, it is determined by its value on $\mathfrak{m}$. We are interested when such a bi-linear form field is (up to scaling) the Ricci curvature of a metric $g \in \mathcal{M}$, i.e., when

$$\text{Ric}(g) = cT \quad \text{for some constant } c.$$

Throughout the paper we will assume that $T$ is positive-definite. We may also assume that $c > 0$ in (1.1) since a compact homogeneous space does not admit any metrics with $\text{Ric} \leq 0$ unless it is a torus (see [5, Theorem 1.84]), which we excluded above.

Define the hypersurface

$$\mathcal{M}_T(G/H) = \{ g \in \mathcal{M} \mid \text{tr}_g T = 1 \} \subset \mathcal{M},$$

where $\text{tr}_g$ is the trace with respect to $g$. We denote it simply by $\mathcal{M}_T$ when the homogeneous space is clear from the context. As shown in [31], a solution to (1.1) can we viewed as a critical point of the functional

$$S : \mathcal{M}_T \to \mathbb{R},$$
where $S(g)$ is the scalar curvature of $g$. More precisely, the following result holds.

**Proposition 1.2.** The Ricci curvature of a metric $g \in \mathcal{M}_T$ equals $c T$ for some $c \in \mathbb{R}$ if and only if $g$ is a critical point of $S|_{\mathcal{M}_T}$.

Here $c$ is the Lagrange multiplier of the variational problem. Our main interest in this paper is to describe the geometry of the functional $S$ and its implications for when (1.1) has a solution. One can compare this to the observation that on $\mathcal{M}_1 = \{g \in \mathcal{M} \mid \text{vol}(g) = 1\}$ the critical points of the scalar curvature are the Einstein metrics on $G/H$, the Einstein constant being the Lagrange multiplier. This problem has also been studied extensively; see, e.g., [34, 6, 8].

We now recall the formulas for the scalar curvature and the Ricci curvature of a homogeneous metric. Given $g \in \mathcal{M}$, we have $Q|_{\mathfrak{m}_i} = x_i Q|_{\mathfrak{m}_i}$ for some constant $x_i > 0$. In general, $\mathfrak{m}_i$ and $\mathfrak{m}_j$ do not have to be orthogonal if some of these summands are equivalent. But we can diagonalize $g$ and $Q$ simultaneously, and hence there exists a decomposition $D \in \mathcal{D}$ such that the metric has the form

$$g = x_1 Q|_{\mathfrak{m}_1} + x_2 Q|_{\mathfrak{m}_2} + \cdots + x_r Q|_{\mathfrak{m}_r}.$$  

We call such metrics diagonal with respect to our choice of $D$ and denote their set by $\mathcal{M}^D(G/H)$, or simply $\mathcal{M}^D$. Thus, $\mathcal{M} = \cup_{D \in \mathcal{D}} \mathcal{M}^D$. We also denote $\mathcal{M}_T^D = \mathcal{M}_T \cap \mathcal{M}^D$. The scalars $x_i$ are simply the eigenvalues of $g$ with respect to $Q$. When these eigenvalues have multiplicity, and some of the modules in the corresponding eigenspace are equivalent under the action of $\text{Ad}_H$, we note that $g \in \mathcal{M}^D$ for a compact infinite family of decompositions $D$. In order to describe all homogeneous metrics on $G/H$, we can thus restrict ourselves to diagonal metrics but allow the decomposition to change. The advantage is that, while the scalar curvature of a homogeneous metric for a fixed decomposition is quite complicated and hence somewhat intractable, for a diagonal metric it has a much simpler form. This idea was first used in [34] to study $G$-invariant Einstein metrics.

We define the structure constants

$$[ijk] = \sum_{\alpha, \beta, \gamma} Q(e_\alpha e_\beta e_\gamma)^2, \quad i, j, k = 1, \ldots, r,$$

where $(e_\alpha)$, $(e_\beta)$ and $(e_\gamma)$ are $Q$-orthonormal bases of $\mathfrak{m}_i$, $\mathfrak{m}_j$ and $\mathfrak{m}_k$. Clearly, $[ijk] \geq 0$, and $[ijk] = 0$ iff $Q(\mathfrak{m}_i, \mathfrak{m}_j, \mathfrak{m}_k) = 0$. We will denote by $B$ the Killing form of $G$. By the irreducibility of $\mathfrak{m}_i$, there exist constants $b_i \geq 0$ such that

$$B|_{\mathfrak{m}_i} = -b_i Q|_{\mathfrak{m}_i},$$

with $b_i = 0$ iff $\mathfrak{m}_i$ lies in the center of $g$. Furthermore, not all of $b_i$ vanish since otherwise $\mathfrak{m}$ is in the center $\mathfrak{z}(g)$ and $G/H$ is a torus. Using this notation, and $d_i = \text{dim} \mathfrak{m}_i$, the scalar curvature is given by

$$S(g) = \frac{1}{2} \sum_i \frac{d_i b_i}{x_i} - \frac{1}{4} \sum_{i,j,k} [ijk] \frac{x_k}{x_i x_j},$$

(see [34]), and the Ricci curvature satisfies

$$\text{Ric}(g)|_{\mathfrak{m}_i} = \left( \frac{b_i}{2} - \frac{1}{2d_i} \sum_{j,k} [ijk] \frac{x_k}{x_j} + \frac{1}{4d_i} \sum_{j,k} [ijk] \frac{x_i^2}{x_j x_k} \right) Q|_{\mathfrak{m}_i}.$$  

For a diagonal metric, the Ricci curvature is not necessarily diagonal, and the off-diagonal terms are given by

$$\text{Ric}(g)(u, v) = \sum_{k,l} \left( \frac{x_k x_l - 2x_k^2 + 2x_k x_l}{4x_k x_l} \right) \sum_{e_\alpha \in \mathfrak{m}_k} Q([u, e_\alpha]_{\mathfrak{m}_i}, [v, e_\alpha]_{\mathfrak{m}_i}), \quad u \in \mathfrak{m}_i, \quad v \in \mathfrak{m}_j,$$
where \( i \neq j \) and the subscript \( m_i \) denotes projection onto \( m_i \); see [20].

For the tensor \( T \), we introduce the constants \( T_i \) such that
\[
T_{|m_i} = T_i Q_{|m_i}.
\]
Varying over all decompositions, these constants determine \( T \) uniquely.

We now recall some formulas for Riemannian submersions which will be useful for us. Let \( K \) be a compact subgroup with Lie algebra \( \mathfrak{k} \) lying between \( H \) and \( G \), i.e., \( H \subset K \subset G \). We then have a homogeneous fibration
\[
K/H \to G/H \to G/K.
\]
We will often consider metrics with respect to which the projection \( G/H \to G/K \) is a Riemannian submersion. Assume that the decomposition \( m = (\mathfrak{g} \cap m) \oplus \mathfrak{g}^\perp \) is orthogonal with respect to both \( Q \) and \( g \). Then \( g \in \mathcal{M} \) is a Riemannian submersion metric if and only if \( g_{\mathfrak{g}^\perp} \) is \( \text{Ad}_K \)-invariant. In this case, \( g_{\mathfrak{g} \cap m} \) can be thought of as a homogeneous metric on the fiber \( F = K/H \), and \( g_{\mathfrak{g}^\perp} \) a homogeneous metric on the base \( B = G/K \). We can introduce a new submersion metric \( g_{s,t} \) on \( G/H \) by scaling the fiber and the base, i.e.,
\[
g_{s,t} = \frac{1}{s} g_F + \frac{1}{t} g_B,
\]
where \( g_F = g_{\mathfrak{g} \cap m} \) and \( g_B = g_{\mathfrak{g}^\perp} \).

One has the following formula for the scalar curvature (see, e.g., [5, Proposition 9.70]):
\[
S(g_{s,t}) = sS_F + tS_B - \frac{t^2}{s} |A|_g,
\]
where \( A \) is the O'Neill tensor of the submersion, while \( S_F \) and \( S_B \) are the scalar curvatures of \( g_F \) and \( g_B \). We will also have use for the Ricci curvature of a submersion metric. According to [5, Proposition 9.70], it is given by
\[
\text{Ric}_{s,t}(u, v) = \begin{cases} 
\text{Ric}_F(u, v) + \frac{t^2}{s} \langle Au, Av \rangle_g & \text{for } u, v \in \mathfrak{g} \cap m, \\
\text{Ric}_B(u, v) - 2 \frac{t}{s} \langle Au, Av \rangle_g & \text{for } u, v \in \mathfrak{g}^\perp, \\
\frac{1}{s} \text{div} A(u, v) & \text{for } u \in \mathfrak{g} \cap m, \ v \in \mathfrak{g}^\perp.
\end{cases}
\]

The proofs of these formulas are pointwise calculations and hence extend to the situation where \( K \) is not compact (i.e., \( G/K \) is not a manifold). In our case it can indeed happen that for some Lie algebra \( \mathfrak{k} \subset \mathfrak{g} \) the connected subgroup \( K \subset G \) with Lie algebra \( \mathfrak{k} \) is not compact. However, this will not affect our discussions.

Let \( N(H) \) denote the normaliser of \( H \). Every element \( n \in N(H) \) acts on \( G/H \) as a diffeomorphism via right translation: \( R_n(gH) = gn^{-1}H \). Combining this with the action by left translation, we obtain an action on the space of metrics via pullback: \( \mathcal{M} \ni g \mapsto (\text{Ad}_n)^*(g) \in \mathcal{M} \). This induces an isometry between \( g \) and \( (\text{Ad}_n)^*(g) \), and hence \( S((\text{Ad}_n)^*(g)) = S(g) \) for the scalar curvature. This also holds for the possibly larger group \( \text{Aut}(G, H) \) of automorphisms of \( G \) that preserve \( H \). Note though that \( S_{|M_L} \) is invariant under \( \text{Aut}(G, H) \), or one of its subgroups, only if \( T \) is as well, in which case the orbit of a critical point consists of further critical points.

The following remark, based on Palais’s principle of symmetric criticality, will be useful for us. If \( T \) is invariant under a subgroup \( L \subset \text{Aut}(G, H) \), and if \( M_L^T \) is the set metrics in \( M_T \) invariant under \( L \), then critical points of the restriction \( S_{|M_L^T} \) are also critical points of \( S_{|M_T} \). Indeed, given \( g \in M_L^T \), the Ricci curvature \( \text{Ric}(g) \) and hence the gradient \( \text{grad} S_{|M_T}(g) \) is invariant under \( L \) (see, e.g., (5.1) below). Consequently, \( \text{grad} S_{|M_T}(g) \) must be tangent to \( M_L^T \).

In the remainder of the paper we will assume for simplicity that \( G, H \) and all the intermediate subgroups are connected. Let us explain why this is in fact not necessary. One only needs to make the following modification. If \( G \) or \( H \) is not connected, we consider only intermediate subalgebras \( \mathfrak{k} \) that are Lie algebras of intermediates subgroups \( H \subset K \subset G \). This is easily seen to be equivalent to saying that \( \mathfrak{k} \) must be invariant under \( \text{Ad}_k \). The proofs of all of our results apply without any changes, and the conclusions are also the same.
Finally, we recall that for a semisimple Lie algebra $\mathfrak{g}$ (and hence for any Lie algebra of a compact non-abelian Lie group) there are only finitely many maximal subalgebras $\mathfrak{k}$ up to conjugacy. In particular, there are only finitely many components of the moduli space of such subalgebras, each one of which is an adjoint orbit $\text{Ad}_G(\mathfrak{k})$ and hence compact.

2. The simplicial complex

In Sections 2–3 we fix the decomposition $D \in \mathcal{D}$ and study the set of metrics $\mathcal{M}^D$ diagonal with respect to $D$.

It will be convenient for us to describe a homogeneous metric in terms of its inverse since this makes the space of metrics precompact. If $g \in \mathcal{M}^D$ is given by (1.3), we set $y_i = \frac{1}{x_i}$ and obtain the following formulas for the scalar curvature and its constraint:

$$\begin{align*}
S(g) &= \frac{1}{2} \sum_i d_i b_i y_i - \frac{1}{4} \sum_{i,j,k} y_i y_j y_k [ijk], \\
\text{tr}_g T &= \sum_i d_i T_i y_i = 1.
\end{align*}$$

We need to study the behavior of $S|_{\mathcal{M}^D}$ at infinity, which means that at least one of the variables $y_i$ goes to 0. It is natural to introduce a simplicial complex and its stratification. Specifically, let

$$\Delta = \Delta^D = \{(y_1, \ldots, y_r) \in \mathbb{R}^r \mid \sum_i d_i T_i y_i = 1 \text{ and } y_i > 0\}.$$  

Notice that the numbers $T_i$, and thus the simplex $\Delta$, depend on the choice of $D$. This simplex is a natural parametrization of the set $\mathcal{M}^D$. We identify a metric $g \in \mathcal{M}^D$ with $y = (y_1, \ldots, y_r) \in \Delta$.

The boundary of $\Delta$ consists of lower-dimensional simplices. For every nonempty proper subset $J$ of the index set $I = \{1, \ldots, r\}$, let

$$\Delta_J = \{y \in \partial \Delta \mid y_i > 0 \text{ for } i \in J, \ y_i = 0 \text{ for } i \in J^c\}.$$  

Thus $\Delta_J$ is a $|J|$-dimensional simplex, which we call a stratum of $\partial \Delta$. The closure of $\Delta_J$ satisfies

$$\bar{\Delta}_J = \bigcup_{J' \subset J} \Delta_{J'},$$  

and we call $\Delta_{J'}$ a stratum adjacent to $\Delta_J$ if $J'$ is a nonempty proper subset of $J$. It will also be useful for us to consider tubular $\epsilon$-neighborhoods of strata for $\epsilon > 0$:

$$T_\epsilon(\Delta_J) = \{y \in \Delta \mid y_i \leq \epsilon \text{ for } i \in J^c\}.$$  

Finally, we associate to each stratum an $\text{Ad}_H$-invariant subspace of $\mathfrak{m}$:

$$\mathfrak{m}_J = \bigoplus_{i \in J} \mathfrak{m}_i.$$

We can fill out the closure $\bar{\Delta}$ with geodesics starting at the center. To this end, consider the unit sphere

$$\mathbb{S} = \{v \in \mathbb{R}^r \mid \sum_i d_i T_i v_i = 0, \ \sum v_i^2 = 1\}$$

of dimension $r - 2$. Define a geodesic $\gamma_v: [0, t_v] \to \bar{\Delta}$ by setting $\gamma_v(t) = v_0 - tv$, where

$$v \in \mathbb{S}, \quad t_v = \frac{1}{r \max_i d_i T_i v_i}, \quad \text{and} \quad v_0 = (v_0, \ldots, v_0) = \left(\frac{1}{rd_1 T_1}, \ldots, \frac{1}{rd_r T_r}\right).$$

The stratification of $\partial \Delta$ induces one of the sphere:

$$\mathbb{S}_J = \{v \in \mathbb{S} \mid \gamma_v(t_v) \in \Delta_J\}.$$  

Our first observation is that we can mark the strata with subalgebras.

**Proposition 2.2.** The functional $S|_{\mathcal{M}^D}$ is bounded from above. Furthermore, for any $v \in \mathbb{S}$ either $S(\gamma_v(t)) \to -\infty$ as $t \to t_v$ or $v \in \mathbb{S}_J$ for some $J$ such that $\mathfrak{h} \oplus \mathfrak{m}_J$ is a subalgebra of $\mathfrak{g}$. 

Proof. Let $A = \frac{\text{max } b_i}{\text{min } T_i}$, which is well-defined since $T_i > 0$ by assumption. We also have $A > 0$ since $G/H$ is not a torus. Then (2.1) implies that $S(g) \leq A$.

For the second claim, let $J$ be the index set with $J^c = \{i \mid v_0 - t_u v_i = 0\}$. Obviously, $J \neq I$ and $t_v > 0$. If $\mathfrak{h} \oplus m_J$ is not a subalgebra, then there exist $i, j \in J$ and $k \in J^c$ such that $[ijk] \neq 0$. Then in formula (2.1), we have a contribution of the form

$$-[ijk] \frac{y_i y_j}{y_k} = -[ijk] \frac{(v_{0i} - t_v v_i)(v_{0j} - t_v v_j)}{v_{0k} - t_v k}$$

with $0 \leq t < t_v$. Since $i, j \in J$ and $k \in J^c$, we know that $(v_{0i} - t_v v_i)(v_{0j} - t_v v_j)$ stays bounded away from 0 and $v_{0k} - t_v k \to 0$ as $t \to t_v$. This implies that $S(\gamma_v(t)) \to -\infty$ as $t \to t_v$. \hfill \Box

We also need to control how fast the scalar curvature goes to $-\infty$. For this purpose we prove the following result.

**Proposition 2.3.** Consider a stratum $\Delta_J$ such that $\mathfrak{h} \oplus m_J$ is not a subalgebra. Then for every $v \in S_J$ and $a > 0$, there exist an open neighbourhood $U(v)$ in $S$ and a positive number $\epsilon(v)$ such that $S(\gamma_v(t)) < -a$ whenever $u \in U(v)$ and $(1 - \epsilon(v))t_u < t < t_u$.

Proof. Let $A = \frac{\text{max } b_i}{\text{min } T_i}$ as before. There exist $i, j \in J$ and $k \in J^c$ such that $[ijk] \neq 0$. Moreover, $(v_{0i} - t_v v_i)(v_{0j} - t_v v_j) > 0$ and $(v_{0k} - t_v k) = 0$. Define

$$\epsilon(v) = \min \left\{ [ijk] \frac{(v_{0i} - t_v v_i^+)(v_{0j} - t_v v_j^+)}{4(A + a)t_v v_k}, \frac{1}{2} \right\},$$

where $v_i^+ = \max\{v_i, 0\}$. Evidently, this quantity is always positive. Choose a neighbourhood $U(v)$ of $v$ in $S$ such that $u_k > 0$ and

$$(v_{0i} - t_u v_i^+)(v_{0j} - t_u v_j^+) > \frac{1}{2}(v_{0i} - t_v v_i^+)(v_{0j} - t_v v_j^+) \quad \text{and}$$

$$v_{0k} - (1 - \epsilon(v))t_u u_k < 2(v_{0k} - (1 - \epsilon(v))t_v v_k) = 2(v_{0k} - t_v v_k + \epsilon(v)t_v v_k) = 2\epsilon(v)t_v v_k$$

for all $u = (u_1, \ldots, u_r) \in U(v)$. This implies

$$S(\gamma_u(t)) \leq A - [ijk] \frac{(v_{0i} - t_u v_i)(v_{0j} - t_v v_j)}{v_{0k} - t_v k} < A - [ijk] \frac{(v_{0i} - t_u v_i^+)(v_{0j} - t_u v_j^+)}{v_{0k} - (1 - \epsilon(v))t_u u_k} \leq -a,$$

provided $u \in U(v)$ and $(1 - \epsilon(v))t_u < t < t_u$. \hfill \Box

If the space $G/H$ has pairwise inequivalent isotropy summands, Proposition 2.3 implies the following result, originally proved in [31].

