Lecture notes for Math 260P: Group actions

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1 Introduction

Given a group G and a set M, a group action (of G on M) is a map $G \times M \to M$ written as $(g, p) \mapsto g \cdot p$ satisfying

1. $e \cdot p = p$ for all $p \in M$ ("the identity acts as the identity")
2. $g_1 \cdot (g_2 \cdot p) = (g_1 g_2) \cdot p$ for all $g_1, g_2 \in G$ and $p \in M$ (associativity)

This notion is familiar from group theory, as some groups are *defined* by their action (e.g., the symmetric group *is what it does* – the elements are permutations). In topology and geometry, one can add structure to G and M and therefore to the group action of G on M. Some examples are the following:

1. If G is a topological group (i.e., a group whose underlying set has a topology such that both group operations are continuous) and M is a topological space, we might ask that the group action is continuous, in which case the action is a *continuous action*.
2. If G is a Lie group (i.e., a group with a smooth manifold structure such that the group operations are smooth) and M is a smooth manifold, then one can study *smooth actions* of G on M.
3. If G is a Lie group and M is a Riemannian manifold, then one can study *isometric actions*.

We will discuss basis properties of group actions in Section 3. In this section, we will discuss two familiar situations in which group actions arise naturally. These are surfaces of revolution and spaces of constant curvature. In both cases, we will start with a well-known Riemannian manifold, and show that it contains a large group of symmetries (called isometries).

1.1 Surfaces of revolution

Given a function $F : [0, \pi] \to \mathbb{R}$ that is positive on the interior of $[0, \pi]$ and zero at the end points, we can rotate the graph

$$\operatorname{graph}(F) = \{(t, F(t)) \mid t \in [0, \pi]\}$$

of F around the x-axis to obtain a (smooth, if we're lucky – see [Pet06, Chapter 1]) surface M^2 in $\mathbb{R}^3$. This surface, as a set, is

$$M = \{(t, F(t)\cos\theta, F(t)\sin\theta) \mid t \in [0, \pi], \theta \in [0, 2\pi]\}.$$

If F is smooth, then the surface is smooth away from $t = 0$ and $t = \pi$. If $F(0) > 0$, then the points on M corresponding to $t = 0$ form a smooth boundary component. If $F(0) = 0$, then M is smooth at the point $(0, 0, 0)$ if and only if the function $f(s)$ defined below satisfies $f'(0) = 1$ and $f^{(r)}(0) = 0$ for all even $r \geq 1$ (i.e., every even derivative vanishes at $t = 0$ – see [Pet06, p. 13]).

Since $\mathbb{R}^3$ comes equipped with the flat metric $g_0 = dx^2 + dy^2 + dz^2$, the inclusion map $i : M \to \mathbb{R}^3$ induces a Riemannian metric $g = i^*(g_0)$ on M. Using $\phi(t, \theta) = (t, F(t)\cos\theta, F(t)\sin\theta)$ as local coordinates, we obtain the following local formula for the metric g:

$$g = (\phi \circ i)^*(dx^2 + dy^2 + dz^2) = \left(\sqrt{1 + F'(t)^2}dt\right)^2 + F(t)^2 d\theta^2.$$

After applying the coordinate change $t \mapsto s(t)$ where

$$s(t) = \int_0^t \sqrt{1 + F'(u)^2}du,$$

letting $s \mapsto t(s)$ denote the inverse map, and defining $f(s) = F(t(s))$, we have the following familiar expression for g:

$$g = ds^2 + f(s)^2 d\theta^2.$$

The fact that g takes such a simple form reflects the fact that it is rotationally symmetric. To formalize this statement, we define an action of $\mathbf{SO}(2)$ on M as follows: We identify $\mathbb{R}^2 \cong \mathbb{C}$ so that elements of $\mathbf{SO}(2)$ are complex numbers of unit length and elements of M are expressed as $(t, f(t)e^{i\theta})$. The action of $\mathbf{SO}(2)$ on M is given by

$$\begin{array}{rcl} \mathbf{SO}(2) \times M & \to & M \\ (e^{i\alpha}, (t, f(t)e^{i\theta})) & \mapsto & e^{i\alpha} \cdot (t, f(t)e^{i\theta}) = (t, f(t)e^{i(\alpha+\theta)}). \end{array}$$

For each $e^{i\alpha} \in \mathbf{SO}(2)$, we claim that $e^{i\alpha}$ acts by isometries on M. That is, the map $A : M \to M$ given by $A(p) = e^{i\alpha} \cdot p$ is an isometry. Another way of saying this is that the action above is an isometric action.

Proof. In local coordinates (for small α, which suffices for the proof), $A \cdot \phi(s, \theta) = \phi(t, \theta + \alpha)$. Hence

$$A_*\left(\frac{\partial}{\partial s}\right) = \left.\frac{d}{du}\right|_{u=0} \phi(s + u, \theta + \alpha) = \left.\frac{\partial}{\partial s}\right|_{A\cdot\phi(s,\theta)}.$$

Similarly, $\partial/\partial\theta$ is preserved by A_*, so $A_* : T_{\phi(s,\theta)}M \to T_{\phi(s,\theta+\alpha)}M$ is the identity map (with respect to the coordinate bases at these two points), which, in particular, implies that A is an isometry.

Alternatively, one can simply compute

$$A^*(g) = ds^2 + f(s)^2 d(\theta + \alpha)^2 = g.$$

□

1.2 Isometry groups of space forms

For a fixed Riemannian manifold M, compositions and inversions of isometries are isometries, hence the set of all isometries of M is a group, denoted $\mathrm{Isom}(M)$ and called the *isometry group*. Moreover, it is a nontrivial fact that this group is a Lie group and therefore has a manifold structure and a dimension. We study isometry groups of the simply connected space forms:

1. In Euclidean space, $\mathbb{R}^n$, it is easy to see that, for all $v \in \mathbb{R}^n$, the translation map $T_v : \mathbb{R}^n \to \mathbb{R}^n$ defined by $T_v(u) = u + v$ is an isometry with respect to the standard metric. Moreover, any element of $O(n)$ preserves the metric, so the set of all isometries of $\mathbb{R}^n$ contains the set

   $$G = \{T_v \circ A \mid v \in \mathbb{R}^n, A \in \mathbf{O}(n)\}.$$

   Moreover, it is easy to see that composing two elements in G yields another element of G and that inverses of elements in G are also in G, so G is a subgroup of $\mathrm{Isom}(\mathbb{R}^n)$.

   We claim that, in fact, $G = \mathrm{Isom}(\mathbb{R}^n)$. That is, we claim that every isometry of $\mathbb{R}^n$ lies in this set.

   Proof. Let $\phi : \mathbb{R}^n \to \mathbb{R}^n$ be an isometry. Set $v = \phi(0)$, and note that $(T_v)^{-1} \circ f$ is an isometry of $\mathbb{R}^n$ that fixes the origin.

   Next, consider the derivative of $T_v^{-1} \circ f$ at 0, which is a map $T_0(\mathbb{R}^n) \to T_0(\mathbb{R}^n)$. This map is a linear isometry, so it is equal to some $A \in \mathbf{O}(n)$.