**Corollary 2.4.** If $\mathfrak{h}$ is maximal in $\mathfrak{g}$, then $S_{|\mathcal{M}T}$ attains its global maximum at a metric $g \in \mathcal{M}T$, and hence Ric$(g) = cT$ for some $c > 0$.

We can add a marking to the strata in $\partial \Delta$. If $\mathfrak{h} \oplus m_J = \mathfrak{k}$ is a subalgebra, we denote the stratum $\Delta_J$ by $(\Delta_J, \mathfrak{k})$ or simply $\Delta_\mathfrak{k}$; if it is not, we denote the stratum by $(\Delta_J, \infty)$ or $\Delta_\infty$.

Next, we need an estimate for $S$ near $(\Delta_J, \mathfrak{k})$. Let $K$ be the connected subgroup of $G$ with Lie algebra $\mathfrak{k}$. Define

$$\alpha_D^T = \sup\{S(h) \mid h \in \mathcal{M}D(K/H) \text{ with } \text{tr}_h T_{|\mathfrak{h} = \mathfrak{m}} = 1\},$$

where the letter $D$ is preserved for the decomposition of the tangent space to $K/H$ induced by $D$. Since a normal homogeneous metric has non-negative scalar curvature, $\alpha_D^T \geq 0$. Also, $\alpha_D^T = 0$ if and only if $K/H$ is a torus. It is important for us to note that, since $\Delta_\mathfrak{k}$ is not closed in general, the supremum in the definition of $\alpha_D^T$ may not be achieved in $\Delta_\mathfrak{k}$, which will complicate our discussion.
Consider a metric in $\mathcal{M}_I^D$ identified with $y = (y_1, \ldots, y_r) \in \Delta$. If $\mathfrak{t} = \mathfrak{h} \oplus \mathfrak{m}_J$ for some $J \subset I$, then

$$y = y|_{\mathfrak{t} \cap \mathfrak{m}} + y|_{\mathfrak{t}^\perp}, \quad y|_{\mathfrak{t} \cap \mathfrak{m}} = \sum_{i \in J} \frac{1}{y_i} Q_{|\mathfrak{m}|}, \quad y|_{\mathfrak{t}^\perp} = \sum_{i \in J^c} \frac{1}{y_i} Q_{|\mathfrak{m}|}.$$ 

We may regard $y|_{\mathfrak{t} \cap \mathfrak{m}}$ as a metric on $K/H$. Its scalar curvature is given by

$$S(y|_{\mathfrak{t} \cap \mathfrak{m}}) = \frac{1}{2} \sum_{i \in J} d_i \bar{b}_i y_i - \frac{1}{4} \sum_{i,j,k \in J} [ijk] \frac{y_i y_j}{y_k},$$

where the Killing form of $K$ restricted to $\mathfrak{m}_i$ equals $-\bar{b}_i Q_{|\mathfrak{m}_i}$. One easily shows that

$$\bar{b}_i = b_i - \sum_{j,k \in J^c} \frac{[ijk]}{d_i}.$$ 

**Proposition 2.5.** Consider a stratum $(\Delta_J, \mathfrak{t})$. If $y \in \mathcal{M}_I^D$ satisfies $\max_{i \in J^c} y_i \leq \epsilon$, then

$$S(y) \leq \alpha^D_\mathfrak{t} + \epsilon \sum_{i \in J^c} \frac{d_i b_i}{2}.$$ 

**Proof.** We break up the formula for the scalar curvature in (2.1) as follows, using the assumption that $[ijk] = 0$ for $i, j \in J$ and $k \in J^c$:

$$S(y) = \frac{1}{2} \sum_{i \in J} d_i b_i y_i + \frac{1}{2} \sum_{i \in J^c} d_i b_i y_i - \frac{1}{4} \sum_{i,j,k \in J} [ijk] \frac{y_i y_j}{y_k}$$

$$- \frac{1}{2} \sum_{i \in J} \sum_{j,k \in J^c} [ijk] \frac{y_i y_j}{y_k} - \frac{1}{4} \sum_{i,j,k \in J} [ijk] \frac{y_j y_k}{y_i} - \frac{1}{4} \sum_{i,j,k \in J} [ijk] \frac{y_i y_j}{y_k}$$

$$\leq \frac{1}{2} \sum_{i \in J} d_i b_i y_i + \frac{1}{2} \sum_{i \in J} \sum_{j,k \in J^c} [ijk] y_i + \frac{1}{2} \sum_{i \in J^c} d_i b_i y_i - \frac{1}{4} \sum_{i,j,k \in J} [ijk] \left( \frac{y_j}{y_k} + \frac{y_k}{y_j} \right)$$

$$\leq S(y|_{\mathfrak{t} \cap \mathfrak{m}}) + \frac{1}{2} \sum_{i \in J^c} d_i b_i y_i,$$

where in the last step we used the estimate $\frac{y_i}{y_k} + \frac{y_k}{y_j} \geq 2$. Now observe that

$$S(y|_{\mathfrak{t} \cap \mathfrak{m}}) = (\text{tr}_{y|_{\mathfrak{t} \cap \mathfrak{m}}} T|_{\mathfrak{t} \cap \mathfrak{m}}) S((\text{tr}_{y|_{\mathfrak{t} \cap \mathfrak{m}}} T|_{\mathfrak{t} \cap \mathfrak{m}}) y|_{\mathfrak{t} \cap \mathfrak{m}}) < S((\text{tr}_{y|_{\mathfrak{t} \cap \mathfrak{m}}} T|_{\mathfrak{t} \cap \mathfrak{m}}) y|_{\mathfrak{t} \cap \mathfrak{m}}) \leq \alpha^D_\mathfrak{t}$$

since $\text{tr}_{y|_{\mathfrak{t} \cap \mathfrak{m}}} T|_{\mathfrak{t} \cap \mathfrak{m}} < \text{tr}_y T = 1$ and the trace of $T|_{\mathfrak{t} \cap \mathfrak{m}}$ with respect to $(\text{tr}_{y|_{\mathfrak{t} \cap \mathfrak{m}}} T|_{\mathfrak{t} \cap \mathfrak{m}}) y|_{\mathfrak{t} \cap \mathfrak{m}}$ equals 1. Consequently,

$$S(y) < \alpha^D_\mathfrak{t} + \frac{1}{2} \sum_{i \in J^c} d_i b_i y_i.$$ 

When $y_i < \epsilon$ for all $i \in J^c$, we get the desired result. \qed

We can reformulate Proposition 2.5 as follows.

**Corollary 2.6.** Let $\Delta_\mathfrak{t}^D$ be a subalgebra stratum. Then for every $a > \alpha^D_\mathfrak{t}$ there exists a constant $\epsilon > 0$ such that the set $\{g \in \mathcal{M}_I^D \mid S(g) \geq a\}$ does not intersect $T_\epsilon(\Delta_\mathfrak{t})$.

Combining Propositions 2.3 and 2.5, we arrive at the following conclusion.
Corollary 2.7. Suppose \( a > \alpha^D_\ell \) for every subalgebra stratum \( \Delta_\ell \). Then \( \{ g \in M^D_T \mid S(g) \geq a \} \) is a (possibly empty) compact subset of \( M^D_T \).

Proposition 2.5 shows that \( \alpha^D_\ell \) is an upper bound for the possible values of the scalar curvature as we approach points in \( \Delta_\ell \). However, it is important to keep in mind that \( S \) does not, in general, extend continuously to the closure of \( \Delta \).

3. Riemannian submersions

In this section we study the behavior of \( S \) near the subalgebra stratum \( \Delta_\ell \) geometrically. It will be more convenient to choose the path \( g_t \) below instead of \( \gamma_\ell(t) \) since we can then use formula (1.7) for Riemannian submersions. The goal is to see if there are metrics near \( \Delta_\ell \) whose scalar curvature is larger than \( \alpha^D_\ell \). As before, suppose \( K \) is an intermediate connected subgroup with Lie algebra \( \mathfrak{g} \) and associated stratum \( \Delta_\ell \). Thus \( \ell = h \oplus \mathfrak{m}_J \) for some \( J \subset I \). Define

\[
\beta^D_\ell = \sup \{ S(h) \mid h \in M^D(G/K) \mbox{ with } \text{tr}_h T_{|\pi|} = 1 \}.
\]

We will show that \( \beta_\ell - \alpha_\ell \) controls the desired behavior.

We have the homogeneous fibration

\[
F = K/H \to G/H \to G/K = B.
\]

Let us consider metrics on \( G/H \) for which this fibration is a Riemannian submersion. We start with a metric of the form

\[
g = g_F + g_B = \sum_{i \in J} \frac{1}{y_i} Q_{|m_i} + \sum_{i \in J^c} \frac{1}{y_i} Q_{|m_i}.
\]

Assume that \( g \) lies in \( M_T \), i.e., \( T_1^* + T_2^* = 1 \), where

\[
T_1^* = \text{tr}_{g_F} T_{|\mathfrak{m}} = \sum_{i \in J} d_i y_i T_i, \quad T_2^* = \text{tr}_{g_B} T_{|\pi} = \sum_{i \in J^c} d_i y_i T_i.
\]

We also require the metric \( g_B \) to be Ad\(_K\)-invariant so that the projection in (3.1) is a Riemannian submersion with \( g_F \) and \( g_B \) the metrics on the fiber and the base.

Consider the two-parameter family

\[
g_{s,t} = \frac{1}{s} g_F + \frac{t}{t} g_B \quad \text{with} \quad sT_1^* + tT_2^* = 1.
\]

Substituting \( s = \frac{1-tT_2^*}{T_1^*} \), we obtain a one-parameter family of metrics

\[
g_t = \frac{T_1^*}{1-tT_2^*} g_F + \frac{1}{t} g_B
\]

lying in \( M^D_T \). We call this the canonical variation associated to \( K \). By (1.7), the scalar curvature of \( g_t \) is

\[
S(g_t) = s S_F + t S_B - \frac{t^2}{s} |A|_g = \frac{S_F}{T_1^*} + T_2^* \left( \frac{S_B}{T_2^*} - \frac{S_F}{T_1^*} \right) t - \frac{t^2 T_2^*}{1-tT_2^*} |A|_g.
\]

Since \( \lim_{t \to 0} g_t = \frac{g_F}{T_1^*} \in \Delta_\ell \), every point in \( \Delta_\ell \) is a limit of such a path \( g_t \). Thus we have

\[
\lim_{t \to 0} S(g_t) = \frac{S_F}{T_1^*} \quad \text{and} \quad \lim_{t \to 0} \frac{dS(g_t)}{dt} = T_2^* \left( \frac{S_B}{T_2^*} - \frac{S_F}{T_1^*} \right).
\]

Notice that

\[
\frac{S_F}{T_1^*} = S(T_1^* g_F) \leq \alpha^D_\ell \quad \text{and} \quad \frac{S_B}{T_2^*} = S(T_2^* g_B) \leq \beta^D_\ell.
\]
We now use these formulas to understand the relationship between the numbers $\alpha^D_t$ corresponding to different strata.

**Proposition 3.5.** If $\Delta_{\mathfrak{t}'}$ is a stratum adjacent to $\Delta_{\mathfrak{t}}$ with $\mathfrak{t}' \subset \mathfrak{t}$, then $\alpha^D_{\mathfrak{t}'} \leq \alpha^D_{\mathfrak{t}}$.

**Proof.** Let $K'$ be the subgroup of $G$ with Lie algebra $\mathfrak{t}'$. Thus $H \subset K' \subset K \subset G$. Given $h \in \mathcal{M}^D(K'/H)$ with $\text{tr}_h T_{|\mathfrak{t}' \cap \mathfrak{m}} = 1$, define a one-parameter family of metrics

$$h_t = \frac{1}{1 - t \sum_i d_i T_i} h + \frac{1}{t} Q_{\mathfrak{t}' \cap \mathfrak{t}} \in \mathcal{M}^D(K/H),$$

where the sum is taken over all $i$ with $m_i \subset \mathfrak{t}' \cap \mathfrak{t}$. Applying (3.4) to the homogeneous fibration $K'/H \to K/H \to K/K'$, we conclude that

$$\lim_{t \to 0} S(h_t) = S(h).$$

This means that, for every $h \in \mathcal{M}^D(K'/H)$ with $\text{tr}_h T_{|\mathfrak{t}' \cap \mathfrak{m}} = 1$, there exists a metric $g \in \mathcal{M}^D(K/H)$ with $\text{tr}_g T_{|\mathfrak{t} \cap \mathfrak{m}} = 1$ and scalar curvature arbitrarily close to $S(h)$. \qed

As we noted in Section 3, it is possible that the supremum in the definition of $\alpha^D_{\mathfrak{t}}$ is not attained by a metric in $\Delta_{\mathfrak{t}}$. In this case, we have the following result.

**Proposition 3.6.** Assume that $\alpha^D_{\mathfrak{t}}$ is not attained. Then there exists an adjacent stratum $\Delta_{\mathfrak{t}'}$ such that $\alpha^D_{\mathfrak{t}'} = \alpha^D_{\mathfrak{t}}$ and $S(h) = \alpha^D_{\mathfrak{t}'}$ for some $h \in \mathcal{M}^D(K'/H)$ with $\text{tr}_h T_{|\mathfrak{t}' \cap \mathfrak{m}} = 1$.

**Proof.** Since $\alpha^D_{\mathfrak{t}}$ is not attained, it is possible to find a sequence $h_i \in \mathcal{M}(K/H)$ with $\lim_{i \to \infty} S(h_i) = \alpha^D_{\mathfrak{t}}$ converging to some $h \in \Delta_{\mathfrak{t}'}$, where the stratum $\Delta_{\mathfrak{t}'}$ is adjacent to $(\Delta_{\mathfrak{t}'}, \mathfrak{t})$. Applying Proposition 2.3 to the homogeneous space $K/H$ and using the nonnegativity of $\alpha^D_{\mathfrak{t}'}$, we conclude that $\Delta_{\mathfrak{t}'} = \Delta_{\mathfrak{t}}$ for some $\mathfrak{t}' \subset \mathfrak{t}$. Similarly, applying Proposition 2.5 to $K/H$ shows that

$$\alpha^D_{\mathfrak{t}} = \lim_{i \to \infty} S(h_i) \leq \alpha^D_{\mathfrak{t}'}.$$ 

In light of Proposition 3.5, this means $\alpha^D_{\mathfrak{t}'} = \alpha^D_{\mathfrak{t}}$. If the supremum $\alpha^D_{\mathfrak{t}'}$ is attained, then we are done. Otherwise, we repeat the argument until we reach a subalgebra $\mathfrak{t}''$ for which $\alpha^D_{\mathfrak{t}''}$ is achieved. By Corollary 2.4, this will be the case at the latest for a subalgebra $\mathfrak{t}''$ in which $\mathfrak{h}$ is maximal. \qed

Finally, we show how the difference $\beta^D_{\mathfrak{t}} - \alpha^D_{\mathfrak{t}}$ controls the behavior of the scalar curvature functional.

**Proposition 3.7.** Consider a subalgebra stratum $\Delta_{\mathfrak{t}}$ such that $\alpha^D_{\mathfrak{t}}$ is attained. If $\beta^D_{\mathfrak{t}} - \alpha^D_{\mathfrak{t}} > 0$, then there exists a metric $g \in \Delta$, arbitrarily close to $\Delta_{\mathfrak{t}}$, with $S(g) > \alpha^D_{\mathfrak{t}}$.

**Proof.** Choose $g_F \in \mathcal{M}^D(K/H)$ such that $\text{tr}_{g_F} T_{|\mathfrak{t} \cap \mathfrak{m}} = 1$ and $S(g_F) = \alpha^D_{\mathfrak{t}}$. If $\beta^D_{\mathfrak{t}} - \alpha^D_{\mathfrak{t}} > 0$, it is possible to find $g_B \in \mathcal{M}^D(G/K)$ with $\text{tr}_{g_B} T_{|\mathfrak{t}' \cap \mathfrak{m}} = 1$ and $S(g_B) - \alpha^D_{\mathfrak{t}} > 0$. Consider the metric $g \in \mathcal{M}^D_T(G/H)$ given by

$$g = 2(g_F + g_B).$$

If we let $g_t$ be the canonical variation as in (3.2), we conclude from (3.4) that

$$\lim_{t \to 0} S(g_t) = S(g_F) = \alpha^D_{\mathfrak{t}} \quad \text{and} \quad \lim_{t \to 0} \frac{dS(g_t)}{dt} = \frac{1}{2} (S(g_B) - \alpha^D_{\mathfrak{t}}) > 0.$$ 

Clearly, $S(g_t) > \alpha^D_{\mathfrak{t}}$ for small $t$. \qed
Combining Propositions 3.6 and 3.7 implies our first main theorem if there exists only one decomposition \( D \) (up to order of summands).

**Proposition 3.8.** Assume that \( G/H \) is a compact homogeneous space such that the modules \( m_i \) are inequivalent. Let \( \ell \) be an intermediate subalgebra of the lowest possible dimension such that 
\[
\alpha^D_\ell = \sup_{\ell'} \alpha^{D'}_{\ell'},
\]
where the supremum is taken over all intermediate subalgebras \( \ell' \). If \( \beta^D_\ell - \alpha^D_\ell > 0 \), then \( S|_{\mathcal{M}_T} \) achieves its maximum at some metric \( g \in \mathcal{M}_T \), and hence \( \text{Ric}(g) = cT \) for some \( c > 0 \).

It is natural to add an additional marking to the strata of \( \partial \Delta \) by labeling a subgroup stratum \((\Delta_\ell, \alpha_\ell, \beta_\ell)\), which encodes the behavior of \( S \) in a neighborhood of \( \Delta_\ell \).

### 4. Global maxima

From now on, we allow the decomposition \( D \) to vary. Clearly, the numbers \( b_i \) and the structure constants \([ijk]\) depend continuously on \( D \). Recall also that the space \( D \) of all decompositions is compact.

Consider an intermediate subgroup \( K \) with Lie algebra \( \mathfrak{k} \). The numbers \( \alpha^D_\ell \) and \( \beta^D_\ell \) introduced above depend on the choice of the decomposition \( D \). Removing this dependence, we define 
\[
\alpha_{G/H} = \sup_{\ell} \alpha^D_\ell,
\]
where the supremum is taken over all intermediate subalgebras \( \ell \).

Our first goal is to extend Propositions 2.3 and 2.5 to all of \( \mathcal{M}_T \). We will use the following parametrisation of the space \( \mathcal{M} \), convenient in our context. Specifically, consider the map \( \sigma : \mathbb{R}^n \times D \to \mathcal{M} \) defined by 
\[
\sigma(y, D) = \frac{1}{y_1} Q_{m_1} + \cdots + \frac{1}{y_t} Q_{m_t},
\]
where \( y = (y_1, \ldots, y_t) \) and \( D \) is the decomposition with irreducible modules \( m_1, \ldots, m_t \). This map is clearly continuous. While it is surjective, it may not be injective. The preimages of some metrics are infinite when some of the isotropy summands of \( G/H \) are equivalent. However, given a metric \( g \in \mathcal{M} \), the preimage \( \sigma^{-1}(g) \) is compact.

To state our next result, we fix a point \((y, D)\) in the boundary of \( \sigma^{-1}(\mathcal{M}_T) \). Let \( \Delta \) be the simplex associated with the decomposition \( D \). Clearly, \( y \) lies in a stratum \( \Delta_J \) for some \( J \subset I \). The following result generalises Proposition 2.5 to all of \( \mathcal{M}_T \).

**Proposition 4.1.** Assume that \( y \) lies in a subalgebra stratum \((\Delta_J, \mathfrak{k})\). Then for every \( \epsilon > 0 \) there exists an open neighbourhood \( U \) of \((y, D)\) in \( \mathbb{R}^n \times D \) such that 
\[
S(\sigma(y', D')) \leq \alpha_\ell + \epsilon
\]
whenever \((y', D') \in U \cap \sigma^{-1}(\mathcal{M}_T)\).

**Proof.** We use the notation \( b_i, T_i \) and \([ijk]\) (respectively, \( b'_i, T'_i \) and \([ijk]'\)) for the constants associated with the decomposition \( D \) (respectively, \( D' \)). Given \( \delta > 0 \), there exists a neighborhood \( U_\delta \) of \((y, D)\) in \( \mathbb{R}^n \times D \) such that 
\[
|y_i - y'_i| + |b_i - b'_i| + |T_i - T'_i| + ||ijk| - |ijk'|| < \delta
\]
for all $i, j, k$ whenever $(y', D') \in U_\delta \cap \sigma^{-1}(M_T)$. Let us choose $\delta$ small enough to ensure that $y'_i > \frac{y_i}{2}$ in this set. The trace constraint implies that $y'_i < \frac{1}{d_i T_i}$. Notice also that there exist common lower and upper bounds for $T_i$ independent of $D$.

If $(y', D') \in U_\delta \cap \sigma^{-1}(M_T)$, we find, as in the proof of Proposition 2.5, that

$$\begin{align*}
S(\sigma(y', D')) &= \frac{1}{2} \sum_i d_i b'_i y'_i - \frac{1}{4} \sum_{i,j,k} y'_i y'_j [ijk]' \\
&\leq \frac{1}{2} \sum_{i \in J} \left( d_i b'_i - \sum_{j,k \in J} [ijk]' \right) y'_i - \frac{1}{4} \sum_{i,j,k \in J} \frac{y'_i y'_j [ijk]'}{y_k} + \frac{1}{2} \sum_{i \in J} d_i b'_i y'_i \\
&\leq \frac{1}{2} \sum_{i \in J} d_i b'_i y'_i - \frac{1}{4} \sum_{i,j,k \in J} \frac{y'_i y'_j [ijk]'}{y_k} + \frac{1}{2} \sum_{i \in J} d_i (b'_i - b_i) y'_i \\
&\quad - \frac{1}{2} \sum_{i \in J} \sum_{j,k \in J} ([ijk]' - [ijk]) y'_i - \frac{1}{4} \sum_{i,j,k \in J} \frac{y'_i y'_j ([ijk]' - [ijk])}{y_k} + \frac{1}{2} \sum_{i \in J} d_i b'_i [y'_i - y_i].
\end{align*}$$

Consequently, for small enough $\delta$, since $y_i = 0$ when $i \in J^c$, we have

$$\begin{align*}
S(\sigma(y', D')) &\leq S(\sigma(y', D)|_{T^\alpha \cap m}) + \frac{1}{2} \sum_{i \in J} \frac{|b'_i - b_i|}{T_i'} \\
&\quad + \frac{1}{2} \sum_{i \in J} \sum_{j,k \in J^c} \frac{|[ijk]' - [ijk]|}{d_i T_i'} + \frac{1}{2} \sum_{i,j,k \in J} \frac{|[ijk]' - [ijk]|}{d_i d_j T_i' T_j y_k} + \frac{1}{2} \sum_{i \in J} d_i b'_i |y'_i - y_i| \\
&\leq S(\sigma(y', D)|_{T^\alpha \cap m}) + \frac{\epsilon}{2}.
\end{align*}$$

Shrinking $\delta$ further if necessary and using the continuity of the scalar curvature, we conclude that

$$S(\sigma(y', D)|_{T^\alpha \cap m}) < S(\sigma(y, D)) + \frac{\epsilon}{2} \leq \alpha_T + \frac{\epsilon}{2},$$

which implies the result.

□

Using a similar (but simpler) proof, we can generalize Proposition 2.3.

**Proposition 4.2.** Assume that $(y, D)$ lies in a stratum $(\Delta, \infty)$. Given $a > 0$, there exists a neighbourhood $U$ of $(y, D)$ in $\mathbb{R}^r \times D$ such that $S(\sigma(y', D')) < -a$ whenever $(y', D') \in U \cap \sigma^{-1}(M_T)$.

The precompactness of $\sigma^{-1}(M_T) \subset \mathbb{R}^r \times D$ and Propositions 4.1 and 4.2 yield the following extension of Corollary 2.7 to all of $M_T$.

**Corollary 4.3.** If $a > \alpha_{G/H}$, then $\{ g \in M_T \mid S(g) \geq a \}$ is a compact subset of $M_T$.

Our next result generalises Proposition 3.6.

**Proposition 4.4.** Assume that $h$ is not maximal in $\mathfrak{g}$. Then there exists an intermediate subgroup $K$ with Lie algebra $\mathfrak{k}$ such that $\alpha_T = \alpha_{G/H}$. If $K$ has the least possible dimension of all such subgroups, then there exists $h \in M(K/H)$ with

$$S(h) = \alpha_T \quad \text{and} \quad \text{tr}_h T|_{T^\alpha \cap m} = 1.$$
We may assume that \( \mathfrak{t}_i \) converge to an intermediate subalgebra \( \mathfrak{k} \) and that \( \dim \mathfrak{t}_i = \dim \mathfrak{k} \) for all \( i \). Our goal is to show that a subsequence of \( (h_i) \) converges to a metric \( h \in \mathcal{M}(K/H) \).

Consider a decomposition \( D_i \in \mathcal{D} \) with modules \( m_1^i, \ldots, m_r^i \) such that
\[
\mathfrak{t}_i = \mathfrak{h} \oplus \bigoplus_{j \in J_i} m_j^i \quad \text{and} \quad h_i(m_k^i, m_l^i) = 0
\]
for some \( J_i \subset I \) and all \( k, l \in J_i \) with \( k \neq l \). Passing to a subsequence if necessary, we may assume that these decompositions converge to some \( D \in \mathcal{D} \) with modules \( m_1, \ldots, m_r \) and that \( J_i \) does not depend on \( i \) (thus we can omit the index \( i \) from the notation \( J_i \)). Let \( K \) be the connected subgroup of \( G \) with Lie algebra \( \mathfrak{k} = \mathfrak{h} \oplus m_J \). There exist positive numbers \( x_{ji} \) such that
\[
h_i = \sum_{j \in J} x_{ji} Q_{|m_j^i|}.
\]
Note that \( \alpha_{G/H} > \sup_\ell \alpha_\ell \), where the supremum is taken over all intermediate subalgebras \( \ell \) between \( \mathfrak{h} \) and \( \mathfrak{k} \). Indeed, if not, there exists a subalgebra \( \ell \) with \( \alpha_\ell > \alpha_{G/H} - \frac{1}{2} \), contradicting the assumption that \( K_i \) is chosen to be of smallest possible dimension. Therefore,
\[
S(h_i) > \alpha_{G/H} - \frac{1}{2} > \sup_\ell \alpha_\ell
\]
for large \( i \). We now claim that the constants \( x_{ji} \) all lie in some compact subset of \( \mathbb{R}_+ \). To see this, we can argue as in Propositions 4.1 and 4.2 and Corollary 4.3 replacing \( G/H \) with the sequence \((K_i/H)\) and using the fact that the structure constants of \( \mathfrak{k}_i \) converge to those of \( \mathfrak{k} \).

Thus, passing to a subsequence if necessary, we may assume that
\[
\lim_{i \to \infty} x_{ji} = x_j \in \mathbb{R}_+
\]
for \( j \in J \). The metric \( h = \sum_{i \in J} x_i Q_{|m_i^j|} \) satisfies \( S(h) = \alpha_\mathfrak{k} = \alpha_{G/H} \) and \( \text{tr}_h T|_{\mathcal{R}^m} = 1 \). \( \square \)

We are now ready to prove our first main theorem.

**Proof of Theorem A.** The existence of the subgroup \( K \) follows from Proposition 4.4. By Proposition 3.7, there exists \( \epsilon > 0 \) such that the superlevel set \( \{ g \in \mathcal{M}_T \mid S(g) \geq \alpha_\mathfrak{t} + \epsilon \} \) is nonempty. Corollary 4.3 implies that this set is also compact. Consequently, \( S|_{\mathcal{M}_T} \) assumes its maximum at a metric \( g \in \mathcal{M}_T \). As a critical point, such a metric satisfies \( \text{Ric}(g) = cT \) for some constant \( c > 0 \). \( \square \)

5. Palais–Smale sequences

In order to find critical points which are not global maxima via mountain pass techniques, one needs certain compactness properties for the functional \( S|_{\mathcal{M}_T} \). Normally, this is achieved by verifying the Palais–Smale condition. However, as we demonstrate in Corollary 5.5 below, this condition cannot be satisfied for \( S|_{\mathcal{M}_T} \) unless \( H \) is maximal in \( G \). Even so, understanding the properties of divergent Palais–Smale sequences in \( \mathcal{M}_T \) is an important ingredient in the variational analysis of \( S|_{\mathcal{M}_T} \). We establish one such property in Theorem 5.7, which will help us prove the existence of saddle points for \( S|_{\mathcal{M}_T} \) on generalised Wallach spaces in Section 6.

A sequence of metrics \( (g_i) \subset \mathcal{M}_T \) is called a **Palais–Smale sequence** if \( S(g_i) \) is bounded and
\[
\lim_{i \to \infty} |\text{grad} S|_{\mathcal{M}_T}(g_i)|_{g_i} = 0,
\]
where the gradient and the norm are taken with respect to the metric on the tensor bundle of \( M \) induced by \( g_i \). The Palais–Smale condition is satisfied if every such sequence has a convergent subsequence. Note that we may also assume, by going to a subsequence, that \( S(g_i) \) converges to some \( \lambda \in \mathbb{R} \) as \( i \to \infty \).
To clarify this condition, we start by computing $\text{grad} S|_{\mathcal{M}_T}$. It is well known (see, e.g., [5, Proposition 4.17]) that the gradient of the functional $S: \mathcal{M} \to \mathbb{R}$ satisfies

$$\text{grad} S(g) = - \text{Ric}(g).$$

The tangent space of $\mathcal{M}_T$ at the metric $g$ consists of those $(0,2)$-tensor fields $h$ on $G/H$ for which $\langle h, T \rangle_g = 0$. Projecting $\text{grad} S$ onto this space, we find

$$\text{grad} S|_{\mathcal{M}_T}(g) = - \text{Ric}(g) + \frac{\langle \text{Ric}(g), T \rangle_g}{\langle T, T \rangle_g} T.$$  

(5.1)

This formula can be used to prove Proposition 1.2: $\text{grad} S|_{\mathcal{M}_T}(g) = 0$ if and only if $\text{Ric}(g) = cT$ for some constant $c$.

Let $(g_i)$ be a Palais–Smale sequence with $\lim_{i \to \infty} S(g_i) = \lambda$. Substituting $g_i$ into (5.1), taking the trace with respect to $g_i$, and using $\text{tr}_{g_i} T = 1$, it follows that

$$\lim_{i \to \infty} \frac{\langle \text{Ric}(g_i), T \rangle_{g_i}}{\langle T, T \rangle_{g_i}} = \lambda.$$  

Altogether, we see that

$$\lim_{i \to \infty} S(g_i) = \lambda \quad \text{and} \quad \lim_{i \to \infty} |\text{Ric}(g_i) - \lambda T|_{g_i} = 0.  $$

(5.2)

Our first result in this section demonstrates that divergent Palais–Smale sequences exist along canonical variations unless $H$ is maximal in $G$. The discussion in Section 7 shows that they can appear along other curves as well.

**Theorem 5.3.** Consider the functional $S|_{\mathcal{M}_T}$ for $T$ positive-definite. Let $K$ be an intermediate subgroup between $H$ and $G$. Assume that the homogeneous space $K/H$ supports a $K$-invariant metric $g_F$ such that

$$\text{Ric}(g_F) = \lambda T|_{K/H} \quad \text{and} \quad \text{tr}_{g_F} T|_{K/H} = \frac{1}{2},$$

(5.4)

for some $\lambda \geq 0$. If we choose a $G$-invariant metric $g_B$ on $G/K$ with $\text{tr}_{g_B} T|_{G/K} = \frac{1}{2}$, then for the canonical variation $g_t$ in (3.2) the metrics $g_{t_1/2}$ form a divergent Palais–Smale sequence with $\lim_{i \to \infty} S(g_{t_1/2}) = \lambda$.

**Proof.** Let $g = g_F + g_B$ be the metric on $G/H$ that gives rise to the canonical variation $g_t$ given by (3.2). Taking the trace of the first equality in (5.4), we find $S(g_F) = \frac{\lambda}{2}$. In light of (3.4),

$$\lim_{t \to 0} S(g_t) = \frac{S(g_F)}{\text{tr}_{g_F} T|_{K/H}} = \lambda.$$  

It remains to show that $|\text{grad} S|_{\mathcal{M}_T}(g_t)|_{g_t}$ goes to 0. Denote by $\text{Ric}_t$ the Ricci curvature of $g_t$. From (1.8) it follows that

$$\text{Ric}_t(u, v) = \begin{cases} 
\text{Ric}_F(u, v) + \frac{t^2}{(2t-1)^2} \langle Au, Av \rangle_g & \text{for } u, v \in \mathfrak{k} \cap \mathfrak{m}, \\
\text{Ric}_B(u, v) - \frac{2t}{2t-1} \langle Au, Av \rangle_g & \text{for } u, v \in \mathfrak{k}^\perp, \\
\frac{t}{2t-1} \text{div} A(u, v) & \text{for } u \in \mathfrak{k} \cap \mathfrak{m}, \ v \in \mathfrak{k}^\perp.
\end{cases}$$

Consequently,

$$\lim_{t \to 0} |\text{Ric}_t - \pi^* \text{Ric}_F|_{g_t} = 0,$$
and Proposition 4.2. It remains to show that the metric $g$ there exists a decomposition

$$
\text{proof.}
$$

Furthermore, the restrictions of the metrics $g$ given by (5.8) satisfies (5.9).

We conclude that $(g_{1/i})$ is a divergent Palais–Smale sequence. □

Choosing an intermediate subgroup $K$ so that $H$ is maximal in $K$, we can use [31, Theorem 1.1] and Theorem 5.3 to conclude the following.

**Corollary 5.5.** Assume that $H$ is not maximal in $G$. Then the functional $S_{\mathcal{M}_T}$ admits a divergent Palais–Smale sequence.

If $(y_i) \subset \mathcal{M}_T$ satisfies

$$
\lim_{i \to \infty} S(y_i) = 0 \quad \text{and} \quad \lim_{i \to \infty} |\text{grad} S_{\mathcal{M}_T}(y_i)|_{y_i} = 0,
$$

it is called a 0-Palais–Smale sequence. The properties of such sequences for the functional $S_{\mathcal{M}_i}$ have been studied extensively; see, e.g., [8, 28]. As a consequence of Theorem 5.3 we have the following result.

**Corollary 5.6.** If the homogeneous space $G/H$ admits an intermediate subgroup $K$ such that $K/H$ is a torus, then $S_{\mathcal{M}_T}$ admits a divergent 0-Palais–Smale sequence.

Our next goal is to establish a useful property of divergent Palais–Smale sequences in $\mathcal{M}_T$. This property will help us prove the existence of saddle points of $S_{\mathcal{M}_T}$ via mountain pass techniques in Section 6. As we demonstrate below, one can also use it to obtain a partial converse of Corollary 5.6.

Let $(g_i) \subset \mathcal{M}_T$ be a divergent Palais–Smale sequence with $\lim_{i \to \infty} S(g_i) = \lambda$. For every $i \in \mathbb{N}$, there exists a decomposition $D_i \in \mathcal{D}$ with modules $m_1^i, \ldots, m_r^i$ such that

$$
g_i = \frac{1}{y_{i1}} Q_{m_1^i} + \frac{1}{y_{i2}} Q_{m_2^i} + \cdots + \frac{1}{y_{ir}} Q_{m_r^i}
$$

for some $y_{i1}, \ldots, y_{ir} > 0$, i.e., $g_i = \sigma((y_{i1}, \ldots, y_{ir}), D_i)$ with the map $\sigma$ defined in Section 4. For brevity, we write simply $g = ((y_{11}, \ldots, y_{1r}), D_i)$. Passing to a subsequence if necessary, we may assume that $D_i$ converges to a decomposition $D$ and the sequence $((y_{i1}, \ldots, y_{ir}), D_i))$ converges to a point $((y_1, \ldots, y_r), D)$ in the closure of $\sigma^{-1}(\mathcal{M}_T)$. Since $(g_i)$ diverges, $(y_1, \ldots, y_r)$ must lie in the boundary of the simplex $\Delta$ associated with $D$.