   We claim that $T_v^{-1} \circ f = A$. Indeed, both maps are isometries of $\mathbb{R}^n$, they both fix the origin, and their derivatives at the origin coincide. (Note that $D(A)_0 = A$ since A is linear.) The claim, and hence the proof, follows since the Riemannian exponential map is surjective and commutes with isometries. □

   It follows from this that the dimension of the isometry group of $\mathbb{R}^n$ is $n + \dim \mathbf{O}(n) = n + \frac{n(n-1)}{2} = \frac{n(n+1)}{2}$.

2. A similar proof shows that $\mathrm{Isom}(\mathbb{S}^n) = O(n+1)$, which also has dimension $\frac{n(n+1)}{2}$.

3. And for hyperbolic space, $\mathrm{Isom}(H^n) = O(n, 1)$, which again has dimension $\frac{n(n+1)}{2}$.

Among simply connected spaces, this “maximal symmetry” condition characterizes these spaces (see [Kob72, Chapter II.3] for a proof):

Theorem 1.2.1 (Maximal symmetry degree)**.** *The isometry group of a Riemannian manifold M^n has dimension at most $\frac{n(n+1)}{2}$. Moreover, if M is simply connected and this dimension is achieved, then M is isometric to the sphere, Euclidean space, or hyperbolic space.*

If M is not simply connected and its isometry group has dimension $\frac{n(n+1)}{2}$, then M is isometric to $\mathbb{S}^n/\mathbb{Z}_2 = \mathbb{RP}^n$.

Observe that $\mathbb{RP}^n$ occurs since its isometry group is $\mathbf{O}(n+1)/\pm \mathrm{id}$, however no other space of constant curvature and nontrivial fundamental group appears.

As a final comment, it can be shown that the standard product metric on the n-manifolds $\mathbb{R} \times M^{n-1}$ and $\mathbf{S}^1 \times M^{n-1}$, where M any of the four constant curvature examples with maximal isometry group, has isometry group of dimension

$$1 + \dim \mathrm{Isom}(M^{n-1}) = 1 + \frac{(n-1)n}{2}.$$

For n-manifolds (with $n > 4$), no closed subgroup of $\mathrm{Isom}(M^n)$ has dimension strictly between $1 + \frac{(n-1)n}{2}$ and $\frac{n(n+1)}{2}$. (See [Kob72] for further statements, a classification of metrics satisfying $\dim \mathrm{Isom}(M^n) = 1 + \binom{n-1}{2}$, and references to further work on related questions.)

1.3 Conclusions and curiosities

Question 1.1. Using the idea of the calculation of $\mathrm{Isom}(\mathbb{R}^n)$, prove that $\mathrm{Isom}(\mathbb{S}^n) = \mathbf{O}(n+1)$ when the metric on $\mathbb{S}^n$ is the standard, or "round", metric.

Question 1.2. Given that $\mathrm{Isom}(\mathbb{S}^n) = \mathbf{O}(n+1)$, prove that $\mathrm{Isom}(\mathbb{RP}^n) = \mathbf{O}(n+1)/\mathbb{Z}_2$, where $\mathbb{Z}_2 \cong \{\pm I\} \subseteq \mathbf{O}(n+1)$.

Question 1.3 (Lens spaces)**.** What are the isometry groups of the lens spaces $L^3_q = \mathbb{S}^3/\mathbb{Z}_q$? If $q = 2$, then $L^3_q = \mathbb{RP}^3$, so assume $q > 2$. By Theorem 1.2.1, the dimension of the isometry group must be less than maximal!

As a first case to consider, suppose that $\mathbb{Z}_q$ acts via a restriction of the (free) Hopf action, i.e., suppose $z \cdot (z_1, z_2) = (zz_1, zz_2)$. Observe that $U(2)$ commutes with this action on $\mathbb{S}^3$, hence the $U(2)$-action descends to L^3_q. (See [McC] for the general case.)

Question 1.4 (Isometry groups of 1-, 2-, and 3-dimensional manifolds)**.**

1. If M is a one-dimensional manifold (i.e., $\mathbb{R}$ or $\mathbf{S}^1$), Theorem 1.2.1 states that $\dim(\mathrm{Isom}(M, g)) \in \{0, 1\}$ for every metric g on M.

   Construct metrics that realize both possibilities.

2. If M is a two-dimensional Riemannian manifold, then the possibilities for the dimension of the isometry group are 0, 1, 2, and 3. Can you construct metrics that realize all of these possibilities?

3. For Riemannian 3-manifolds M, we have $\dim(M) \in \{0, 1, 2, 3, 4, 5, 6\}$. Again, are all of these possible? (The answer is no. There is no metric on any 3-manifold with 5-dimensional isometry group – again see [Kob72, Chapter II.3].)

2 Lie groups: a crash course

[[[I mostly referred to Ziller's notes on Lie Groups and Symmetric Spaces. In class, we covered just the basics, up to the Lie group exponential map, which is needed to defined action fields.]]]

Example 2.0.1 ($\mathbf{S}^3$ as unit quaternions)**.** The Lie group $\mathbf{SU}(2) \cong \mathbf{S}^3$, as can be seen by parameterizing the matrices in $\mathbf{SU}(2)$. However, we wish to think if $\mathbf{S}^3$ in a different way. Just as we define

$$\mathbb{C} = \{a + bi \mid a, b \in \mathbb{R}\}$$

where i is a new object not in $\mathbb{R}$, and just as we can define multiplication on $\mathbb{C}$ using the relation $i^2 = -1$, we can similarly define the quaternions $\mathbb{H} \cong \mathbb{R}^4$. In fact, let i, j, and k be distinct elements (not in $\mathbb{R}$) such that

$$i^2 = j^2 = k^2 = -1 \quad \text{and} \quad ij = -ji = k.$$

Define

$$\mathbb{H} = \{a + bi + cj + dk \mid a, b, c, d \in \mathbb{R}\}$$

and define multiplication simply by distributing terms and using the relations on i, j, and k.

Define the norm $|\cdot| : \mathbb{H} \to \mathbb{R}_{\geq 0}$ by

$$|a + bi + cj + dk|^2 = a^2 + b^2 + c^2 + d^2.$$

One can check that $|q_1 q_2| = |q_1||q_2|$ for all $q_1, q_2 \in \mathbb{H}$ (i.e., that the norm is multiplicative). It follows that

$$\mathbf{S}^3 = \{q \in \mathbb{H} \mid |q| = 1\}$$

is closed under multiplication. Moreover, the inverse of $q = a + bi + cj + dk$ is the conjugate

$$\bar{q} = a - bi - cj - dk,$$

so $\mathbf{S}^3$ is a group. It has the usual smooth structure given by identifying it as a hypersurface in $\mathbb{R}^4 \cong \mathbb{H}$, and the multiplication and inversion maps are smooth, hence $\mathbf{S}^3$ is a Lie group.

3 Group actions on manifolds

Throughout this chapter, we use the following notation:

- G denotes a Lie group. For example, G will be the cyclic group $\mathbb{Z}_k = \mathbb{Z}/k\mathbb{Z}$, the circle group $\mathbf{S}^1$, the torus $T^r = \mathbf{S}^1 \times \cdots \times \mathbf{S}^1$, the additive group $\mathbb{R}$, or any of the many nonabelian Lie groups (including products of $\mathbf{O}(n)$, $\mathbf{SO}(n)$, $\mathbf{U}(n)$, $\mathbf{Sp}(n)$, etc.)
- M denotes a connected smooth manifold.
- $\alpha : G \times M \to M$ denotes a smooth group action. Another way of saying this is that the induced map $\phi : G \to \mathrm{Diff}(M)$ given by $\phi(g) = \alpha(g, \cdot)$ is a group homomorphism.

We will often consider *isometric* group actions on Riemannian manifolds. For these actions, the map $\phi : G \to \mathrm{Diff}(M)$ has image inside the isometry group of M. As we will see, every smooth action by a compact group induces an isometric group action on M. On the other hand, we will often assume that M already has a Riemannian metric and that some group acts isometrically on M. In this case, averaging the metric over a larger subgroup of diffeomorphisms might not be possible while preserving the preexisting curvature condition.

In fact, given a smooth action by a compact group G on a smooth manifold M, we can always define a Riemannian metric on M such that $G \subseteq \mathrm{Isom}(M)$. We prove this now, since we will use it frequently.

Theorem 3.0.2 (Invariant metrics)**.** *If G is a compact Lie group acting smoothly on a smooth manifold M, then there exists a G-invariant Riemannian metric on M.*

Proof. We first recall that compact Lie groups admit bi-invariant measures, called Haar measures, such that G has unit volume (see, for example, [**?**, Exercise 1.7]).

Given such a measure $d\mu$ on G, we proceed to the proof. We begin with any Riemannian metric $Q(\cdot, \cdot)$ on M, then we define a new metric $\tilde{Q}(\cdot, \cdot)$ on M by declaring, for all $p \in M$ and $X, Y \in T_pM$ that

$$\tilde{Q}(X, Y) = \int_G Q(g_*(X), g_*(Y)) d\mu_g.$$

Since $d\mu$ is left-invariant, this defines a G-invariant Riemannian metric on M. $\square$

3.1 Basic concepts and examples

One should keep in mind the following examples as we define fundamental concepts in the next subsection:

Example 3.1.1 ($\mathbf{S}^1$ on $\mathbb{S}^2$)**.** $\mathbf{SO}(2) = \mathbf{S}^1$ acts on $\mathbb{S}^2 = \{(w,t) \in \mathbb{C} \times \mathbb{R} \mid |w|^2 + t^2 = 1\}$ by rotation about the z-axis. One way to write this action is as $z \cdot (w,t) = (zw, t)$.

We could define another action by $z \cdot (w,t) = (z^k w, t)$. Notice however that k-th roots of unit act trivially (i.e., as the identity), so the homomorphism $\phi : \mathbf{S}^1 \to \mathrm{Diff}(\mathbb{S}^2)$ has kernel $\ker(\phi) \cong \mathbb{Z}_k$. We therefore get an induced map

$$\bar{\phi} : \mathbf{S}^1 / \ker(\phi) \to \mathrm{Diff}(\mathbb{S}^2).$$

Since the kernel is a normal subgroup, and since quotients of Lie groups by normal subgroups are again Lie groups (see Theorem 3.2.1). In this case, we easily see that $\mathbf{S}^1 / \ker(\phi) \cong \mathbf{S}^1/\mathbb{Z}_k \cong \mathbf{S}^1$. Moreover, under this identification, the action $\bar{\phi}$ is takes the form

$$z \cdot (w,t) = (zw, t) \quad \text{or} \quad z \cdot (w,t) = (z^{-1}w, t).$$

Example 3.1.2 ($\mathbb{Z}_2$ on $\mathbb{S}^n$)**.** If $g \in \mathbb{Z}_2$ denotes the generator, we define the free antipodal action of $\mathbb{Z}_2$ on $\mathbb{S}^n$ via

$$g \cdot (x_0, \ldots, x_n) = (-x_0, \ldots, -x_n).$$

The quotient is, by definition, real projective space $\mathbb{RP}^n$.

Example 3.1.3 ($\mathbf{S}^1$ on $\mathbb{S}^3$)**.** We consider $\mathbf{S}^1 \subseteq \mathbb{C}$ and $\mathbb{S}^3 \subseteq \mathbb{C}^2$, we let $k, l \in \mathbb{Z}$, and we define an action of $\mathbf{S}^1$ on $\mathbb{S}^3$ by

$$z \cdot (z_1, z_2) = (z^k z_1, z^l z_2).$$

As in the previous example, this action has nontrivial kernel if $\gcd(k,l) > 1$. One can see that dividing by $\mathbb{Z}_{\gcd(k,l)} \subseteq \mathbf{S}^1$ yields a new action with $\gcd(k,l) = 1$. We will make this assumption.

Note that special case of $k = l = 1$ is a free circle action on $\mathbb{S}^3$, called the Hopf action. In fact, the space of orbits is $\mathbb{S}^3/\mathbf{S}^1 \cong \mathbb{CP}^1 \approx \mathbb{S}^2$, and the map $\mathbb{S}^3 \to \mathbb{S}^2$ is called the Hopf fibration.

Also note that the subaction of a $\mathbb{Z}_q \subseteq \mathbf{S}^1$, by which we mean simply the restriction of ϕ to $\mathbb{Z}_q$, yields quotient spaces $\mathbb{S}^3/\mathbb{Z}_q$ that include the lens spaces. (One should check that $\gcd(k,q) = 1$ and $\gcd(l,q) = 1$ are required for the quotient to be a manifold. The requirement on the action is that the action is free – see Theorem 3.2.1)

Example 3.1.4 ($\mathbf{S}^3$ on $\mathbb{S}^{4n+3}$)**.** Writing $\mathbf{S}^3$ as the unit quaternions, and writing

$$\mathbb{S}^{4n+3} = \{(q_0, \ldots, q_n) \in \mathbb{H}^{n+1} \mid \sum |q_i|^2 = 1\},$$

we can define an action of $\mathbf{S}^3$ on $\mathbb{S}^{4n+3}$ by

$$q \cdot (q_0, \ldots, q_n) = (qq_1, \ldots, qq_n).$$

(Question: What other actions are possible? Be careful since $\mathbf{S}^3$ is not commutative!) This action can be seen to be free, and the orbit space is quaternionic projective space $\mathbb{HP}^n$.

Elements of the orbit space are denoted by equivalence classes $[q_0, \ldots, q_n]$, and we can define new group actions on this space. For example, the torus T^{n+1} acts by

$$(z_0, \ldots, z_n) \cdot [q_0, \ldots, q_n] = [z_0^{k_0} q_0, \ldots, z_n^{k_n} q_n].$$

Question: Do some group elements act trivially?

So far, the groups have been pretty small relative to the manifold, at least in terms of dimension. Here are some examples where the group is very large relative to the manifold:

Example 3.1.5. The standard actions of $\mathbf{O}(n)$ on $\mathbb{R}^n$, $\mathbf{U}(n)$ on $\mathbb{C}^n$, and $\mathbf{Sp}(n)$ on $\mathbb{H}^n$ take unit vectors to unit vectors, hence we obtain actions of

- $\mathbf{O}(n)$ on $\mathbb{S}^{n-1}$,
- $\mathbf{U}(n)$ on $\mathbb{S}^{2n-1}$, and
- $\mathbf{Sp}(n)$ on $\mathbb{S}^{4n-1}$.