**Theorem 5.7.** Let $(g_i)$ be a divergent Palais–Smale sequence with $\lim_{i \to \infty} S(g_i) = \lambda$. Then there exists an intermediate subgroup $K$ with Lie algebra $\mathfrak{k}$ such that the limit point $(y_1, \ldots, y_r)$ lies in the stratum $(\Delta_J, \mathfrak{k})$. If the homogeneous space $G/K$ is isotropy irreducible, then the metric $g_F$ on $K/H$ given by

$$
g_F = \sum_{j \in J} \frac{1}{y_j} Q_{m_j}
$$

satisfies

$$
\text{Ric}(g_F) = \lambda T_{|K/H}|.
$$

Furthermore, the restrictions of the metrics $g_i$ to the space $G/K$ converge, after appropriate re-normalisation, to a $G$-invariant metric on $G/K$.

**Proof.** The existence of $K$ such that $(y_1, \ldots, y_r)$ lies in $(\Delta_J, \mathfrak{k})$ follows from the convergence of $S(g_i)$ and Proposition 4.2. It remains to show that the metric $g_F$ given by (5.8) satisfies (5.9).
Denote by $b_{ji}$, $[jkl]_i$ and $T_{ji}$ the constants associated with the decomposition $D_i$ which converge to the constants $b_j$, $[jkl]$ and $T_j$ associated to $D$. Without loss of generality, assume that the dimension $d_j = \dim m^*_j$ does not depend on $i$. According to (1.5),

\begin{equation}
\text{Ric}(g_i)|_{m^*_j} = \left( \frac{b_{ji}}{2} - \frac{1}{2d_j} \sum_{k<l=1}^r [jkl]_i \frac{y_{li}}{y_{ki}} + \frac{1}{4d_j} \sum_{k<l=1}^r [jkl]_i \frac{y_{ki}y_{li}}{y_{lj}^2} \right)Q|_{m^*_j} \quad \text{for all } j \in I.
\end{equation}

Thus (5.2) implies that

\[ \frac{b_{ji}y_{ji}}{2} - \frac{1}{2d_i} \lim_{i \to \infty} \left( \sum_{k<l=1}^r [jkl]_i \frac{y_{li}y_{lj}}{y_{ki}} - \frac{1}{2} \sum_{k<l=1}^r [jkl]_i \frac{y_{ki}y_{li}}{y_{lj}} \right) = \lambda T_{ji}y_{ji} \quad \text{for } j \in I. \]

Recall also that the limit $\lim_{i \to \infty} y_{ki}$ is $y_k = 0$ if $k \in J^c$ and is positive if $k \in J$. Furthermore, $\lim_{i \to \infty} [jkl]_i = [jkl] = 0$ if $k, l \in J$ and $j \in J^c$. Thus

\begin{equation}
\lim_{i \to \infty} \sum_{k<l=1}^r [jkl]_i \frac{y_{li}}{y_{ki}} < \infty \quad \text{for } j \in J,
\end{equation}

and

\begin{equation}
\lim_{i \to \infty} \sum_{k<l=1}^r [jkl]_i \left( \frac{y_{li}y_{lj}}{y_{ki}} + \frac{y_{ki}y_{li}}{y_{lj}} - \frac{y_{ki}y_{li}}{y_{lj}} \right) = 0 \quad \text{for } j \in J^c.
\end{equation}

In order to show that the metric $g_F$ satisfies (5.9), we need the following lemma. Passing to a subsequence if necessary, we can assume that for all $j, k, l \in I$, the sequences

\[ [jkl] \frac{y_{ki}}{y_{li}}, \quad \frac{[jkl]}{y_{ki}}, \quad \frac{y_{ki}}{y_{li}} \]

are monotone as $i \to \infty$.

**Lemma 5.13.** Assume that $G/K$ is isotropy irreducible. If $u, v \in J$ and $p, q \in J^c$, then

\[ \lim_{i \to \infty} \frac{[puv]}{y_{pi}} = 0 \quad \text{and} \quad \lim_{i \to \infty} \frac{y_{pi}}{y_{qi}} = 1. \]

**Proof.** Using formula (5.11) with $j = u$ and isolating the term with $k = p$ and $l = v$, we see that $\frac{[puv]}{y_{pi}}$ converges to some $\delta_{pav} \geq 0$. Since $G/K$ is isotropy irreducible, there exists, for each fixed $p, q \in J^c$, an index $j_0 \in J$ such that $[j_0pq] \neq 0$. Using (5.11) with $j = j_0$, we conclude that $\frac{y_{pi}}{y_{qi}}$ converges to some $\gamma_{pq} > 0$, i.e.,

\begin{equation}
\lim_{i \to \infty} \frac{[puv]}{y_{pi}} = \delta_{pav} \geq 0 \quad \text{and} \quad \lim_{i \to \infty} \frac{y_{pi}}{y_{qi}} = \gamma_{pq} > 0.
\end{equation}

Our first goal is to show that $\delta_{pav} = 0$. Applying (5.12) with $j = \mu \in J^c$ and using (5.14) yields

\[ \lim_{i \to \infty} \left( 2 \sum_{k \in J} \sum_{a \in J^c} \frac{y_{ki}[\mu ak]}{y_{ai}} (\frac{y_{ai}}{y_{pi}} - \frac{y_{ai}}{y_{pi}}) - \sum_{k,l \in J} [\mu kl] \frac{y_{ki}y_{li}}{y_{pi}} \right) = 0. \]

Consequently

\begin{equation}
2 \sum_{k \in J} \sum_{a \in J^c} y_k [\mu ak] (\gamma_{a\mu} - \gamma_{a\mu}) - \sum_{k,l \in J} y_k y_\delta_{\mu kl} = 0.
\end{equation}

Let us add up these expressions over $\mu \in J^c$. Clearly,

\[ \sum_{\mu, a \in J^c} [\mu ak] (\gamma_{a\mu} - \gamma_{a\mu}) = 0, \quad \text{and hence} \quad \sum_{\mu \in J^c} \sum_{k,l \in J} y_k y_\delta_{\mu kl} = 0. \]

Since $\delta_{\mu kl} \geq 0$, this implies $\delta_{\mu kl} = 0$ for all $\mu \in J^c$ and $k, l \in J$. 

Our second goal is to show that $\gamma_{qp} = 1$. Assume that $\gamma_{qp} < 1$. Choose $m, n \in J^c$ such that $\gamma_{mn}$ is the smallest possible, i.e.,

$$\gamma_{mn} = \min_{a, b \in J^c} \gamma_{ab} \leq \gamma_{qp} < 1.$$  

Using (5.15) with $\mu = m$, we obtain

$$\sum_{k \in J} \sum_{a \in J} y_k[mak](\gamma_{am} - \gamma_{ma}) = 0.$$  

Since $G/K$ is isotropy irreducible, there exists $j_1 \in J$ such that $[mnj_1] \neq 0$. Therefore, the above sum contains the term

$$y_{j_1}[mnj_1](\gamma_{mn} - \gamma_{mn}) = y_{j_1}[mnj_1](\frac{1}{\gamma_{mn}} - \gamma_{mn}) > 0.$$  

Since this sum equals zero, there exist $j_2 \in J$ and $\nu \in J^c$ such that $[\nu j_2] \neq 0$ and

$$y_{j_2}[\nu j_2](\gamma_{\nu m} - \gamma_{\nu m}) = y_{j_2}[\nu j_2](\gamma_{\nu m} - \frac{1}{\gamma_{\nu m}}) < 0.$$  

This inequality implies that $\gamma_{\nu m} < 1$ and hence

$$\gamma_{\nu m} = \gamma_{mn} \gamma_{\nu m} < \gamma_{mn},$$  

which contradicts the minimality of $\gamma_{mn}$. Thus, $\gamma_{qp}$ cannot be less than 1. Analogous arguments show that $\gamma_{qp}$ is not greater than 1.

We now demonstrate that the metric $g_F$ given by (5.8) satisfies (5.9). Denote the modules in the decomposition $D$ by $m_1, \ldots, m_r$. The Killing form of $K$ restricted to $m_j$ equals $-\bar{b}_j Q_{m_j}$ for all $j \in J$, and recall that

$$\bar{b}_j = b_j - \frac{1}{d_j} \sum_{k, l \in J^c} [jkl].$$  

Using (5.2), (5.10) and Lemma 5.13, we see that for all $j \in J$

$$\lambda T|_{m_j} = \lim_{i \to \infty} \left( \frac{b_{ji}}{2} - \frac{1}{2d_j} \sum_{k, l = 1}^r [jkl] \frac{y_{ki}}{y_{yi}} + \frac{1}{4d_j} \sum_{k, l = 1}^r [jkl] \frac{y_{ki} y_{yi}}{y_{ji}} \right) Q|_{m_j}$$

$$= \left( \frac{b_{ji}}{2} - \frac{1}{2d_j} \sum_{k, l \in J} [jkl] \frac{y_{ki}}{y_{yi}} + \frac{1}{4d_j} \sum_{k, l \in J} [jkl] \frac{y_{ki} y_{yi}}{y_{ji}} \right) Q|_{m_j} = \text{Ric}(g_F)|_{m_j}.$$  

It remains to show that the off-diagonal components of $\text{Ric}(g_F)$ coincide with those of $\lambda T$. For this, let $(e_{ji}^l)$ be a $Q$-orthonormal basis of $m_i^l$, and let the subscript $m_l^i$ denotes projection onto $m_l^i$. We may assume that $(e_{ji}^l)$ converges, as $i \to \infty$, to a basis $(e_{ji}^l)$ of $m_i$ for every $l \in I$.

Formula (1.6) implies that

$$\text{Ric}(g_i)(e_{ji}^l, e_{kj}^l) = \sum_{l, m = 1}^r \left( \frac{1}{2} - \frac{y_{mi} y_{yi}}{2y_{yj} y_{yi}} \right) \sum_{\mu = 1}^d Q([e_{ji}^l, \bar{e}_{\mu}^l]_{m_i^l}, [e_{ki}^l, \bar{e}_{\mu}^l]_{m_i^l})$$

for all $j, k \in I$ with $j \neq k, p = 1, \ldots, d_j$ and $q = 1, \ldots, d_k$, and hence

$$\lim_{i \to \infty} \sqrt{y_{yi} y_{ki}} \sum_{l, m = 1}^r \left( \frac{1}{2} - \frac{y_{mi} y_{yi}}{2y_{yj} y_{yi}} \right) \sum_{\mu = 1}^d Q([e_{ji}^l, \bar{e}_{\mu}^l]_{m_i^l}, [e_{ki}^l, \bar{e}_{\mu}^l]_{m_i^l}) = \lambda T(e_{ji}^l, e_{kj}^l) \sqrt{y_{yi} y_{ki}}.$$  

The Cauchy–Schwarz inequality yields

$$|Q([e_{ji}^l, \bar{e}_{\mu}^l]_{m_i^l}, [e_{ki}^l, \bar{e}_{\mu}^l]_{m_i^l})| \leq |[e_{ji}^l, \bar{e}_{\mu}^l]_{m_i^l}| |Q| \leq \sqrt{|jlm|} \sqrt{|klm|} \leq \max(|jlm|, |klm|)$$,

where $|Q|$ is the norm of the Killing form $Q$.
If \( j, k, l \in J \) and \( m \in J^c \), then
\[
\lim_{i \to \infty} |Q([e^{ji}_p, e^{li}_\mu]_{m_i}, [e^{ki}_q, e^{li}_\mu]_{m_i})| \leq \lim_{i \to \infty} \max\{[jlm]_i, [klm]_i\} = \max\{[jlm], [klm]\} = 0.
\]
Moreover, by Lemma 5.13,
\[
\lim_{i \to \infty} \frac{|Q([e^{ji}_p, e^{li}_\mu]_{m_i}, [e^{ki}_q, e^{li}_\mu]_{m_i})|}{y_{mi}} \leq \lim_{i \to \infty} \frac{\max\{[jlm]_i, [klm]_i\}}{y_{mi}} = 0.
\]
Using (5.16) along with these formulas, we find, for \( j, k \in J \) with \( j \neq k \), that
\[
\lambda T(e^j_p, e^k_q) = \lim_{i \to \infty} \sum_{l, m = 1}^r \left( \frac{1}{2} \cdot \frac{y_{mi}}{2y_i} + \frac{y_{lj}y_{mi}}{4y_jy_k} \right) \sum_{r = 1}^{d_i} Q([e^{ji}_p, e^{li}_\mu]_{m_i}, [e^{ki}_q, e^{li}_\mu]_{m_i})
= \sum_{l, m \in J} \left( \frac{1}{2} \cdot \frac{y_{mi}}{2y_i} + \frac{y_{lj}y_{mi}}{4y_jy_k} \right) \sum_{r = 1}^{d_i} Q([e^{ji}_p, e^{li}_\mu]_{m_i}, [e^{ki}_q, e^{li}_\mu]_{m_i})
= \text{Ric}(g_F)(e^j_p, e^k_q).
\]
Thus \( \text{Ric}(g_F) = \lambda T_{|K/H} \).

**Remark 5.17.** Theorems 5.3 and 5.7 show that the existence of Palais–Smale sequences is closely related to the prescribed Ricci curvature problem for \( K/H \), where \( K \) is an intermediate subgroup. In particular, \( \lambda \) may be the scalar curvature of any critical point of \( S \) restricted to \( M_T(K/H) \). Of course, such critical points might not exist for specific intermediate subgroups and Ricci candidates.

As a consequence of Theorem 5.7 we obtain the following property of 0-Palais–Smale sequences, which is a partial converse to Corollary 5.6.

**Corollary 5.18.** Consider a divergent Palais–Smale sequence \((g_i)\) and a subgroup \( K \) as in Theorem 5.7. If the limit of \( S(g_i) \) equals 0 and the homogeneous space \( G/K \) is isotropy irreducible, then \( K/H \) is a torus.

Finally, we state a property of \( S_{|M_T} \) in a neighborhood of a stratum \( \Delta_t \) with negative “derivative”. This will be useful for the construction of saddle points.

**Proposition 5.19.** Consider a compact homogeneous space \( G/H \) such that the modules \( m_r \) are pairwise inequivalent. Let \( K \) be an intermediate subgroup with Lie algebra \( \mathfrak{k} \) such that \( G/K \) is isotropy irreducible. Assume that \( \alpha_\mathfrak{k} < \alpha_t \) for every intermediate subalgebra \( \mathfrak{k} \) contained in \( \mathfrak{k} \) as a proper subset. Furthermore, assume that \( \beta_\mathfrak{k} - \alpha_\mathfrak{k} < 0 \). Given \( \delta > 0 \), there exist \( \delta_-, \delta_+, \epsilon > 0 \) such that \( \delta_- < \delta_+ \leq \delta \) and \( S(y) \leq \alpha_t - \epsilon \) for all \( y \in T_{\delta_-}(\Delta_t) \setminus T_{\delta_+}(\Delta_t) \).

**Proof.** Take \( y = (y_1, \ldots, y_r) \in \Delta \) and let \( \Delta_t = (\Delta_t, \mathfrak{k}) \). We have
\[
S(y) = S(y|_{r \cap m}) + \frac{1}{2} \sum_{i \in J^c} d_i b_i y_i - \frac{1}{4} \sum_{i, j, k \in J^c} [ijk] \frac{y_i y_j}{y_k}
- \frac{1}{4} \sum_{i \in J} \sum_{j, k \in J^c} [ijk] \left( \frac{y_k y_j}{y_i} + y_i \left( \frac{y_j}{y_k} + \frac{y_k}{y_j} - 2 \right) \right)
\leq \alpha_t \text{tr}_{y|_{r \cap m}} T|_{r \cap m} + \frac{1}{2} \sum_{i \in J^c} d_i b_i y_i - \frac{1}{4} \sum_{i, j, k \in J^c} [ijk] \frac{y_i y_j}{y_k}
- \frac{1}{4} \sum_{i \in J} \sum_{j, k \in J^c} y_i [ijk] \left( \frac{y_j}{y_k} + \frac{y_k}{y_j} - 2 \right).
\]
Choose \( p, q \in J^c \) so that
\[
    y_p = \min_{i \in J^c} y_{i} \quad \text{and} \quad y_q = \max_{i \in J^c} y_{i}.
\]
If \( y_p = y_q \), then \( y_i \) are the same for all \( i \in J^c \). In this case, the restriction \( y_{|\tau} \) defines a \( K \)-invariant metric on \( G/K \) with scalar curvature
\[
    S(y_{|\tau}) = \frac{1}{2} \sum_{i \in J^c} d_i b_i y_i - \frac{1}{4} \sum_{i,j,k \in J^c} \frac{y_k y_j}{y_k} \sum_{i \in J^c} y_p \left( \frac{d_i b_i}{2} - \frac{1}{4} \sum_{j,k \in J^c} [ijk] \right).
\]
Moreover,
\[
    S(y_{|\tau}) = \beta_t \operatorname{tr}_{y_{|\tau}} T_{|\tau} = \beta_t \sum_{i \in J^c} d_i T_i y_p.
\]
The equality \( y_p = y_q \) does not hold in general. One may view the quantity \( \frac{y_q}{y_p} - 1 \) as a measure of the deviation of \( y_{|\tau} \) from being a \( K \)-invariant metric on \( G/K \). First, we consider the situation where this deviation is small. More precisely, assume that
\[
    \frac{y_q}{y_p} - 1 \leq \Theta, \quad \text{where} \quad \Theta = \frac{\alpha_t - \beta_t}{\sum_{i \in J^c} d_i T_i + 1} \min_{i \in J^c} d_i T_i > 0.
\]
Since \( G/K \) is isotropy irreducible,
\[
    S(y) \leq \alpha_t \operatorname{tr}_{y_{|\tau}} T_{|\tau} + \frac{1}{2} \sum_{i \in J^c} d_i b_i y_i - \frac{1}{8} \sum_{i,j,k \in J^c} y_i [ijk] \left( \frac{y_j}{y_k} + \frac{y_k}{y_j} \right) \leq \alpha_t \operatorname{tr}_{y_{|\tau}} T_{|\tau} + \sum_{i \in J^c} y_p \left( \frac{d_i b_i}{2} - \frac{1}{4} \sum_{j,k \in J^c} [ijk] \right) \leq \alpha_t \operatorname{tr}_{y_{|\tau}} T_{|\tau} + \beta_t \sum_{i \in J^c} d_i T_i y_p + \sum_{i \in J^c} y_p \left( \frac{y_i}{y_p} - 1 \right) \left( \frac{d_i b_i}{2} - \frac{1}{4} \sum_{j,k \in J^c} [ijk] \right) \leq \alpha_t \operatorname{tr}_{y_{|\tau}} T_{|\tau} + \beta_t \operatorname{tr}_{y_{|\tau}} T_{|\tau} + \sum_{i \in J^c} d_i b_i y_p \left( \frac{y_i}{y_p} - 1 \right).
\]
If \( y \in T_{\delta_+}(\Delta_t) \setminus T_{\delta_-}(\Delta_t) \) for some \( \delta_- > \delta_+ > 0 \) such that \( \delta_- < \delta_+ \), then \( \delta_- < y_q \leq \delta_+ \) and
\[
    S(y) \leq \alpha_t - (\alpha_t - \beta_t) \operatorname{tr}_{y_{|\tau}} T_{|\tau} + \sum_{i \in J^c} \frac{d_i b_i y_p}{2} \left( \frac{y_i}{y_p} - 1 \right) \leq \alpha_t - (\alpha_t - \beta_t) d_q T_q y_q + \frac{1}{2} \sum_{i \in J^c} \Theta d_i b_i y_p \leq \alpha_t - (\alpha_t - \beta_t) d_q T_q \delta_- + \frac{\alpha_t - \beta_t}{2} \min_{i \in J^c} d_i T_i \delta_+ \leq \alpha_t - (\alpha_t - \beta_t) \min_{i \in J^c} d_i T_i \left( \delta_- - \frac{\delta_+}{2} \right).
\]
Setting \( \delta_- = \frac{3\delta_+}{4} \), we conclude that
\[
    (5.20) \quad S(y) \leq \alpha_t - \frac{\delta_+}{4} \frac{2}{\alpha_t - \beta_t} \min_{i \in J^c} d_i T_i.
\]
Now, let us consider the situation where the deviation of \( y_{|\tau} \) from being a \( K \)-invariant metric on \( G/K \) is large. More precisely, assume that
\[
    \frac{y_q}{y_p} - 1 > \Theta.
\]
Because $G/K$ is isotropy irreducible, there exists $m \in J$ such that $[mpq] > 0$. If $y \in T_{\delta_+}(\Delta_t) \setminus T_{\delta_-}(\Delta_t)$, then