Since these actions commute with the actions described above of $\mathbb{Z}_2$, $\mathbf{S}^1$, and $\mathbf{S}^3$ on $\mathbb{S}^{n-1}$, $\mathbb{S}^{2n-1}$, and $\mathbb{S}^{4n-1}$, respectively, we obtain the following induced actions on the corresponding projective spaces:

- $\mathbf{O}(n)$ on $\mathbb{RP}^{n-1}$,
- $\mathbf{U}(n)$ on $\mathbb{CP}^{n-1}$, and
- $\mathbf{Sp}(n)$ on $\mathbb{HP}^{n-1}$.

Finally, it is important to note the Lie subgroups $H \subseteq G$ induce natural actions:

Example 3.1.6. Given a subgroup $H \subseteq G$, we obtain an action of H on G by left multiplication: $h \cdot g = hg$. Another action is $h \cdot g = gh^{-1}$. A third action is by conjugation: $h \cdot g = hgh^{-1}$.

We will discuss further example at the end of the section.

As we proceed through the basic definitions and properties of group actions, the reader is encouraged to see what each new term means in a few of the examples above.

Throughout, we will keep the notation $\phi : G \to \mathrm{Diff}(M)$ for a group action of G on M.

Definition 3.1.7. The action ϕ is

- *effective* if $\ker(\phi) = \{e\}$ (that is, only the identity acts as the identity),
- *almost effective* if $\ker(\phi)$ is finite.

Since $\ker(\phi) \subseteq G$ is a closed normal subgroup, the quotient group $\bar{G} = G/\ker(\phi)$ is another Lie group, and we get an induced, effective action $\bar{G} \to \mathrm{Diff}(M)$. For this reason, we almost exclusively consider effective actions.

Definition 3.1.8 (Two things called "G-p"). The *orbit* through $p \in M$ is denoted by Gp, $G(p)$, or $G \cdot p$, and it is the subset

$$G \cdot p = \{g \cdot p \mid g \in G\}.$$

The *isotropy group* at $p \in M$ is denoted by G_p and is defined by

$$G_p = \{g \in G \mid g \cdot p = p\}.$$

Observe the following:

Lemma 3.1.9. *For all $g \in G$ and $p \in M$, $G_{g \cdot p} = gG_pg^{-1}$, so isotropy groups along an orbit lie in the same conjugacy class.*

Proof. Easy. Just write down the set definitions. □

Definition 3.1.10 (Orbit space). We define M/G as a set to be the set of G-orbits. Equivalently, we identify $p \sim q$ iff $p = gq$ for some $g \in G$.

The topology on M/G is the quotient topology, i.e., $U \subseteq M/G$ is declared to be open iff its inverse image under the quotient map $M \to M/G$ is open.

Sometimes, the orbit space is a manifold. A sufficient condition for this to occur in the case of compact group actions is that the action is free, as in the following definition:

Definition 3.1.11 (Free actions). The action of G on M is

- *free* if $G_p = \{e\}$ for all $p \in M$ (that is, the only thing that fixes anything is that which fixes everything).
- *almost free* if G_p is finite for all $p \in M$.

- *semifree* if G_p is $\{e\}$ or G for all $p \in M$.
- *trivial* if $G_p = G$ for all $p \in M$.

Observe that, actions by finite groups are automatically almost free. Also note that we have seen examples of free and semifree actions. (Which ones?)

Also note that these definitions are given in terms of the isotropy groups. Other important definitions are in terms of the orbits:

Definition 3.1.12 (Transitive actions)**.** The action of G on M is *transitive* if $G \cdot p = M$ for some (equivalently, for all) $p \in M$.

A final basic concept is that of *action fields*:

Definition 3.1.13 (Action fields)**.** Recall that $\mathfrak{g}$ is the Lie algebra of left-invariant vector fields on G, and that $\mathfrak{g} \cong T_e G$. Let $\mathcal{X}(M)$ denote the space of all smooth vector fields on M. We define a map

$$\begin{array}{ccc} \mathfrak{g} & \longrightarrow & \mathcal{X}(M) \\ X & \mapsto & X^* \end{array}$$

where $X^*(p) = \frac{d}{dt}\big|_{t=0} (\exp(tX) \cdot p)$.

The following lemma is an important fact about action fields:

Lemma 3.1.14. *For all $X \in \mathfrak{g}$, the action field X^* is smooth. Additionally, for all $p \in M$, the map $\mathfrak{g} \to T_p M$ given by $X \mapsto X^*(p)$ is linear with kernel $\mathfrak{g}_p$ and image $T_p(G \cdot p)$.*

In other words, the tangent space to the orbits are spanned by action fields, and $X^*(p) = 0$ if and only if $\exp(tX) \in G_p$ for all $t \in \mathbb{R}$.

Proof. Exercise. (To get injectivity, use uniqueness of flows and the fact that $\exp(tX)$ is the flow of X^*. Surjectivity can be shown using a dimension count. For linearity, one can apply the Baker-Campbell-Hausdorff formula.) □

3.2 Quotients by free group actions

As mentioned in the previous section, free actions by compact Lie groups admit smooth quotient spaces. We prove this now:

Theorem 3.2.1 (Quotients by free actions are manifolds)**.** *If G is compact and acts freely on M, there exists a smooth structure on M/G such that $\pi : M \to M/G$ is a principal G-bundle (and, in particular, a submersion).*

Proof of theorem. By Theorem 3.0.2, we can endow M with a G-invariant Riemannian metric. Hence we assume without loss of generality that M is a Riemannian manifold and that G acts by isometries.

Our task is to construct coordinate charts on M/G. Fix any $p \in M$, and define the map $G \to M$ by $g \mapsto g \cdot p$. This map is an embedding. (Proof: It is *injective* because the G-action is free, it is an *immersion* by Theorem 3.1.14 since the differential at $e \in G$ is $X \mapsto X^*(p)$ and at other g is a left translate of this, and therefore it is an *embedding* because G is compact.)

Since the orbit (i.e., the image of the map $G \to M$ above) is an embedding, there exists G-invariant tubular neighborhood

$$N_\epsilon(G \cdot p) = \exp(\nu^{<\epsilon}(G \cdot p)) = \exp(\{v \in \nu(G \cdot p) \mid |v| < \epsilon\})$$

around $G \cdot p$. Here we are using the Riemannian exponential map. Define the *slice* at p to be the set

$$S_p = \exp(\nu_p^{<\epsilon}(G \cdot p)).$$

Observe that $\pi(S_p) \subseteq M/G$ is open since its preimage is the open set $N_\epsilon(G \cdot p)$.

We define charts on M/G as follows: For $\bar{p} \in M/G$, choose any $p \in \pi^{-1}(\bar{p})$, choose an ϵ-tubular neighborhood as above, then define coordinates on $\pi(S_p)$ by the composition

$$\mathbb{R}^k \xrightarrow{f} \nu_p^{<\epsilon}(G \cdot p) \xrightarrow{\exp} S_p \xrightarrow{\pi} \pi(S_p),$$

where $k = \dim(M) - \dim(G)$ is the codimension of the orbit $G \cdot p$ and where f is any diffeomorphism identifying $\mathbb{R}^k$ and $\nu_p^\epsilon(G \cdot p)$.

Assuming for a moment that these charts are smooth, it follows from the definition of π that it is a submersion. Indeed, $T_{\pi(p)}(M/G)$ is identified via π_* with $T_p(S_p)$, both of which have dimension $\dim(M) - \dim(G)$.

These coordinate charts clearly cover M/G, so it suffices to show that the coordinate interchanges are smooth. For this, it will be important to note the following

Fact: $g \cdot S_p \cap S_p$ is empty for all $g \in G \setminus \{e\}$.