$$S(y) \leq \alpha_t \operatorname{tr}_{y \in \mathfrak{m}} T_{[\mathfrak{t} \cap \mathfrak{m}]} + \frac{1}{2} \sum_{i \in J^c} d_i b_i y_i - \frac{1}{4} \sum_{i,j,k \in J^c} [ijk] \frac{y_i y_j y_k}{2}$$

$$- \frac{1}{4} \sum_{i \in J} \sum_{j,k \in J^c} y_i[ijk] \left( \frac{y_j y_k}{y_i} - 2 \right)$$

$$\leq \alpha_t + \frac{1}{2} \sum_{i \in J^c} d_i b_i - \frac{1}{4} y_m [mpq] \left( \frac{y_p y_q}{y_l} - 2 \right)$$

$$\leq \alpha_t + \frac{\delta_+}{2} \sum_{i \in J^c} d_i b_i - y_m [mpq] \Theta^2 \frac{4(1 + \Theta)}{4(1 + \Theta)}.$$

Proposition 2.3 and Corollary 2.6, together with the hypothesis that $\alpha' < \alpha_t$ for $\mathfrak{t}' \subset \mathfrak{t}$, imply the existence of $\delta_0, \epsilon_0 > 0$ such that $S(y) < \alpha_t - \epsilon_0$ whenever $\min_{i \in J} y_i < \delta_0$. On the other hand, if

$$\min_{i \in J} y_i \geq \delta_0 \quad \text{and} \quad \delta_- \leq \frac{\delta_0 [mpq] \Theta^2}{4(1 + \Theta)} \left( \sum_{i \in J^c} d_i b_i + 1 \right),$$

then

$$S(y) \leq \alpha_t - \delta_0 [mpq] \Theta^2 \frac{8(1 + \Theta)}{4(1 + \Theta)}.$$

Recalling (5.20), we conclude that the assertion of Proposition 5.19 holds if $\delta_+$ is sufficiently small, $\delta_- = \frac{3\delta_0}{4}$, and

$$\epsilon = \min \left\{ \frac{\delta_+}{4} (\alpha_t - \beta_t) \min_{i \in J^c} d_i T_i, \epsilon_0, \frac{\delta_0 [mpq] \Theta^2}{8(1 + \Theta)} \right\}.$$

\[ \square \]

**Remark 5.21.** We do not know if Theorem 5.7, Corollary 5.18 and Proposition 5.19 hold without the assumption that $G/K$ is isotropy irreducible. This assumption was clearly crucial in our proofs. Results such as Theorem 5.7 and Proposition 5.19 are required if one wants to develop a general theory for the existence of saddle points, as was done in the Einstein case.

### 6. Saddle points on generalised Wallach spaces

In this section, we prove the existence of critical points of the functional $S_{|\mathcal{M}_T}$ with co-index at most 1 for a large class of Ricci candidates $T$ on generalised Wallach spaces. In a sense, our main result, Theorem B, is the generalised Wallach space analogue of the graph theorem for Einstein metrics obtained in [8]. The proof will require the estimate for $S_{|\mathcal{M}_T}$ in Proposition 5.19.

Recall that $G/H$ is called a *generalized Wallach space* if $G$ and $H$ are connected and $\mathfrak{m}$ splits into three irreducible modules

$$\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3$$

such that $[123]$ is, up to permutations, the only non-vanishing structure constant. This implies, in particular, that we have three intermediate subgroups $K_i$ with Lie algebras $\mathfrak{k}_i = \mathfrak{h} \oplus \mathfrak{m}_i$, where $i = 1, 2, 3$. Furthermore, the homogeneous spaces $G/K_i$ and $K_i/H$ are all isotropy irreducible. We assume that the modules $\mathfrak{m}_i$ are inequivalent, in particular, there exists only one decomposition $D$ up to order of summands.

We are now ready to prove our next main result.
Proof of Theorem B. One easily computes the constants $\alpha_t$ and $\beta_t$ explicitly. They are given by

$$
\alpha_t = \frac{d_i - 2[123]}{2d_i T_i} \quad \text{and} \quad \beta_t = \frac{d_j + d_k}{2(d_j T_j + d_k T_k)},
$$

where $(i, j, k)$ is a permutation of $(1, 2, 3)$; cf. [32, Section 4].

The simplex $\Delta$ associated with the decomposition (6.1) is two-dimensional, with vertices

$$
V_1 = \left( \frac{1}{d_1 T_1}, 0, 0 \right), \quad V_2 = \left( 0, \frac{1}{d_2 T_2}, 0 \right) \quad \text{and} \quad V_3 = \left( 0, 0, \frac{1}{d_3 T_3} \right).
$$

These vertices are the only subalgebra strata $\Delta_{\epsilon_1}, \Delta_{\epsilon_2}$ and $\Delta_{\epsilon_3}$. The metric

$$
g = \frac{1}{y_1} Q_{|m_1} + \frac{1}{y_2} Q_{|m_2} + \frac{1}{y_3} Q_{|m_3}
$$

lies in $\mathcal{M}_T$ if and only if

$$
\text{tr}_y T = d_1 T_1 y_1 + d_2 T_2 y_2 + d_3 T_3 y_3 = 1,
$$

and its scalar curvature satisfies

$$
S(g) = \frac{d_1}{2} y_1 + \frac{d_2}{2} y_2 + \frac{d_3}{2} y_3 - \frac{123}{2} \left( \frac{y_1 y_2}{y_3} + \frac{y_2 y_3}{y_1} + \frac{y_3 y_1}{y_2} \right).
$$

Without loss of generality, assume that $\beta_{\epsilon_i} - \alpha_{\epsilon_i} < 0$ for $i = 1, 2$ and that $\alpha_{\epsilon_i} \geq \alpha_{\epsilon_2}$. One easily sees that this implies $\alpha_{\epsilon_3} > \alpha_{\epsilon_2}$ and that $\beta_{\epsilon_3} - \alpha_{\epsilon_3} > 0$; see Figure 2 for a typical configuration.

We first claim that for every $a, b \in \mathbb{R}$ such that $\alpha_{\epsilon_i} \notin [a, b]$ for $i = 1, 2, 3$, the scalar curvature $S_{|\mathcal{M}_T}$ satisfies the Palais–Smale condition on $S^{-1}((a, b))$. Indeed, since $G/K_i$ and $K_i/H$ are isotropy irreducible, Theorem 5.7 implies that $\lim_{t \to \infty} S(g_t) = \alpha_{\epsilon_i}$ with $i \in \{1, 2, 3\}$ for every divergent Palais–Smale sequence $(g_t)$, contradicting our assumption.

We will use a mountain pass argument to prove the existence of a critical point of $S_{|\mathcal{M}_T}$. For this we choose a curve $\gamma : \mathbb{R} \to \Delta$ such that $\gamma((-\infty, 0])$ is the canonical variation converging to $V_1$ and $\gamma([0, \infty))$ the canonical variation converging to $V_2$. These two canonical variations meet at the center $y_0 = (w, w, w)$ of $\Delta$ with $w = 1/(d_1 T_1 + d_2 T_2 + d_3 T_3)$. Since $\beta_{\epsilon_i} - \alpha_{\epsilon_i} < 0$ for $i = 1, 2$, one easily sees that

$$
S(y_0) = \frac{d_1 + d_2 + d_3 - 3[123]}{2(d_1 T_1 + d_2 T_2 + d_3 T_3)} > \alpha_{\epsilon_3}.
$$

Formula (3.3) implies that for a subgroup $K$ with $\beta - \alpha < 0$ the canonical variation has strictly monotone scalar curvature. This means that $\inf_{t \in \mathbb{R}} S(\gamma(t)) > \alpha_{\epsilon_3}$.

Let $\phi$ be the gradient flow of the functional $S_{|\mathcal{M}_T}$. Applying $\phi$ to $\gamma$, we obtain a family of paths $\gamma_s(t) = \phi_s(\gamma(t))$. Let

$$
c = \sup_{s \geq 0} \inf_{t \in \mathbb{R}} S(\gamma_s(t)).
$$

Since the gradient flow increases the scalar curvature, we have

$$
c \geq \inf_{t \in \mathbb{R}} S(\gamma(t)) > \alpha_{\epsilon_3}.
$$

We now claim that $c < \alpha_{\epsilon_2}$. To prove this, use Proposition 5.19 to obtain $\epsilon > 0$ such that $S(y) < \alpha_{\epsilon_2} - \epsilon$ as long as $y$ lies in the difference of tubular neighbourhoods $T_{\delta_+}(\Delta_{\epsilon_2}) \setminus T_{\delta_-}(\Delta_{\epsilon_2})$. We may assume that $\delta_+$ is small enough to ensure that the closure of $T_{\delta_+}(\Delta_{\epsilon_3})$ does not contain $\Delta_{\epsilon_1}$. Along the canonical variations, we have

$$
\lim_{t \to -\infty} S(\gamma(t)) = \alpha_{\epsilon_2} \geq \alpha_{\epsilon_2} \quad \text{and} \quad \lim_{t \to \infty} S(\gamma(t)) = \alpha_{\epsilon_2}.
$$

This implies that, for sufficiently small $\epsilon$, the set $\{y \in \Delta \mid S(y) > \alpha_{\epsilon_2} - \epsilon\}$ has at least two connected components. One of these components is contained in $T_{\delta_+}(\Delta_{\epsilon_2})$ and another in the complement $\Delta \setminus T_{\delta_-}(\Delta_{\epsilon_2})$. Each of the curves $\gamma_s$ connects them. This means that each $\gamma_s$ must pass through $T_{\delta_+}(\Delta_{\epsilon_2}) \setminus T_{\delta_-}(\Delta_{\epsilon_2})$. Thus $c \leq \alpha_{\epsilon_2} - \epsilon$. Making $\epsilon$ smaller if necessary, we may assume that $c \in (\alpha_{\epsilon_2} + \epsilon, \alpha_{\epsilon_2} - \epsilon)$.
Since \( S_{M_T} \) satisfies the Palais–Smale condition on \( S^{-1}((\alpha t_1 + \epsilon, \alpha t_2 - \epsilon)) \), a standard argument now shows that there exists a critical point \( g \) of co-index at most 1 with \( S(g) = c \). Indeed, if we have no critical point at level \( c \), then the Palais–Smale condition implies that there exist constants \( \eta, \delta > 0 \) such that
\[
|\text{grad} \, S_{|M_T}(h)|_h > \eta \quad \text{for all } h \in S^{-1}((c - \delta, c + \delta)) \cap M_T.
\]
But then \( S(\gamma_s(t)) > c + \frac{\delta}{2} \) for all \( t \in \mathbb{R} \) if \( s \) is sufficiently large, contradicting the definition of \( c \). If none of these critical points at level \( c \) has co-index \( \leq 1 \), then we can deform \( \gamma_s \) into a curve \( \tilde{\gamma} \) with
\[
\inf_{t \in \mathbb{R}} S(\tilde{\gamma}(t)) > c.
\]

Generalized Wallach spaces were classified in [14, 27]. The simplest example is the flag manifold \( SU(3)/T^2 \), which we discuss in the next section in more detail. Some higher-dimensional generalized Wallach spaces are \( U(p + q + r)/(U(p) \times U(q) \times U(r)) \) and their analogues with \( SO(n) \) and \( Sp(n) \). There are two cases where the modules \( m_i \) are equivalent, namely, \( SO(n + 2)/SO(n) \) and the Ledger–Obata spaces \( H^4/\text{diag}(H) \) with \( H \) simple (here we have spaces of homogeneous metrics of dimensions up to six). In both cases the Ricci tensor of a diagonal metric is diagonal as well, as can be seen using \( \text{Aut}(G, H) \). The above proof of Theorem B yields a critical point in the space of diagonal metrics and thus a critical point of \( S_{|M_T} \).

It is natural to ask whether one can obtain an analogue of the graph theorem in [8] for Einstein metrics in the context of the prescribed Ricci curvature problem on general homogeneous spaces. The presence of divergent Palais–Smale sequences, proved in Section 5, is an obvious obstacle. Nevertheless, as Theorem B shows, this obstacle can be circumvented at least in some situations.

7. Examples

In this section we consider a few examples that illustrate our theorems. Our main purpose is to show that the possible behavior of critical points is richer than previously observed in the literature. In particular, we demonstrate the existence of saddles. We also show that critical points can be degenerate and that they are not necessarily isolated.

7.1. The Wallach space \( SU(3)/T^2 \). Let \( G = SU(3) \) and \( H = T^2 \subset SU(3) \) embedded as a maximal torus. Then \( G/H \) is the Wallach space \( SU(3)/T^2 \). It is the model example for the type of manifolds considered in Section 6. On \( G \) we choose as the biinvariant metric \( Q \) the negative of the Killing form \( B(X, Y) = -6 \text{tr}(XY) \). The structure constant [123] equals \( \frac{1}{3} \). The group \( G \) has exactly three maximal connected Lie subgroups \( K_1, K_2 \) and \( K_3 \) containing \( H \), namely,
\[
H = T^2 \subset K_i = U(2) \subset SU(3) = G, \quad i = 1, 2, 3,
\]
where the inclusion of \( U(2) \) in \( SU(3) \) is given by one of the three block embeddings. According to (6.2),
\[
\alpha_{t_i} = \frac{1}{3T_i} \quad \text{and} \quad \beta_{t_i} = \frac{1}{T_j + T_k},
\]
where \( (i, j, k) \) is a permutation of \( \{1, 2, 3\} \). These formulas, together with Theorem A and Theorem B, imply the following result.

**Proposition 7.1.** Let \( G/H \) be the flag manifold \( SU(3)/T^2 \). Then we have:
(a) The functional \( S_{|M_T} \) attains its global maximum if
\[
\frac{T_j + T_k}{3T_i} < 1, \quad \text{where} \quad T_i = \min\{T_1, T_2, T_3\}.
\]
(b) $S_{i|\mathcal{M}_T}$ has a critical point of co-index 0 or 1 if
\[
\frac{T_j + T_k}{3T_i} > 1
\]
for two distinct $i$.

**Remark 7.2.** (a) The sufficient condition for the attainment of the global maximum in part (a) of the proposition was first obtained in [32]. Maple experiments suggest that the critical points in part (b) are typically non-degenerate and hence saddles.

(b) While Theorem 5.3 shows that the canonical variation for $K_i$ gives rise to a divergent Palais–Smale sequence, not all divergent Palais–Smale sequences arise in this way. For instance, consider a curve of metrics $g(t) = (y_1(t), y_2(t), y_3(t)) \in \mathcal{M}_T$ with $\lim_{t \to \infty} \frac{y_2(t)}{y_3(t)} = 1$. It is easy to see that $(g(i))$ is a divergent Palais–Smale sequence. It comes from a canonical variation only if $y_2(t) = y_3(t)$. On the other hand, Corollary 5.18 implies non-existence of 0-Palais–Smale sequences in $\mathcal{M}_T$.

Given a metric $g = (y_1, y_2, y_3)$, the constraint $\text{tr}_g T = 1$ takes the form
\[2T_1y_1 + 2T_2y_2 + 2T_3y_3 = 1,
\]
and the scalar curvature satisfies
\[S(g) = y_1 + y_2 + y_3 - \frac{1}{6} \left( \frac{y_1y_2}{y_3} + \frac{y_2y_3}{y_1} + \frac{y_1y_3}{y_2} \right).
\]
The presence of the constant $c$ in the equation $\text{Ric}(g) = cT$ allows us to assume that $T_1 + T_2 + T_3 = 1$. Choosing the coordinates
\[x = \frac{4\sqrt{3}}{3}(T_1 - T_2) \quad \text{and} \quad y = 4T_3 - \frac{4}{3}
\]
in the space of so normalised Ricci candidates, we mark points in the $(x, y)$-plane that correspond to different behaviors of $S_{i|\mathcal{M}_T}$. The large triangle in Figure 1 is the set of $(x, y)$ representing positive-definite $T$. The dark-grey triangle consists of those $(x, y)$ for which Theorem A guarantees the existence of a global maximum. In the light-grey regions, Theorem B yields a critical point of co-index 0 or 1. Notice that $N(H)/H$ is isomorphic to the permutation group on three letters. It acts on the tangent space $\mathfrak{m}$ by permuting the modules $\mathfrak{m}_i$, and hence on the space of tensors $T$ by permuting $T_i$. Thus the space of Ricci candidates exhibits a natural threefold symmetry.