(To prove this, use the fact that $\exp : \nu^\epsilon(G \cdot p) \to N_\epsilon(G \cdot p)$ is a diffeomorphism and hence injective. In particular, the slices S_p and $S_{g \cdot p}$ are disjoint whenever $g \cdot p \neq p$. Since the action is free, S_p and $S_{g \cdot p}$ are disjoint for all $g \neq e$.)

Suppose therefore that $\pi(S_p)$ and $\pi(S_q)$ are overlapping coordinate charts on M/G. Upstairs, in M, this means that the G-orbit through some $r_0 \in S_p$ intersects S_q. Choose $g \in G$ such that $g \cdot r_0 \in S_q$. CLAIM: For all $r \in S_p \cap \pi^{-1}(\pi(S_q))$, we have $g \cdot r \in S_q$

Given this, we see that the coordinate interchange is given by the *smooth* composition

$$\mathbb{R}^k \cong \nu_p^{<\epsilon} \supseteq \exp_p^{-1}(S_p \cap G \cdot S_q) \xrightarrow{\exp_p} S_p \cap G \cdot S_q \xrightarrow{g \cdot} G \cdot S_p \cap S_q \xrightarrow{exp_q^{-1}} \nu_q^{<\epsilon} \cong \mathbb{R}^k.$$

This concludes the proof that M/G admits a smooth structure such that $M \to M/G$ is a submersion. □

Proof that $M \to M/G$ is a principal G-bundle. We define the local trivializations by using the open sets $\pi(S_p)$ from above, and the trivialization diffeomorphisms

$$h : \pi(S_p) \times G \longrightarrow \pi^{-1}(\pi(S_p))$$

where $(q, g) \mapsto g \cdot q$.

Given two local trivializations $(\pi(S_{p_1}), h_1)$ and $(\pi(S_{p_2}), h_2)$, we choose $h \in G$ such that $h \cdot p_1 = p_2$ and conclude that $h_2^{-1} \circ h_1$ maps

$$\begin{array}{rcl} (\pi(q), g) & \mapsto & h_2^{-1}(g \cdot \exp_{p_1}(v)) \\ & = & h_2^{-1}(gh^{-1} \cdot \exp_{p_2}(h_* v)) \\ & = & (\pi(\exp_{p_2}(h_*(v))), gh^{-1}) \\ & = & (\pi(q), gh^{-1}). \end{array}$$

In other words, the transition function

$$G \stackrel{h_1(\bar{q},\cdot)}{\longrightarrow} \pi^{-1}(\pi(q)) \stackrel{h_2(\bar{q},\cdot)^{-1}}{\longrightarrow} G$$

is given by $g \mapsto h \cdot g = gh^{-1}$. □

With this theorem in hand, we obtain a couple of important corollaries.

Corollary 3.2.2. *If G is compact and if $H \subseteq G$ is a closed submanifold, then G/H is a manifold and $G \to G/H$ is a principal H-bundle.*

Proof. Let H act on G on the right. This action is clearly free, so the theorem implies G/H admits a smooth manifold structure. Moreover, this structure satisfies the property that $\pi : G \to G/H$ is a principal H-bundle. □

A corollary of this corollary is the following:

Corollary 3.2.3. *Orbits of compact Lie group actions are embedded submanifolds.*

Proof. We already saw that orbits $G \cdot p$ are homeomorphic to G/G_p and hence admit a smooth structure. But we can now show that the inclusion $G \cdot p \to M$ is a smooth embedding.

The map $G \times M \to M$ is smooth, hence the restriction to the embedded submanifold $G \times \{p\} \subseteq G \times M$ is a smooth map

$$G = G \times \{p\} \to M$$

whose image is the orbit $G \cdot p$ through p. Observe that the restriction of this map to G_p is the constant map, hence we get an induced map

$$G/G_p \to M.$$

Now the theorem states that $G \to G/G_p$ is a submersion, hence this induced map is smooth ([Lee02, Chapter 7]).

Finally, this induced map is an injective (easy), an immersion (see Lemma ?? below – if $X^*(p) = 0$ in $T_p(G \cdot p)$, then $X \in \mathfrak{g}_p$ and hence $X = 0$ in $\mathfrak{g}/\mathfrak{g}_p$), and therefore an embedding (because G, and hence G/G_p is compact – see [Lee02, Chapter 7]). □

3.3 The slice theorem

In the case of free actions, we saw in the previous section that the orbit space is smooth. Moreover, the orbits were embedded submanifolds with G-invariant tubular neighborhoods whose slices were diffeomorphic to the open sets of the orbit space.

For smooth actions which are not free, we lose the property that M/G is smooth. However, we still obtain "nice" G-invariant tubular neighborhoods of the orbits. This is crucial for studying local properties of group actions. The result is the following:

Theorem 3.3.1 (Slice Theorem). *Let G be a compact group acting isometrically on a Riemannian manifold M. For all $p \in M$, the orbit $G \cdot p$ is embedded in M.*

Moreover, for all $p \in M$, there exists an $\epsilon > 0$ so that the slices

$$S_x = \exp_x(\nu_x^{<\epsilon}(G \cdot p))$$

at $x \in G \cdot p$ and the tubular neighborhood

$$G \cdot S_p = N_\epsilon(G \cdot p) = \bigcup_{x \in G \cdot p} S_x$$

about $G \cdot p$ satisfy all of the following:

1. *The slices S_x are pairwise disjoint.*
2. *$g \cdot S_x = S_{g \cdot x}$ for all $g \in G$ and $x \in G \cdot p$.*
3. *G_x acts on S_x, and the action is G-equivariant via $\exp_x$ to the isotropy representation $G_x \to \mathbf{O}(\nu_x(G \cdot p))$.*
4. *For $q \in S_p$, the isotropy group G_q is a subgroup of G_p, and, in general, for $q \in N_\epsilon(G \cdot p)$, G_q is conjugate to a subgroup of G_p.*
5. *There map $[g, q] \mapsto g \cdot q$ is a well defined diffeomorphism*

   $$G \times_{G_p} S_p \longrightarrow N_\epsilon(G \cdot p),$$

   where $G \times_{G_p} S_p$ is the quotient space $(G \times S_p)/G_p$ of the right action $h \cdot (g, q) = (gh, h^{-1} \cdot q)$. In fact, this diffeomorphism is G-equivariant if we let G act on $G \times_{G_p} S_p$ by $h \cdot [g, q] = [hg, q]$ and on $N_\epsilon(G \cdot p)$ by simply restricting the action of M to $N_\epsilon(G \cdot p)$.
6. *If the isotropy representation at p is trivial (i.e., G_p acts trivially on S_p), then clearly $N_\epsilon(G \cdot p)$ is diffeomorphic to $(G/G_p) \times S_p$.*
7. *The map $[g, q] \mapsto [g]$ defines a G-equivariant fiber bundle projection*

   $$G \times_{G_p} S_p \longrightarrow G/G_x \cong G \cdot p$$

   with fiber S_p and group G_x. In fact, it is the (S_p)-fiber bundle associated to the (G_p)-principal bundle $G \to G/G_p$.