A computer-assisted experiment indicates that the gradient of $S_{i|\mathcal{M}_T}$ is never 0 in the white regions, i.e., Theorems A and B are optimal on $SU(3)/T^2$. More precisely, we choose one million points $g$ in the space of metrics and compute $\text{Ric}(g)$ for each of them. We then normalize the obtained values of $\text{Ric}(g)$ and mark them in the $(x, y)$-plane as in Figure 1. Notice that the normalization factor, equal to the sum of the components of $\text{Ric}(g)$, turns out to be always positive. The values of $\text{Ric}(g)$ that are positive-definite fill up the grey regions in Figure 1, while the indefinite values fill up the unbounded blue regions. Thus the image of the Ricci map is the union of all these regions.

We now illustrate the behavior in some explicit examples. After choosing specific values for $T_2$ and $T_3$, we can solve the constraint for $y_1$, substitute the result into the formula for $S(g)$ and sketch the graph of $S$ as a function of $y_2$ and $y_3$. In Figure 2 one sees instances of a global maximum, a saddle, and a case with no critical points. As the following example shows, moving from the interior of the dark-grey triangle in Figure 1 to a light-grey region, one finds that the transition is achieved by creating a curve of critical points.

**Example 1.** It is well known that on the Wallach space the Kähler metrics $g = (y_1, y_2, y_3) \in \mathcal{M}_T$ are characterized by the equation $\frac{1}{y_k} = \frac{1}{y_i} + \frac{1}{y_j}$ for some permutation $(i, j, k)$ of $\{1, 2, 3\}$. They are Kähler–Einstein if, in addition, $y_i = y_j$. Of course, such Einstein metrics are critical points of $S_{i|\mathcal{M}_T}$. 
when $T = \frac{1}{6}g$. To consider a specific example, we set $T_1 = T_2 = T_3 = \frac{1}{4}$. This choice of $T$ and its permutations are marked by the red dots in Figure 1. As explained in the introduction, all the Kähler metrics associated with the same complex structure as $T$ have the same Ricci curvature. A straightforward computation shows that these metrics satisfy

$$y_1 = -2y_2 + 1 + \sqrt{3y_2^2 - 2y_2 + 1}, \quad 2y_3 = y_2 + 1 - \sqrt{3y_2^2 - 2y_2 + 1}, \quad 0 < y_2 < 2,$$

and thus form a curve at $S = \frac{4}{3}$ containing the Kähler–Einstein metric at $y_2 = \frac{2}{3}$. This curve approaches the boundary of the simplex $\Delta$ at both ends at the strata $\Delta_{t_1}$ and $\Delta_{t_2}$ with zero derivatives. One easily shows that the Hessian of $S_{|M_T}$ has a zero and a negative eigenvalue. Thus the curve is a non-degenerate critical submanifold. By the Morse–Bott lemma, it is isolated and consists of local maxima. In Figure 3, we depict the graph of $S_{|M_T}$ with the red dot at the Kähler–Einstein metric and the black plane just below $S = \frac{4}{3}$. This makes the curve easily visible and indicates that the critical points on it are, in fact, global maxima.
Each metric with \( y_2 \in (0, \frac{2}{3}) \) is isometric to one with \( y_2 \in \left( \frac{2}{3}, 2 \right) \). The isometry is given by the element of the normalizer that takes \((y_1, y_2, y_3)\) to \((y_2, y_1, y_3)\). On the interval \((0, \frac{2}{3})\), however, the volume is strictly decreasing, and hence there are no further isometries.

It was proven in [17] that if \( g_1 \) and \( g_2 \) are two metrics on a compact de Rham-irreducible manifold such that \( g_1 \) is Einstein with non-negative sectional curvature and \( \text{Ric}(g_1) = \text{Ric}(g_2) \), then \( g_1 \) and \( g_2 \) must be the same up to scaling. Example 1 shows that the non-negativity assumption on the sectional curvature cannot be dropped. This example also underscores the difference between \( S|_{\mathcal{M}_T} \) and the functional \( S|_{\mathcal{M}_1} \) associated with the Einstein equation. Indeed, as shown in [8], the set of critical points of \( S|_{\mathcal{M}_1} \) always has compact connected components. Moreover, by the well-known finiteness conjecture, one expects this set to be finite. Another difference is that the Kähler–Einstein metrics are saddle points of \( S|_{\mathcal{M}_1} \).

Remark 7.3. (a) Apart from the three Kähler–Einstein metrics discussed above, the only other Einstein metric on the Wallach space is induced by \( Q \). It is a strict local maximum of \( S|_{\mathcal{M}_T} \) with \( T = \frac{1}{6}Q \), the Ricci candidate represented by the origin in Figure 1.

(b) It is interesting to note that in Example 1 there exists a second curve of critical points consisting of Kähler metrics with signature \((+, +, -)\) obtained by changing the signs in front of the square root. It is also a non-degenerate critical submanifold, but now it is a local minimum.

7.2. The space \((SU(2) \times SU(2))/S^1_{p,q} \) with \( p \neq q \). Let \( G \) be the product group \( SU(2) \times SU(2) \) and \( H \) the subgroup

\[
S^1_{p,q} = \left\{ \begin{pmatrix} e^{2\pi pt} & 0 \\ 0 & e^{-2\pi qt} \end{pmatrix}, \begin{pmatrix} e^{2\pi pt} & 0 \\ 0 & e^{-2\pi qt} \end{pmatrix} \right\} \quad t \in \mathbb{R}
\]

with \( p, q \in \mathbb{Z} \). We may assume that \( p, q \neq 0 \) since otherwise \( G/H \) is a product. Transforming \( G \) by an automorphism if necessary, we may also assume that \( p, q > 0 \) with \( \gcd(p, q) = 1 \). The biinvariant metric \( Q \) we choose on \( G \) is such that \( |(X, Y)|_Q = -\frac{1}{2}(\text{tr}(X^2) + \text{tr}(Y^2)) \). Finally, suppose that \( p \neq q \). This guarantees that the irreducible summands in the isotropy representation of \( G/H \) are pairwise inequivalent. The remaining case with \( p = q \) will be discussed in the next subsection.

Let us identify \( SU(2) \) and \( \text{su}(2) \) with the group of unit quaternion and the Lie algebra of purely imaginary quaternions in the standard way. Then

\[
H = \{ (e^{\eta pi}, e^{\eta qi}) \mid \eta \in \mathbb{R} \} \quad \text{and} \quad \mathfrak{h} = \text{span}\{ (p \mathbf{i}, q \mathbf{i}) \}.
\]
The space $D$ contains the decomposition
\[ m = m_0 \oplus m_1 \oplus m_2 \]
with
\[ m_0 = \text{span}\{(q i, -p i)\}, \quad m_1 = \{(z j, 0) \mid z \in \mathbb{C}\} \quad \text{and} \quad m_2 = \{(0, z j) \mid z \in \mathbb{C}\}. \]

On $m_0$ the representation $\text{Ad}_H$ is trivial. On $m_1$ and $m_2$ the element $(e^{q i}, e^{p i}) \in H$ sends $(z j, 0)$ and $(0, z j)$ to $(e^{2 pi}z j, 0)$ and $(0, e^{2 pi}z j)$, respectively. Since we assume that $p \neq q$, the modules $m_0$, $m_1$ and $m_2$ are pairwise inequivalent. Hence there is only one decomposition in $D$ up to the order of summands. The tensor $T$ has the form
\[ T = T_0 Q|m_0 + T_1 Q|m_1 + T_2 Q|m_2 \]
with $T_i > 0$.

Our next result provides a classification of the critical points of $S|_{M_T}$. In fact, the conditions in this result agree with those obtained from Theorem A.

**Proposition 7.5.** Let $G/H$ equal $(SU(2) \times SU(2))/S^1_{p,q}$ with $p \neq q$. The functional $S|_{M_T}$ has a critical point if and only if
\[ T_1 \geq T_2 \quad \text{and} \quad T_0 T_1 > \frac{2(p^2 + q^2)}{p^2} (T_1 - T_2)^2 \]
or
\[ T_2 \geq T_1 \quad \text{and} \quad T_0 T_2 > \frac{2(p^2 + q^2)}{q^2} (T_1 - T_2)^2. \]

When it exists, the critical point is unique and is a global maximum.

**Proof.** We first show that the conditions in the proposition are necessary and sufficient and that the critical point is unique. Indeed, it is not difficult to solve the equation $\text{Ric}(g) = cT$ directly. To prove that the critical point is a global maximum, however, we need to apply Theorem A.

We set $r = \frac{p}{q}$ and $\alpha = \frac{2}{1 + r^2}$. One easily finds
\[ d_0 = 1, \quad d_1 = d_2 = 2, \quad b_0 = b_1 = b_2 = 8, \quad [011] = 4\alpha \quad \text{and} \quad [022] = 4\alpha r^2. \]
The structure constants unrelated to $[011]$ and $[022]$ by permutation are equal to 0. According to (1.5), the metric
\[ g = \frac{1}{y_0} Q|m_0 + \frac{1}{y_1} Q|m_1 + \frac{1}{y_2} Q|m_2 \]
has Ricci curvature $cT$ if and only if
\[ \frac{y_1}{y_0} = \frac{1}{\alpha} (4 - cT_1), \quad \frac{y_2}{y_0} = \frac{1}{\alpha r^2} (4 - cT_2), \quad cT_0 = \frac{1}{\alpha} (4 - cT_1)^2 + \frac{1}{\alpha r^2} (4 - cT_2)^2. \]
This system has positive solutions for some $c \in \mathbb{R}$ if and only if the line $y = cT_0$ intersects the curve
\[ y = \frac{1}{\alpha} (4 - cT_1)^2 \quad \text{and} \quad \frac{1}{\alpha r^2} (4 - cT_2)^2 \]
in the $(c, y)$-plane above the interval $(0, \frac{4}{\text{max}(T_1, T_2)})$. Elementary analysis shows that the conditions in the proposition are necessary and sufficient for this to happen. Moreover, there is at most one intersection. Thus, when it exists, the critical point is unique. We now show that it must be a global maximum.

Let $S^1_{lt}$ and $S^1_{rt}$ be the projections of $H$ onto the left and the right $SU(2)$ factor in $G$. As follows from the structure constants, there are three connected subgroups of $G$ containing $H$. The first one is a maximal torus $K_0 = T^2$. The other two are $K_1 = SU(2) \times S^1_{lt}$ and $K_2 = S^1_{lt} \times SU(2)$. Hence $\mathfrak{t}_0 = \mathfrak{h} \oplus m_0$ and $\mathfrak{t}_i = \mathfrak{h} \oplus m_0 \oplus m_i$ for $i = 1, 2$. The abelian subgroup $K_0$ is contained in each of the two maximal connected subgroups $K_1$ and $K_2$. Our plan is to apply Theorem A, and we hence
need to compute the constants \( \alpha_t \) and \( \beta_t \) with \( i = 1, 2 \). We carry out the computation for \( \xi_1 \), the one for \( \xi_2 \) being similar.

The metric \( Q \) makes each factor in \( SU(2) \times SU(2) \) a sphere of radius 1. The space \( G/K_1 = SU(2)/S^1 \), where \( S^1 \) denotes a circle, is the base of the Hopf fibration. Therefore, equipped with the restriction of the metric \( Q \), it becomes a sphere of radius \( \frac{1}{2} \) and scalar curvature 8. Normalising to satisfy the constraint \( 2T_2y_2 = 1 \), we find \( \beta_{t_1} = \frac{4}{T_2} \). The fibre \( K_1/H \) is the lens space \( S^3/Z_{2p+q} \). An \( H \)-invariant metric

\[
h = \frac{1}{y_0} Q_{[m_0]} + \frac{1}{y_1} Q_{[m_1]}, \quad y_0, y_1 > 0,
\]

on this space is a Berger-type metric with scalar curvature

\[
S(h) = 8y_1 - \alpha \frac{y_1^2}{y_0}.
\]

Under the constraint

\[
\text{tr}_h T|_{K_1/H} = T_0y_0 + 2T_1y_1 = 1,
\]

we have

\[
S(h) = 8y_1 - \alpha \frac{T_0y_1^2}{1 - 2T_1y_1}.
\]

One easily shows that the maximum of the expression on the right-hand side is achieved at

\[
y_1 = \frac{8}{16T_1 + \alpha T_0 + \sqrt{\alpha^2 T_0^2 + 16\alpha T_0 T_1}}
\]

with value

\[
\frac{4}{T_1} + \frac{\alpha T_0}{2T_1^2} - \frac{\sqrt{\alpha^2 T_0^2 + 16\alpha T_0 T_1}}{2T_1^2}.
\]

Thus

\[
\alpha_{t_1} = \frac{4}{T_1} + \frac{\alpha T_0}{2T_1^2} - \frac{\sqrt{\alpha^2 T_0^2 + 16\alpha T_0 T_1}}{2T_1^2} \quad \text{and} \quad \beta_{t_1} = \frac{4}{T_2}.
\]

Elementary analysis proves that \( \beta_{t_1} - \alpha_{t_1} > 0 \) when the condition of the proposition is satisfied. By analogous arguments, \( \beta_{t_2} - \alpha_{t_2} > 0 \). Using Theorem A, we conclude that the critical point must be a global maximum.

**Remark 7.8.** The condition in Proposition 7.5 can be expressed without case distinction:

\[
T_2 - \frac{q \sqrt{2(p^2 + q^2)T_0 T_2}}{2(p^2 + q^2)} < T_1 < T_2 + \frac{p^2 T_0 + p \sqrt{p^2 T_0^2 + 8(p^2 + q^2)T_0 T_2}}{4(p^2 + q^2)}.
\]

If \( g \) is given by (7.6), the constraint \( \text{tr}_g T = 1 \) takes the form

\[
T_0y_0 + 2T_1y_1 + 2T_2y_2 = 1,
\]

and the scalar curvature satisfies

\[
S(g) = 8y_1 + 8y_2 - \frac{2q^2}{p^2 + q^2} y_1 \frac{y_1^2}{y_0} - \frac{2p^2}{p^2 + q^2} y_2 \frac{y_2^2}{y_0}.
\]

As in the previous subsection, we can choose specific values for the components of \( T \) and sketch the graph of \( S \) as a function of \( y_1 \) and \( y_2 \). Figure 4 shows an example of a global maximum and a case with no critical points. The black planes are drawn at \( S = 0 \), and one can clearly see the direction of the abelian subgroup. The markings of \( \partial \Delta \) are as follows: two sides are the strata \( \Delta_{t_1} \) and \( \Delta_{t_2} \), the vertex adjacent to both of these side is \( \Delta_{t_0} \), and all the other strata are of type \( \Delta_{\infty} \). In the picture, one can recognize the markings and the maximal curvatures at the subalgebra strata.
7.3. The Stiefel manifold $V_2(\mathbb{R}^4)$. We now study the space $G/H = (SU(2) \times SU(2))/S_{p,q}$ as in the previous subsection but with $p = q = 1$. Thus, the circle group is embedded diagonally into $SU(2) \times SU(2)$. Since the twofold cover $SU(2) \times SU(2) \to SO(4)$ sends the diagonal embedding $\text{diag}(SU(2)) \subset SU(2) \times SU(2)$ to $SO(3) \subset SO(4)$, the space $G/H$ coincides with the Stiefel manifold $V_2(\mathbb{R}^4) = SO(4)/SO(2)$.

This example is special in that the set of decompositions is not discrete; in fact, as we will see below, $D$ is two-dimensional. As in the previous example, we identify $SU(2)$ and $\mathfrak{su}(2)$ with the group of unit quaternions and the Lie algebra of purely imaginary quaternions. Now

$$H = \{(e^{\eta i}, e^{\eta j}) \mid \eta \in [0, 2\pi)\} \quad \text{and} \quad \mathfrak{h} = \text{span}\{(i, i)\}.$$  

Setting

\begin{equation}
(7.9) \quad m_0 = \text{span}\{(i, -i)\}, \quad m_1 = \{(zj, 0) \mid z \in \mathbb{C}\} \quad \text{and} \quad m_2 = \{(0, zj) \mid z \in \mathbb{C}\},
\end{equation}

we obtain a decomposition as in (7.4). We saw that, under $\text{Ad}_H$, the element $(e^{\eta i}, e^{\eta j}) \in H$ takes $(zj, 0)$ and $(0, zj)$ to $(e^{2\eta i}zj, 0)$ and $(0, e^{2\eta i}zj)$, respectively. This implies that the restrictions of $\text{Ad}_H$ to $m_1$ and $m_2$ are equivalent complex representations. Using the $Q$-orthonormal bases $\{\frac{1}{\sqrt{2}}\}$ of $m_0$, $\{(j, 0), (k, 0)\}$ of $m_1$, and $\{(0, j), (0, k)\}$ of $m_2$, we can thus represent the metric $g \in \mathcal{M}$ and the tensor field $T$ by the matrices

\begin{equation}
(7.10) \quad g = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 1 & 0 & x_3 \\
0 & 0 & x_1 & -x_4 \\
0 & x_4 & x_3 & x_2
\end{pmatrix}
\quad \text{and} \quad T = \begin{pmatrix}
T_0 & 0 & 0 & 0 \\
0 & T_1 & T_3 & T_4 \\
0 & 0 & T_1 & -T_4 \\
0 & T_3 & -T_4 & T_2
\end{pmatrix},
\end{equation}

which must be positive-definite. This is the case if $x_0, x_1, x_2 > 0$ and $x_1x_2 - x_3^2 - x_4^2 > 0$ and similarly for $T$. The outer automorphism of $G = SU(2) \times SU(2)$ that switches the two factors preserves $H$ and hence induces an isometry of $(V_2(\mathbb{R}^4), Q)$. It acts on $g$ by taking $(x_1, x_2, x_3, x_4)$ to $(x_2, x_1, x_3, -x_4)$, and similarly on $T$. Conjugation by $(j, j)$ preserves $H$ as well. It acts on $g$ by taking $(x_1, x_2, x_3, x_4)$ to $(x_1, x_2, x_3, -x_4)$. Thus the composition takes $(x_1, x_2, x_3, x_4)$ to $(x_2, x_1, x_3, x_4)$, which shows that we may assume $T_1 \geq T_2$. It will be convenient for us to denote

$$\gamma(T_0, T_2, T_3, T_4) = \frac{T_0 + \sqrt{T_0^2 + 16T_0T_2} + 16\sqrt{T_0^2 + T_2^2}}{8}.$$  

As we will see shortly, the maximal subgroups of $G$ containing $H$ are the subgroups $K_1$ and $K_2$ from the proof of Proposition 7.5 and a one-parameter family of three-dimensional subgroups $K^\theta$, where
\( \theta \in [0, 2\pi) \), isomorphic to SU(2). Theorem A leads to the following result. We distinguish the case where the supremum \( \alpha_{G/H} \) is attained by one of the four-dimensional subgroups from the case where it is attained by a three-dimensional subgroup.

**Proposition 7.11.** Suppose that \( G/H = V_2(\mathbb{R}^4) \) and \( T_1 \geq T_2 \). Then we have:

(a) \( \alpha_{G/H} = \alpha_{t_2} \) and \( \beta_{t_2} - \alpha_{t_2} > 0 \) if and only if

\[
\gamma(T_0, T_2, T_3, T_4) < T_1 < T_2 + \frac{T_0 + \sqrt{T_0^2 + 16T_0T_2}}{8}.
\]

(b) \( \alpha_{G/H} = \alpha_{t^\theta} \) and \( \beta_{t^\theta} - \alpha_{t^\theta} > 0 \) for some \( \theta \in [0, 2\pi) \) if and only if

\[
\frac{T_0}{2} - T_2 + 4\sqrt{T_3^2 + T_4^2} < T_1 \leq T_0, T_2, T_3, T_4).
\]

In both cases, \( S_{\mathcal{M}_T} \) attains its global maximum.

**Proof.** We start by describing the space of all decompositions. Since the representations of \( H \) on \( m_1 \) and \( m_2 \) are complex equivalent representations, \( D \) is the two-parameter family of decompositions \( D_{s,\theta} \) with modules

\[
(7.12) \quad m_0 = \text{span}\{(i, -i)\}, \quad m_1^{s,\theta} = \{(se^{\theta}z, -tz, j) \mid z \in \mathbb{C}\} \quad \text{and} \quad m_2^{s,\theta} = \{(te^{\theta}z, j, sz, j) \mid z \in \mathbb{C}\},
\]

where \( s^2 + t^2 = 1 \). Let us fix a \( Q \)-orthonormal basis in \( m_1^{s,\theta} \) consisting of the vectors

\[
(7.13) \quad v_1 = (s \cos \theta, j, s \sin \theta, k, -t, j) \quad \text{and} \quad v_2 = (-s \sin \theta, j, s \cos \theta, k, -t, k)
\]

and one in \( m_2^{s,\theta} \) consisting of

\[
(7.14) \quad w_1 = (t \cos \theta, j, t \sin \theta, k, s, j) \quad \text{and} \quad w_2 = (-t \sin \theta, j, t \cos \theta, k, s, k).
\]

Using these bases, for the decomposition \( D_{s,\theta} \), one easily computes

\[
(7.15) \quad [011] = [022] = 4(s^2 - t^2)^2 \quad \text{and} \quad [012] = 8s^2t^2.
\]

The structure constants unrelated to these by permutation are 0.

The decomposition corresponding to \( s = 1 \) coincides with (7.9) and gives rise to the intermediate subgroups

\[
K_0 = T^2, \quad K_1 = SU(2) \times S^1_{s=t} \quad \text{and} \quad K_2 = S^1_{s=t} \times SU(2)
\]

with Lie algebras

\[
\mathfrak{k}_0 = \mathfrak{h} \oplus m_0, \quad \mathfrak{k}_1 = \mathfrak{h} \oplus m_0 \oplus m_1 \quad \text{and} \quad \mathfrak{k}_2 = \mathfrak{h} \oplus m_0 \oplus m_2.
\]

If \( s = t = \frac{\sqrt{2}}{2} \), we have, for each \( \theta \), the intermediate subgroup

\[
K^\theta \simeq SU(2) \quad \text{with Lie algebra} \quad \mathfrak{k}^\theta = \mathfrak{h} \oplus m_1^{s,\theta}.
\]

The remaining decompositions do not produce any subgroups. Thus, there are three isolated intermediate subgroups, \( K_0, K_1 \) and \( K_2 \), as well as a one-parameter family of subgroups \( K^\theta \).

The identity component of the normalizer of \( H \) is given by \( N_0(H) = T^2 \subset SU(2) \times SU(2) \), and hence \( N_0(H)/H \simeq S^1 \), represented by elements of the form \((e^{\theta}, 1) \in N_0(H)\). These elements act via right translation on \( G/H \) and via conjugation on \( m \). Thus they also act on \( D \) and, via pullback, on \( \mathcal{M} \). It is easy to see that \((e^{\theta}, 1)\) takes \( D_{s,\theta} \) to \( D_{s,\theta+2\eta} \). This implies, in particular, that the subalgebras \( \mathfrak{k}^\theta \) are all conjugate to each other by elements of \( N_0(H) \). Since \( N_0(H) \) acts on \( G/H \) by isometries in \( Q \), it follows that \((G/K^\theta, Q)\), as well as \((K^\theta/H, Q)\), are all isometric to each other.
We now compute the constants $\alpha_t$ and $\beta_t$ for the intermediate maximal subalgebras. Let us begin with the four-dimensional ones. Arguing as in the proof of Proposition 7.5, we find

$$\alpha_t = \frac{8T_i + T_0 - \sqrt{T_0^2 + 16T_0T_i}}{2T_i^2} \quad \text{and} \quad \beta_t = \frac{4}{T_0},$$

where $(i, j)$ is a permutation of $\{1, 2\}$. The assumption $T_1 \geq T_2$ implies that $\alpha_{t_2} \geq \alpha_{t_1}$. Furthermore, $\beta_{t_2} - \alpha_{t_2} > 0$ if and only if

$$T_1 < T_2 + \frac{T_0 + \sqrt{T_0^2 + 16T_0T_2}}{8}. \tag{7.16}$$

For the three-dimensional subalgebras $\mathfrak{k}^\theta$, we use the bases (7.13) and (7.14). Now, the parameters $s$ and $t$ both equal $\frac{1}{\sqrt{2}}$. The tensor field $T$ satisfies

\begin{align*}
2T(v_1, v_1) &= 2T(v_2, v_2) = T_1 + T_2 - 2(\cos \theta T_3 - \sin \theta T_4), \\
2T(w_1, w_1) &= 2T(w_2, w_2) = T_1 + T_2 + 2(\cos \theta T_3 - \sin \theta T_4), \\
2T(v_1, w_1) &= 2T(v_2, w_2) = T_1 - T_2,
\end{align*}

and $T(v_1, v_2) = T(w_1, w_2) = 0$. The action of the quotient $N_0(H)/H$ establishes an isometry between $(K^\theta/H, Q)$ and

$$(\text{diag}(SU(2))/\text{diag}(S^4), Q) = (SU(2)/S^4, 2Q'),$$

where $Q'(X, Y) = -\frac{1}{2} \text{tr}(XY)$. Equipped with the biinvariant metric $Q'$, the group $SU(2)$ is a sphere of radius 1, and hence $(SU(2)/S^4, Q')$ is a sphere of radius $\frac{1}{2}$ since it is the base of the Hopf fibration. Thus $(SU(2)/S^4, 2Q')$ has scalar curvature 4. The trace constraint means that $g(v_1, v_1) = 2T(v_1, v_1)$, which implies

$$\alpha_{t^\theta} = \frac{4}{T_1 + T_2 - 2(\cos \theta T_3 - \sin \theta T_4)}.$$

The action of the normalizer shows that $(G/K^\theta, Q)$ are all isometric to the symmetric space $((SU(2) \times SU(2))/\text{diag}(SU(2)), Q)$, and hence their sectional curvatures are given by

$$\sec \left( \frac{(i, -i)}{\sqrt{2}}, \frac{(j, -j)}{\sqrt{2}} \right) = \frac{1}{2} \left| \left| (i, -i), (j, -j) \right| \right|^2_Q = |(k, k)|^2_Q = 2.$$

Consequently, $(G/K^\theta, Q)$ have scalar curvature 12. The trace constraint for $K^\theta$-invariant metrics on $G/K^\theta$ takes the form

$$g\left( \frac{(i, -i)}{\sqrt{2}}, \frac{(i, -i)}{\sqrt{2}} \right) = g(w_1, w_1) = g(w_2, w_2) = T\left( \frac{(i, -i)}{\sqrt{2}} \right) + T(w_1, w_1) + T(w_2, w_2)
= T_0 + T_1 + T_2 + 2(\cos \theta T_3 - \sin \theta T_4).$$

Thus

$$\beta_{t^\theta} = \frac{12}{T_0 + T_1 + T_2 + 2(\cos \theta T_3 - \sin \theta T_4)}.$$

We now choose an angle $\theta_0$ such that $\alpha_{t^\theta_0}$ is maximal, i.e., $\alpha_{t^\theta_0} = \sup_{\theta} \alpha_{t^\theta}$. Observe that

$$\cos \theta T_3 - \sin \theta T_4 = \sqrt{T_3^2 + T_4^2} \sin(\theta - \eta)$$
for some phase shift $\eta$. The largest possible value of this quantity is $\sqrt{T_3^2 + T_4^2}$, which means

$$\alpha_{\phi_0} = \frac{4}{T_1 + T_2 - 2\sqrt{T_3^2 + T_4^2}}$$

and

$$\beta_{\phi_0} = \frac{12}{T_0 + T_1 + T_2 + 2\sqrt{T_3^2 + T_4^2}}.$$ 

Clearly, $\beta_{\phi_0} - \alpha_{\phi_0} > 0$ if and only if

(7.18) \[ T_0 < 2T_1 + 2T_2 - 8\sqrt{T_3^2 + T_4^2}. \]

Let us determine $\alpha_{G/H}$. As noted above, the assumption $T_1 \geq T_2$ implies that $\alpha_{\phi_2} \geq \alpha_{\phi_1}$. Elementary analysis shows that $\alpha_{\phi_2} \geq \alpha_{\phi_0}$ if and only if $T_1 \geq \gamma(T_0, T_2, T_3, T_4)$. Combining this condition with (7.16), we arrive at statement (a) of the proposition. Reversing the inequality and taking note of (7.18), we obtain statement (b). \(\square\)

**Remark 7.19.** The action of $N_0(H)/H$ fixes the diagonal metrics, i.e., those $g$ of the form (7.10) for which $x_3 = x_4 = 0$. Consequently, the Ricci curvature of a diagonal metric must be diagonal as well. In particular, if $T_3 = T_4 = 0$, then every solution to the system (7.7) defines a diagonal metric with $\text{Ric}(g) = cT$. On the other hand, given a critical point of $S|\mathcal{M}_T$ with $T_3 = T_4 = 0$ and $x_3 \neq 0$, one obtains a circle of further critical points by applying the normalizer.

For a metric $g$ as in (7.10), the constraint $\text{tr}_g T = 1$ becomes

$$\frac{T_0}{x_0} + \frac{2x_1T_2 + 2x_2T_1 - 4x_3T_3 - 4x_4T_4}{x_1x_2 - x_3^2 - x_4^2} = 1,$$

and the scalar curvature satisfies

$$S(g) = \frac{8}{x_0} + \frac{8(x_1 + x_2)}{\Lambda} - \frac{8x_1x_2}{x_0\Lambda} - \frac{x_0(x_1 - x_2)^2}{\Lambda^2} - \frac{2x_0}{\Lambda},$$

where $\Lambda = x_1x_2 - x_3^2 - x_4^2$; see, e.g., [8]. We may assume that $T_0 = 1$. If $T_3^2 + T_4^2 = 0$, we show in Figure 5 points in the $(T_1, T_2)$-plane that correspond to various behaviors of $S|\mathcal{M}_T$. In the union of the three grey regions Proposition 7.5 and Remark 7.19 guarantee the existence of a (diagonal) critical point. In the dark-grey region Proposition 7.11 (a) yields a global maximum, and in the middle-grey region Proposition 7.11 (b) applies. A computer-assisted experiment suggests that this global maximum is always a diagonal metric; see Example 4.

If $T_3^2 + T_4^2 \neq 0$, the picture is the same but with the line $2T_1 = 1 - 2T_2$ and the curve $8T_1 = 1 + \sqrt{1 + 16T_2}$ shifted to the right and the reflection of $8T_1 = 1 + \sqrt{1 + 16T_2}$ shifted up. In this case, however, one cannot use Proposition 7.5 to find critical points in the light-grey region.

Next, we give several examples that illustrate various behaviors.

**Example 2.** Suppose that $T_0 = 1$, $T_1 = T_2 = t > 0$ and $T_3^2 + T_4^2 = 0$. A straightforward computation using (7.7) shows that the diagonal metrics with

(7.20) \[ x_0 = \sqrt{32t + 1}, \quad x_1 = x_2 = \frac{32t + 1 + \sqrt{32t + 1}}{8} \quad \text{and} \quad x_3 = x_4 = 0 \]

are critical points of the functional $S|\mathcal{M}_T$, when restricted to the set of diagonal metrics. By Remark 7.19, they are also critical points of the scalar curvature functional on all of $\mathcal{M}_T$. Maple shows that all four eigenvalues of the Hessian are negative for $t > \frac{1}{4}$, i.e., (7.20) defines a local maximum. On the other hand, if $t < \frac{1}{4}$, we have two negative and two positive eigenvalues, which means (7.20) is a saddle. Thus global maxima among the set of diagonal metrics in $\mathcal{M}_T$ can, in fact, be saddles on all of $\mathcal{M}_T$. By continuity, there exists a neighborhood in $\mathcal{M}_T$ of the green line segment $T_1 = T_2 < \frac{1}{4}$ in Figure 5 where $S|\mathcal{M}_T$ has a saddle. We point out that (7.20) is the Jensen Einstein metric when $t = \frac{1}{4}$, and a local maximum. It is marked with the blue dot in Figure 5.
Also notice that the metrics (7.20) with $t \in \left( -\frac{1}{32}, 0 \right]$ still have Ricci curvature $T$, but now $T$ is indefinite.

Our next example is similar in spirit to Example 1, obtained by going from a region with local maxima into one with saddles. It exhibits a two-dimensional set of critical points.

**Example 3.** From the previous example we see that the metric with $t = \frac{1}{4}$ must be special. Thus we choose $T_0 = 1$, $T_1 = T_2 = \frac{1}{4}$ and $T_3^2 + T_4^2 = 0$. This choice of $T$ is marked by the red dot in Figure 5. Direct verification shows that the metrics $g \in \mathcal{M}_T$ with

\begin{equation}
  \begin{align*}
    x_0 &= 2t, & x_1 &= x_2 = t, & x_3 &= t \sqrt{\frac{2t - 3}{2t - 1}} \cos \psi & \text{and} & x_4 &= t \sqrt{\frac{2t - 3}{2t - 1}} \sin \psi
  \end{align*}
\end{equation}

are critical points of $S_{|\mathcal{M}_T}$ for $t \in \left[ \frac{3}{2}, \infty \right)$ and $\psi \in [0, 2\pi)$. These metrics all have scalar curvature 8. They form a surface diffeomorphic to $\mathbb{R}^2$, described in the coordinates $(t, \psi)$. The normalizer $N_0(H)/H$ acts on this surface via $(t, \psi) \rightarrow (t, \psi + 2\eta)$. Thus metrics with the same value of $t$ are isometric. On the other hand, the squared volume of the metric (7.21) equals $\frac{8t^5}{(2t - 1)}$, and hence metrics with different values of $t$ are not isometric. To determine the critical point type of (7.21), we compute the eigenvalues of the Hessian of $S_{|\mathcal{M}_T}$. Two of them are always negative, and the other two vanish. The 0-eigenspace is tangent to the surface of critical points. Consequently, this surface is a non-degenerate critical submanifold with index 2. Using the Morse–Bott lemma, we conclude that it is isolated and consists of local maxima. We suspect that they are in fact global maxima. Notice that in this example the critical submanifold does not contain any Einstein metrics.

There are also arcs of critical points on $V_2(\mathbb{R}^4)$ consisting of non-diagonal metrics. Indeed, recall that this space supports two circles of Einstein metrics isometric to the canonical product Einstein metric on $V_2(\mathbb{R}^4) \simeq S^3 \times S^2$; see [8]. Scaling each factor does not change the Ricci curvature and hence yields an arc of critical points. These arcs turn out to be non-degenerate critical submanifolds with index 3 consisting of local maxima. In contrast, Einstein metrics in the two circles are saddles of the functional $S_{|\mathcal{M}_1}$ with co-index 1.
Example 4. Assume that $T_3^2 + T_4^2 = 0$. A computer-assisted experiment indicates that $S_{|\mathcal{M}_T}$ has a non-diagonal critical point if and only if $T$ lies in the pink region in Figure 6. Since the action of $N_0(H)/H$ leaves $T$ unchanged, we obtain a circle of non-diagonal critical points for each $T$. It is always a non-degenerate critical submanifold of index 2 and co-index 1. In addition, Proposition 7.5 guarantees the existence of a diagonal critical point for $T$ between the thick curves. By Proposition 7.11, this diagonal critical point must be a global maximum when $T$ lies in the dark-grey or the middle-grey region in Figure 5. Computation of the eigenvalues of the Hessian indicates that it is a local maximum for $T$ in the rest of the dotted region and a saddle with index and co-index 2 for $T$ in the yellow region. The transition from the pink to the yellow region is achieved across the curve

$$T_1(t) = \frac{4l^2(1-t)}{16t^4 + 1}, \quad T_2(t) = \frac{4(t - 1)}{16t^4 + 1}, \quad t \in (\frac{1}{4}, 1).$$

This is precisely the curve where non-diagonal critical points merge with diagonal ones. The Hessian at each of these merged critical points has two negative and two zero eigenvalues. However, we do not know whether these merged critical points lie on critical submanifolds as in Example 3. Moving out of the pink region on the other side, the non-diagonal critical points cannot become diagonal and hence disappear. As this happens, the Hessian must have two zero eigenvalues.