Using these neighborhoods, we can generalize the statement that M/G is smooth if G acts freely:

Corollary 3.3.2. *If G acts on M with a unique isotropy type, then M/G is a smooth manifold.*

Here, we say that isotropy subgroups are of the same type if they are conjugate to each other. The proof of this corollary mimics that of the proof of Theorem 3.2.1.

The proof of the slice theorem is sketched as follows:

Proof of the slice theorem. We begin by assuming that G acts isometrically on M, and we choose tubular neighborhoods $N_\epsilon(G \cdot p) = \exp(\nu^{<\epsilon}(G \cdot p))$ as in the proof of Theorem 3.2.1. Properties (1) and (2) follow immediately. Property (3) follows from (2) together with the fact that the Riemannian exponential commutes with isometries.

Property (4) also follows immediately from (1) and (2). In fact, if $q = \exp_p(v)$ for some $v \in \nu_p^{<\epsilon}(G \cdot p)$, then

$$G_q = \{g \in G_p \mid g_*(v) = v\}.$$

Observe that, in fact, the subrepresentation $G_q \to \mathbf{O}(\nu_p(G \cdot p))$ has an invariant subspace (spanned by v). One can push this analysis further to conclude that, up to conjugacy, there are only finitely many isotropy subgroups $G_{p'}$ for $p' \in S_p$. Moreover, since isotropy groups along a G-orbit are conjugate (see Theorem 3.1.9), this proves that $N_\epsilon(G \cdot p)$ contains only finitely many isotropy types. Since $p \in M$ was arbitrary, we conclude that, when M is compact, that there exist only finitely many isotropy types of M. (See Corollary 3.3.3 below.)

Checking that the maps in (5) and (7) are as claimed is straightforward. To check smoothness, one can use Theorem 3.2.1 together with the fact that G_p acts freely on $G \times S_p$. Property (6) is straightforward, so this completes the proof sketch. □

We proceed to a number of other important corollaries.

Corollary 3.3.3. *If G and M are compact, then any action of G on M has only finitely many isotropy types.*

Proof sketch. We induct over the dimension of M. Clearly, if $\dim M = 0$, then M is a disjoint union of a finite number of points (by compactness), hence the set of isotropy groups $\{G_p \mid p \in M\}$ is already finite.

Assume the result for manifolds of dimension less than n, and suppose $\dim(M) = n$. Since each point is contained in an open slice neighborhood, we may use compactness to cover M by finitely many slice neighborhoods. It suffices to show that each slice neighborhood has finitely many isotropy types.

Fix a slice neighborhood $N_\epsilon(G \cdot p) = \bigcup_{x \in G \cdot p} S_x$. Since every $q \in N_\epsilon(G \cdot p)$ is equal to $g \cdot r$ for some $r \in S_p$ (which implies $G_q = gG_rg^{-1}$), all isotropy types are repesented in the slice S_p at p. Hence it suffices to show that the action of G_p on the slice S_p has only finitely many isotropy types.

To prove this, we recall that the slice theorem implies that the G_p action on S_p is (G_p)-equivariant (via $\exp_p$) to the G_p action on $\nu_p^{<\epsilon}(G \cdot p)$. Since this action is linear, the isotropy groups at tv for unit vectors v and $t \in (0, \epsilon)$ are constant. Hence we see that it suffices to count the number of isotropy types of the restriction action of G_p on the sphere of radius $\epsilon/2$ inside $\nu_p^{<\epsilon}(G \cdot p)$. Since this is a compact group action on a compact manifold of dimension less than n, we conclude by the induction hypothesis that it has only finitely many isotropy types. This concludes the proof. □

Exercise 3.3.4. Prove directly (without inducting over the dimension of M) that circle actions on compact manifolds have finitely many isotropy types.

As a consequence of the previous corollary, one has the following special, useful result about torus actions. I can't remember the proof, so I'm stating it as an exercise:

Exercise 3.3.5 (Free circle actions from torus actions)**.** If T is a torus acting on a compact manifold M such that every isotropy subgroup has codimension greater than one, then there exists a circle inside T that acts freely on M.

We will come back to the slice theorem repeatedly. For now, we proceed to an important consequence of the slice theorem. It is important enough to merit its own section.

3.4 Isotropy types and the principal orbit theorem

At every point $p \in M$, we have the orbit $G \cdot p$ through p and an isotropy subgroup $G_p \subseteq G$. However, many of these orbits are diffeomorphic, and many of these groups are isomorphic. To better understand this picture, we make the following definition:

Definition 3.4.1 (Orbit types and isotropy types)**.** We define a relation on the set $\{G_p \mid p \in M\}$ of isotropy groups by declaring that G_p and G_q have the same type if and only if they are conjugate in G. We call the equivalence classes *isotropy types*.

Similarly, we define a relation on the set $\{G \cdot p \mid p \in M\}$ of orbits by declaring that $G \cdot p$ and $G \cdot q$ have the same type if and only if the isotropy subgroups at p and q are conjugate in G (i.e., of the same type).

More generally, we define the following:

Definition 3.4.2 (Partial ordering on isotropy and orbit types)**.** If H and K are isotropy subgroups of G, we denote their isotropy types by (H) and (K), and we say that $(H) \leq (K)$ if and only if H is conjugate to a subgroup of K. This defines a partial ordering on the set of isotropy types.

Similarly, we say that one orbit type is less than or equal to another if and only if the first orbit's isotropy type is greater than or equal to the second orbit's.

The principal orbit theorem below states, in particular, that a minimal isotropy type exists for compact group actions on compact manifolds. Before proving the theorem, we give a few examples.

Example 3.4.3 (Torus actions have trivial principal isotropy group)**.** If G is an abelian Lie group, then distinct isotropy subgroups belong to distinct isotropy types. In particular, the closed subgroup $\bigcap_{p\in M} G_p$ of G is a Lie subgroup when G is compact. On the other hand, this subgroup is equal to the kernel of the action. Dividing the action by this ineffective kernel yields a new action with trivial principal isotropy subgroup.

Example 3.4.4 (Principal isotropy groups of some transitive actions)**.** Consider the standard action of $\mathbf{O}(n+1)$ on $\mathbb{S}^n$. The isotropy subgroup at every point is easily seen to be a subgroup *conjugate* to a standard block embedding of $\mathbf{O}(n)$ into $\mathbf{O}(n+1)$. The principal isotropy group for this action is therefore $\mathbf{O}(n)$.

Another transitive action is given by any group G acting on itself by left translation. In this case, the action is free, which implies that the principal isotropy group is trivial.

Example 3.4.5 (Finite but nontrivial principal isotropy group)**.** Let $G = \mathbf{SO}(n)$. Identify $\mathbb{R}^{n^2}$ with the set $\mathrm{Mat}(n, \mathbb{R})$ of n-by-n matrices with real entries, and let G act on $\mathbb{R}^{n^2}$ by conjugation. Observe that the linear subspace $\mathrm{Sym}(n, \mathbb{R})$ of symmetric matrices is invariant under this action, as is the linear hyperplane $M^{n(n+1)/2-1}$ of $\mathrm{Sym}(n, \mathbb{R})$ given by matrices with trace zero. Finally, define the usual (G-invariant) norm on $\mathbb{R}^{n^2}$ by declaring $|A|^2 = \mathrm{tr}(A^T A)$, and consider the unit sphere $\mathbb{S}^{n(n+1)/2-2}$ inside M. In short, identify

$$\mathbb{S}^{n(n+1)/2-2} = \{A \in \mathrm{Mat}(n, \mathbb{R}) \mid A^T = A, \mathrm{tr}\, A = 0, \mathrm{tr}(A^T A) = 1\}.$$

For example, this construction produces an action of $\mathbf{SO}(2)$ on itself, of $\mathbf{SO}(3)$ on $\mathbb{S}^4$, and of $\mathbf{SO}(5)$ on $\mathbb{S}^{13}$.

Since symmetric matrices are orthogonally diagonalizable (over the real numbers), every orbit passes through a diagonal matrix. Since isotropy groups are conjugate along orbits, we can classify the isotropy types by examining those at diagonal matrices. Moreover, at least for $n \geq 3$, we see that every orbit passes through a diagonal matrix $p = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$ with $\lambda_1 \leq \cdots \leq \lambda_n$. The isotropy subgroup at p depends on (and only on) the number of distinct eigenvalues. If there are k distinct eigenvalues, the isotropy subgroup will be the subgroup

$$\mathrm{S}(\mathbf{O}(m_1) \times \cdots \times \mathbf{O}(m_k)) \subseteq \mathbf{O}(m_1) \times \cdots \times \mathbf{O}(m_k) \subseteq \mathbf{O}(n)$$

of block-diagonal matrices with determinant one, where $m_1, \ldots, m_k$ are the multiplicities of the eigenvalues. Clearly the minimal isotropy type corresponds to the case where $\lambda_1 < \cdots < \lambda_n$. (Observe that this is the generic case, i.e., this condition holds on an open and dense subset.) The principal isotropy group is therefore

$$\mathbb{Z}_2^{n-1} \cong \mathrm{S}\left(O(1) \times \cdots \times \mathbf{O}(1)\right),$$

the subgroup of diagonal matrices with entries ± 1 subject to the condition that the determinant is one.

The following theorem is of great importance when studying the orbit space of a group action:

Theorem 3.4.6 (Principal Orbit Theorem)**.** *Let G be a compact Lie group acting isometrically on a Riemannian manifold M. The following hold:*

1. *There exists a unique maximal orbit type.*

2. *The union M_0 of maximal orbits is open and dense in M.*

3. *The G-action on M restricts to M_0, and $M_0 \to M_0/G$ is a Riemannian submersion. It is also a fiber bundle with fiber G/H, where H is a principal isotropy group.*

4. *The quotient M_0/G of the principal part is open, dense, and connected in M/G.*

The first conclusion, of course, is equivalent to there existing a unique minimal isotropy type. The orbits of maximal type are called the *principal orbits*, and the isotropy groups of minimal type are called *principal isotropy groups.*

Points whose orbits are maximal are called *regular points.* Other points are called *exceptional points* or *singular points*, according to whether their orbits have dimension equal to or less than, respectively, the dimension of the principal orbits. For example, for the circle action on $\mathbb{S}^2$, the poles are singular points since their orbits have dimension less than one, while all other points are regular points. Examples of exceptional points arise any time the isotropy at a point is a (nontrivial) finite group.

The union of the principal orbit types is also denoted by M_{reg}. Since the projection $\pi : M \to M/G$ is continuous and open (since M/G has the quotient topology), we conclude from the principal orbit theorem that $\pi(M_0)$ is also connected, open, and dense in M/G. Moreover, since the G-action on M restricts to a G-action on M_0, and since this action has a unique isotropy type, we have from Theorem 3.3.2 that $\pi(M_0) = M_0/G$ admits a smooth manifold structure such that $M_0 \to \pi(M_0)$ is a Riemannian submersion and, in fact, a fiber bundle projection with fiber G/H, where H is a principal isotropy group.

In general, for any isotropy type (H), one can define $M_{(H)}$ to be the union of orbits with isotropy type (H) and conclude the following:

- $M_{(H)}$ is a (disjoint) union of embedded submanifolds (the orbits – see Theorem 3.2.3).

- $M_{(H)}$ is G-invariant (i.e., the G-action restricts to a G-action on $M_{(H)}$).
- $M_{(H)} \to M_{(H)}/G$ is a Riemannian submersion and a fiber bundle projection with fiber G/H (since (H) is the unique isotropy type – see Theorem 3.3.2).

We can say more. Let M^H denote the fixed-point set of H. Observe that $M^H \subseteq M_{(H)}$. Points in the latter subset are fixed by subgroups conjugate to H but perhaps not by H itself. In fact, the following holds:

$$M_{(H)} = \left(M^H \cap M_{(H)}\right) \times_{N(H)/H} (G/H)$$

where $N(H) = \{g \in G \mid gHg^{-1} = H\}$ is the normalizer of H. That is, $M_{(H)}$ can be expressed as a fiber bundle with fiber G/H over the points in M fixed by H but not by anything else.[1]

Other important facts hold for these G-invariant subsets of M, however we return to our analysis of the regular part of M, that is, M_0, the union of principal orbits. The proof of the principal orbit theorem uses the following simple, but important, lemma:

Lemma 3.4.7 (Kleiner's lemma)**.** *Assume G acts isometrically on M. If $c : [0,1] \to M$ is a minimal length curve curve from $G \cdot c(0)$ to $G \cdot c(1)$, then*

$$G_{c(t)} = \{g \in G \mid g \cdot c(s) = c(s) \text{ for all } s \in [0,1]\} = \bigcap_{s\in[0,1]} G_{c(s)}.$$

In particular, $G_{c(t)}$ is constant along the interior of $\operatorname{im}(c)$ and is a subgroup of both $G_{c(0)}$ and $G_{c(1)}$.

Proof. Define $G_c = \{g \in G \mid g \cdot c(t) = c(t) \text{ for all } t \in [0,1]\}$. Let $t \in (0,1)$. We claim that $G_{c(t)} = G_c$ for all $t \in (0,1)$. If this is not the case, there exists $s \in (0,1)$ and $g \in G$ such that g fixes $c(s)$ but not all of c. Since G acts isometrically on M, the curve $t \mapsto g \cdot c(s)$ is another minimal geodesic from $G \cdot c(0)$ to $G \cdot c(1)$. Since $g \cdot c \neq c$, the derivative of this curve at s is not equal to $c'(s)$. Hence the curve

$$\tilde{c}(t) = \begin{cases} c(t) & \text{if} \quad t \in [0,s] \\ g \cdot c(t) & \text{if} \quad t \in [s,1] \end{cases}$$

is a minimizing curve from $G \cdot c(0)$ to $G \cdot c(1)$ that fails to be smooth. This contradicts the first variation of energy formula. □

Proof of the principal orbit theorem. To prove the first statement, we show existence and uniqueness separately. To begin, suppose

$$G \geq K_1 \geq K_2 \geq \cdots$$

[1] Is this correct?

is any decreasing chain of isotropy subgroups. It suffices to show that this chain stabilizes (i.e., that the K_i have the same type for all sufficiently large i). Observe that, since isotropy subgroups are closed subgroups, each K_i is a Lie subgroup of K_{i-1}. In particular, each K_i has a dimension. We first claim that the dimensions $\dim(K_i)$ are decreasing in i. Indeed, $K_i \geq K_{i+1}$ implies that K_{i+1} is conjugate to a subgroup of K_i. Conjugation is a Lie group isomorphism of K_{i+1} and its conjugate, and the inclusion a Lie subgroup in and immersion, hence $\dim(K_i) \geq \dim(K_{i+1})$. It follows that the dimensions $\dim(K_i)$ stabilize and hence that the identity components of the K_i are isomorphic for all sufficiently large i. (Use that injective vector space homomorphisms are, in fact, isomorphisms, together with the fact that an isomorphism of Lie algebras induces an isomorphism of the indentity components of Lie groups.) By the compactness of G and the fact that isotropy groups are *closed* subgroups of G, we see that these K_i have only finitely many components, hence the isotropy type stabilizes as well.