To conduct the experiment, we first observe that $\text{Ric}(g)$ is diagonal if and only if $x_3 = x_4 = 0$ or $x_5 = 4x_1x_2$. Since the action of $N_0(H)/H$ leaves diagonal Ricci curvature unchanged, it suffices to consider only metrics $g$ with $x_4 = 0$. We take two million non-diagonal $g$ satisfying $x_0^2 = 4x_1x_2$ and $x_4 = 0$ and calculate $\text{Ric}(g)$. After checking that $\text{Ric}(g)$ is positive-definite and normalizing so that its $m_0$ component becomes 1, we mark the $m_1$ and $m_2$ components in the $(T_1, T_2)$-plane as in Figure 6. The obtained values fill up the pink region. Finally, we compute the eigenvalues of the Hessian to determine the critical point types.

For a specific example, choose $T_0 = 1$, $T_1 = \frac{13}{172}$ and $T_2 = \frac{15}{172}$, the green dot in Figure 6. Direct verification shows that the gradient of $S_{|\mathcal{M}_T}$ vanishes at the circle of metrics with $x_0 = \frac{181}{59}$, $x_1 = \frac{905}{472}$, $x_2 = \frac{302}{205}$ and $x_3^2 + x_4^2 = \frac{32761}{55096}$. These metrics are all saddles with index 2 and co-index 1. At the same time, the diagonal metric with

$$x_0 = \frac{\sqrt{20866}}{59}, \quad x_1 = \frac{62598 + 277\sqrt{20866}}{54044} \quad \text{and} \quad x_2 = \frac{83464 - 439\sqrt{20866}}{22892}$$

is a strict local maximum. It cannot be a global maximum because its scalar curvature is less than $\alpha_\phi$ and there are no further critical points.

Example 5. Since the Stiefel manifold is a generalized Wallach space, Theorem B implies the existence of critical points of $S_{|\mathcal{M}_T}$ for a large set of candidates $T$. According to (7.15), the structure constant $[123]$ is the only nonzero structure constant for the decomposition (7.12) when $s = t = \frac{\sqrt{2}}{2}$. To apply Theorem B, we need $T$ to be diagonal with respect to this decomposition. According to (7.17), we can achieve this, as long as $T_1 = T_2$, by choosing $\theta = \arctan \frac{T_2}{T_3}$. The three intermediate subalgebras are $\mathfrak{e}_0$, $\mathfrak{e}_\phi$ and $\mathfrak{e}_{\phi + \pi}$. Using the formulas for $\alpha_\phi$ and $\beta_\phi$ in the proof of Proposition 7.11, we see that $\beta_\phi - \alpha_\phi < 0$ and $\beta_{\phi + \pi} - \alpha_{\phi + \pi} < 0$ if and only if

$$T_1 < \frac{1}{4} - \frac{2|T_3^2 - T_4^1|}{\sqrt{T_3^2 + T_4^2}},$$

in which case $S_{|\mathcal{M}_T}$ has a critical point. When $T_4 = 0$, this inequality simplifies to $T_1 < \frac{1}{4} - 2|T_3|$, and when $T_3 = T_4 = 0$, it becomes $T_1 < \frac{1}{4}$.

7.4. The Ledger–Obata space $H^3/\text{diag}(H)$ and $\text{Spin}(8)/G_2$. Our final example is motivated by the following observations. Let the homogeneous space $G/H$ be such that $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2$ with both summands irreducible. When $\mathfrak{m}_1$ and $\mathfrak{m}_2$ are inequivalent, one can solve the equation $\text{Ric}(g) = cT$
directly; see [31]. There are only two cases where \( m_1 \) and \( m_2 \) are equivalent representations. If \( G \) is simple, then \( G/H \) must equal \( \text{Spin}(8)/G_2 = S^7 \times S^7 \); see the classification in [18, 22]. If it is not, one easily checks that the only possibility is that \( G/H \) is the Ledger–Obata space \( H^3/\text{diag}(H) \), where \( H \) is simple and the diagonal embedding is given by

\[
H \ni h \mapsto \text{diag}(h) = (h, h, h) \in H^3.
\]

We discuss here the latter case, the former one being quite similar; see Remark 7.29. In what follows, \((e_i)\) is a basis of \( \mathfrak{h} \) orthonormal with respect to \(-B_H\), the negative of the Killing form of \( H \).

Let \( G/H = H^3/\text{diag}(H) \). There are exactly three intermediate subgroups, namely,

\[
K_1 = \{(a, b, b) \in G \mid a, b \in H\},
K_2 = \{(a, b, a) \in G \mid a, b \in H\} \quad \text{and} \quad K_3 = \{(a, a, b) \in G \mid a, b \in H\}.
\]

The outer automorphism \( R : G \to G \) given by \( R(a, b, c) = (c, a, b) \) interchanges these subgroups. Choose the biinvariant metric \( Q \) on \( G \) to be \( Q = -B_H - B_H - B_H \). Then \( R \) is an isometry of \((G, Q)\).

Fix an \( \text{Ad}_H \)-invariant, \( Q \)-orthogonal decomposition of \( m \) by setting

\[
(7.22) \quad m_1 = \{(-2X, X, X) \in g \mid X \in \mathfrak{h}\} \quad \text{and} \quad m_2 = \{(0, X, -X) \in g \mid X \in \mathfrak{h}\}.
\]

Then \( \mathfrak{k} = \mathfrak{h} \oplus m_1 \). Furthermore, \( \text{Ad}_H \) is an irreducible representation of real type on each \( m_i \) since \( H \) is simple. The collections \( \left( \frac{1}{\sqrt{2}} (-2e_j, e_j, e_j) \right) \) and \( \left( \frac{1}{\sqrt{2}} (0, e_j, -e_j) \right) \) constitute \( Q \)-orthonormal bases of \( m_1 \) and \( m_2 \). With respect to these bases, the metric and \( T \) have the form

\[
(7.23) \quad g = \begin{pmatrix} x_1 & x_3 \\ x_3 & x_2 \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} T_1 & T_3 \\ T_3 & T_2 \end{pmatrix}
\]

with \( \det g > 0 \) and \( \det T > 0 \). (Each entry in these matrices represents a scalar \( a \times a \) matrix, where \( a = \dim H \).) Applying Theorem A, we obtain the following result.

**Proposition 7.24.** Suppose that \( G/H = H^3/\text{diag}(H) \) and \( T_1T_2 > T_3^2 \). The functional \( S_{|M_T} \) attains its global maximum if

\[
(7.25) \quad \frac{3}{4} T_2 < T_1 < \frac{9}{5} T_2 - \frac{14\sqrt{3}}{5} |T_3|.
\]
Proof. We need to compute $\alpha_t$ and $\beta_t$. A well-known formula for the sectional curvature of a normal homogeneous metric (see [5, Proposition 7.87b]) implies
\[
S(Q_{G/K}) = \sum_{j,k} \sec(Q_{G/K}) \left( \frac{(0, e_j, -e_j)}{\sqrt{2}}, \frac{(0, e_k, -e_k)}{\sqrt{2}} \right) = \frac{1}{2} \sum_{j,k} |e_j, e_k|^2,
\]
where $|\cdot|$ is the norm corresponding to $-B_H$. Similarly,
\[
S(Q_{K_1/H}) = \sum_{j,k} \sec(Q_{K_1/H}) \left( \frac{(-2e_j, e_j, e_j)}{\sqrt{6}}, \frac{(-2e_k, e_k, e_k)}{\sqrt{6}} \right) = \frac{3}{8} \sum_{j,k} |e_j, e_k|^2.
\]
In addition, $(K_1/H, Q)$ is an isotropy irreducible symmetric space, and hence $[m_1, m_1] \subset \mathfrak{h}$. It follows from (1.4) that
\[
S(Q_{G/K}) = \frac{a}{2} \quad \text{and, by the above,} \quad S(Q_{K_1/H}) = \frac{3a}{8}
\]
with $a = \dim H$. The trace constraints on $G/K_1$ and $K_1/H$ have the form $x_2 = aT_2$ and $x_1 = aT_1$, respectively. We conclude that
\[
\alpha_t = \frac{3}{8T_1} \quad \text{and} \quad \beta_t = \frac{1}{2T_2},
\]
which means $\beta_t - \alpha_t > 0$ if and only if $4T_1 > 3T_2$.

As observed above, the automorphism $R$ takes $K_1$ to $K_2$. Since $R^3$ is the identity, the matrix of the pullback of $T$ by $R$ with respect to our fixed bases of $m_1$ and $m_2$ is
\[
\begin{pmatrix}
-\frac{1}{2} & \frac{\sqrt{3}}{2} \\
\frac{\sqrt{3}}{2} & -\frac{1}{2}
\end{pmatrix}
\begin{pmatrix}
T_1 & T_3 \\
T_3 & T_2
\end{pmatrix}
\begin{pmatrix}
-\frac{1}{2} & -\frac{\sqrt{3}}{2} \\
\frac{\sqrt{3}}{2} & -\frac{1}{2}
\end{pmatrix}
= \begin{pmatrix}
\frac{T_1 + 3T_2 - 2\sqrt{3}T_3}{\sqrt{3}(T_1 - T_2) - 2T_3} & \frac{\sqrt{3}(T_1 - T_2) - 2T_3}{\sqrt{3}(T_1 - T_2) - 2T_3} \\
\frac{\sqrt{3}(T_2 - T_1) - 2T_3}{\sqrt{3}(T_2 - T_1) - 2T_3} & \frac{3T_1 + T_2 + 2\sqrt{3}T_3}{3T_1 + T_2 + 2\sqrt{3}T_3}
\end{pmatrix}.
\]
The maps from $(G/K_1, Q)$ to $(G/K_2, Q)$ and from $(K_1/H, Q)$ to $(K_2/H, Q)$ induced by $R$ are isometries. They preserve the scalar curvature, so
\[
\alpha_t = \frac{3}{2(T_1 + 3T_2 - 2\sqrt{3}T_3)} \quad \text{and} \quad \beta_t = \frac{2}{3T_1 + T_2 + 2\sqrt{3}T_3}.
\]
Consequently, $\beta_t - \alpha_t > 0$ if and only if $5T_1 < 9T_2 - 14\sqrt{3}T_3$.

To obtain the third subgroup, we need to apply $R$ to $K_2$, or $R^2$ to $K_1$. The matrix of the pullback of $T$ by $R^2$ with respect to (7.22) is
\[
\begin{pmatrix}
\frac{T_1 + 3T_2 + 2\sqrt{3}T_3}{\sqrt{3}(T_2 - T_1) - 2T_3} & \frac{\sqrt{3}(T_2 - T_1) - 2T_3}{\sqrt{3}(T_2 - T_1) - 2T_3} \\
\frac{\sqrt{3}(T_2 - T_1) - 2T_3}{\sqrt{3}(T_2 - T_1) - 2T_3} & \frac{3T_1 + T_2 + 2\sqrt{3}T_3}{3T_1 + T_2 + 2\sqrt{3}T_3}
\end{pmatrix}.
\]
This implies
\[
\alpha_t = \frac{3}{2(T_1 + 3T_2 + 2\sqrt{3}T_3)} \quad \text{and} \quad \beta_t = \frac{2}{3T_1 + T_2 - 2\sqrt{3}T_3}.
\]
Thus $\beta_t - \alpha_t > 0$ if and only if $5T_1 < 9T_2 + 14\sqrt{3}T_3$.

There are two scenarios for the condition of Theorem A to be satisfied. One is that $\alpha_t \geq \max\{\alpha_t, \alpha_t\}$ (equivalently, $3T_1 \leq 3T_2 - 2\sqrt{3}|T_3|$), in which case it suffices to demand that $4T_1 > 3T_2$. The other is that $\alpha_t = \max\{\alpha_t, \alpha_t\}$. Then we need $5T_1 < 9T_2 - 14\sqrt{3}|T_3|$. Combining these conditions, we arrive at (7.25).

For $g$ and $T$ of the form (7.23), the constraint $\text{tr}_g T = 1$ is
\[
\frac{a(x_1T_2 + x_2T_1 - 2x_3T_3)}{x_1x_2 - x_3^2} = 1,
\]
and a computation shows that the scalar curvature is given by

\[ S(g) = \frac{a(9x_1x_2^2 + 12x_1^2x_2 - 6x_1x_2^2 - 18x_2x_3^2 - x_3^3)}{24(x_1x_2 - x_3^2)^2}. \]

Thus we have the same critical points, up to scaling, no matter what group $H$ we choose. The formulas

\[ x = \frac{1}{2}(T_1 - T_2) \quad \text{and} \quad y = T_3 \]

define coordinates in the space of Ricci candidates $T$ normalised so that $T_1 + T_2 = 1$. Figure 7 shows points in the $(x, y)$-plane that correspond to different behaviors of $S_{|\mathcal{M}_T}$. The tensor $T$ is positive-definite if and only if $x^2 + y^2 < \frac{1}{4}$. The order-three automorphism $R$ yields a natural symmetry on the space of Ricci candidates. Its action rotates the picture in Figure 7 by $\frac{2\pi}{3}$. The inside of the large triangle is the set of $(x, y)$ that satisfy the condition of Proposition 7.24. The equalities $\alpha_{t_1} = \alpha_{G/H}$, $\alpha_{t_2} = \alpha_{G/H}$ and $\alpha_{t_3} = \alpha_{G/H}$ hold in the dark-grey, the middle grey and the light-grey triangle, respectively. A computer-assisted experiment with one million metrics, similar to the one explained in Section 7.1, indicates that $S_{|\mathcal{M}_T}$ also has a critical point for $(x, y)$ in the pink regions. Indefinite tensors that are Ricci curvatures of metrics up to scaling fill up the blue regions. Thus the image of the Ricci map is the union of the grey, pink and blue areas.

Maple is able to solve the Euler–Lagrange equations for $S_{|\mathcal{M}_T}$ for any specific choice of $T$. It suggests that the solution representing a metric, when it exists, is unique and is always a global maximum.

**Example 6.** Assume that $T_1 = t$, $T_2 = 1 - t$ and $T_3 = 0$. One easily checks that the formulas

\[ x_1 = a\sqrt{-20t^2 + 30t - 9}, \quad x_2 = a\frac{-20t^2 + 30t - 9 + t\sqrt{-20t^2 + 30t - 9}}{3(7t - 3)} \quad \text{and} \quad x_3 = 0 \]

define a metric with positive Ricci curvature for $\frac{3}{4} < t < 1$, and this metric is a critical point of $S_{|\mathcal{M}_T}$. Computing the Hessian, we conclude that it is a strict local maximum unless $t = \frac{3}{4}$. In
Figure 7 the Ricci candidates with \( T_1 = t \in (\frac{3}{4}, 1) \), \( T_2 = 1 - t \) and \( T_3 = 0 \) occupy the segment of the \( x \)-axis with \( x \in (-\frac{1}{14}, \frac{1}{2}) \). The orbit of this segment under \( R \), indicated by the green lines, gives two further segments with the same behavior. Continuity of \( S_{|\mathcal{M}_T} \) implies that the orbit of \( \{(x, 0) \mid x \in (-\frac{1}{14}, \frac{1}{2}) \setminus \{\frac{1}{2}\}\} \) has a neighborhood in the space of positive-definite Ricci candidates where \( S_{|\mathcal{M}_T} \) admits a local maximum. The computer experiment mentioned above indicates that this neighbourhood coincides with the union of the pink and grey regions.

The case of \( t = \frac{3}{4} \), respectively \( x = \frac{1}{2} \), is special. The orbit of the corresponding tensor \( T \) under \( R \) is depicted by the red dots in Figure 7. The three tensors in this orbit are, in fact, Einstein metrics. The \( H^3 \)-equivariant diffeomorphisms from \( H^3/\text{diag}(H) \) to \( H \times H \) given by

\[
(a, b, c) \text{ diag}(H) \mapsto (ac^{-1}, bc^{-1}),
\]

\[
(a, b, c) \text{ diag}(H) \mapsto (ab^{-1}, cb^{-1}) \quad \text{and} \quad (a, b, c) \text{ diag}(H) \mapsto (ba^{-1}, ca^{-1})
\]

are isometries between these Einstein metrics and the canonical Einstein product metric on \( H \times H \). Scaling the factors in the product metric, we obtain arcs of metrics with Ricci curvature equal (up to a scalar multiple) to the tensor at each of the red dots. As it turns out, these arcs are non-degenerate critical submanifolds with index 1 and co-index 0 and hence consist of local maxima.

The only other Einstein metric on \( H^3/\text{diag}(H) \) is the normal homogeneous metric induced by \( Q \); see [15]. It corresponds to the origin in Figure 7 and is a strict local maximum of the associated functional \( S_{|\mathcal{M}_T} \).

**Remark 7.28.** It is interesting to note that formulas (7.27) define a mixed-signature metric with positive-definite Ricci curvature for \( t \in \left(\frac{3}{4} - \frac{3\sqrt{3}}{20}, \frac{3}{7}\right) \), a Riemannian metric with positive-definite Ricci curvature for \( t \in \left(\frac{2}{7}, 1\right) \), and a Riemannian metric with indefinite Ricci curvature for \( t \in \left(1, \frac{3}{4} + \frac{3\sqrt{3}}{20}\right) \).

**Remark 7.29.** The remaining homogeneous space with two equivalent summands is \( Spin(8)/G_2 \). It is, in fact, quite similar. The metric and \( T \) take the same form. There are three intermediate subgroups isomorphic to \( Spin(7) \), which are permuted by the triality automorphism of \( Spin(8) \). The only difference is that the fibres \( Spin(7)/G_2 \) all have scalar curvature \( \frac{63}{7} \) and the bases all have scalar curvature \( 42 \). One now easily sees that the conditions for a global maximum on \( Spin(8)/G_2 \) are, in fact, the same as on the Ledger–Obata space, i.e., Theorem A guarantees the attainment of the global maximum if (7.25) holds. Moreover, the scalar curvature on \( Spin(8)/G_2 \) is given by the same formula as (7.26) with \( \alpha = 84 \); see [23]. The behavior of the critical points is thus the same as for the Ledger–Obata space. The set of Einstein metrics consists of three metrics isometric to the canonical product Einstein metric on \( S^7 \times S^7 \), and the normal homogeneous metric.

**References**


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