To see that the minimal isotropy type is unique, suppose for a moment that there were two minimal isotropy types, G_p and G_q. Connect the corresponding maximal orbit types by a minimal geodesic c. Using the notation of Kleiner's lemma, we conclude that G_c is conjugate to a subgroup of both G_p and G_q. However minimality implies that G_c is conjugate to all of G_p and to all of G_q, so G_p and G_q are conjugate to each to other and hence have the same isotropy type.

To prove the second statement, let M_0 denote the union of the principal orbits. We claim that M_0 is open and dense in M

- (M_0 is open.) Let $p \in M_0$ and chose a slice neighborhood $G \cdot p$. By the slice theorem, points in the slice S_p at p are conjugate to subgroups of G_p and hence are also minimal. Other points in the (open) slice neighborhood around p are obtained by conjugating groups G_q with $q \in S_p$, hence they are also minimal.

- (M_0 is dense.) Let $p \in M$. Choose any minimal geodesic $c : [0, 1] \to M$ connecting $G \cdot p$ to any principal orbit $G \cdot q$. Kleiner's lemma implies that $G_{c(t)}$ is constant along c and that it is conjugate to a subgroup of $G_{c(1)}$, which as the same type as G_q. Since G_q is a principal isotropy type, so is $G_{c(t)}$ for all t. Since $t \in (0, 1)$ can be chosen arbitrarily small, we see that there are points with minimal istropy type arbitrarily close to p.

To prove the third statement, we first note that G preserves M_0 since isotropy groups along orbits are of the same type. We then apply Theorem 3.2.1 to conclude that M_0/G admits a smooth structure and a Riemannian metric such that $M_0 \to M_0/G$ is a Riemannian submersion. The fact that it is a fiber bundle with fiber G/H follows from the slice theorem, generalizing the proof in the case where the action is free.

To prove the last statement, note that M_0/G is automatically open and dense in M/G since the quotient map $M \to M/G$ is open and continuous. It suffices to show that M_0/G is connected. For this, we work upstairs. Let $G \cdot p$ and $G \cdot q$ be any two distinct orbits, and connect them by a minimal geodesic. Since the isotropy type at points in $G \cdot p$ and $G \cdot q$ are minimal, Kleiner's lemma implies that the isotropy type along the minimal geodesic are also minimal. Projecting to M/G, we see that the points in M/G corresponding to $G \cdot p$ and $G \cdot q$ are in the same path component of M/G. This concludes the proof. □

We remark that, if the orbits of G are connected (e.g., if G is connected), then the proof actually shows that M_0 is connected in M. Indeed, if $p, q \in M_0$, then so are all of the points in the path connected orbits $G \cdot p$ and $G \cdot q$. Applying Kleiner's lemma as in the proof, the claim follows. For general actions, however, the regular part might not be connected:

Example 3.4.8 (M_0 need not be connected)**.** Let $\mathbb{Z}_2$ act on $\mathbb{S}^2$ (or any even-dimensional sphere) by reflection across the equator. Points away from the equator have trivial isotropy, while points on the equator are fixed and hence singular points. The regular part therefore is the disjoint union of the open northern and southern hemispheres.

A related, but less trivial example is given by starting with the linear action of $\mathbf{SO}(3)$ on $\mathbb{S}^2$, and considering the subaction of $\mathbf{O}(2)$, where we identify $\mathbf{O}(2)$ as a subgroup of $\mathbf{SO}(3)$ by the map $A \mapsto \begin{pmatrix} A & 0 \\ 0 & \det(A) \end{pmatrix}$. Here, the equatorial points no longer singular since their orbits have dimension one, however they are exceptional. Indeed, points away from the equator have trivial isotropy while those on the equator have isotropy $\mathbb{Z}_2$. Once again, the regular part is the complement of the equator and hence disconnected.

Exercise 3.4.9. For the specific actions of $\mathbf{SO}(2)$ on $\mathbb{S}^2$ by rotation, or of $\mathbf{SO}(3)$ on $S^5 = \{A \in \mathrm{Mat}(3, \mathbb{R}) \mid A^T = A, \operatorname{tr} A = 0, \operatorname{tr}(A^T A) = 1\}$ by conjugation, prove "by hand" the principal orbit theorem.

3.5 Fixed point sets and induced actions

Definition 3.5.1. For $H \subseteq G$, the *fixed point set* M^H of H is the set of points in M fixed by every element of H.

The following is a basic, but important structure result for fixed point sets of isometries:

Theorem 3.5.2. *For an isometric G-action on M, the components of M^G are totally geodesic, embedded submanifolds of M.*

Moreover, if G contains an element of order greater than two, then M^G has even codimension. If G contains an element of order greater than two and M is oriented, then M^G is oriented as well.

Proof. Let $p \in M^G$ and choose $U \subseteq T_pM$ such that $\exp_p : U \to \tilde{U}$ is a normal neighborhood around p. Observe that points $\exp_p(v)$ in $\tilde{U}$ lie in M^G if and only if

$$\exp_p(v) = g \cdot \exp_p(v) = \exp_p(g_*(v)).$$

Since $\exp_p : U \to \tilde{U}$ is injective, we see that $\tilde{U} \cap M^H = \exp_p(U \cap V)$ where V is the set of vectors in T_pM fixed by every element of G. Since $V \subseteq T_pM$ is an embedding and $\exp_p : U \to \tilde{U}$ is a diffeomorphism, we conclude that $U \cap M^H \subseteq U$ is an embedding and, in fact, totally geodesic, at p. Since $p \in M^G$ was arbitrary, this proves the first statement.

For the second statement, consider the action of G on the normal space $\nu_p(M^G)$ to M^G at a point $p \in M^G$. By assumption, G contains a cyclic subgroup $\langle g \rangle$ of order greater than two. Since the irreducible representations of such groups are of complex dimension one acting by multiplication by a root of unity (of order greater than two), $\nu_p(M^G)$ has even (real) dimension and an orientation induced by g. This proves that M^G has even codimension and, in the case where M is oriented, that M^G has an induced orientation. □

Next, we study induced group actions on fixed point sets.

Definition 3.5.3 (Normalizer)**.** For a subgroup $H \subseteq G$, the normalizer $N(H)$ is the subgroup of G given by

$$N(H) = \{g \in G \mid Hg = gH\}.$$

The importance of the normalizer is the following basic fact: Elements of $N(H)$ act on M^H. That is, given $g \in N(H)$ and $p \in M^H$, the point $g \cdot p \in M^H$ because, for all $h \in H$, there exists h' such that $hg = gh'$ and hence

$$h \cdot (g \cdot p) = (hg) \cdot p = (gh') \cdot p = g \cdot (h' \cdot p) = g \cdot p.$$

This fact is especially useful for abelian group actions, since, in that case, the normalizer of any subgroup is the entire group.

Note however that the induced action of $N(H)$ on M^H need not be effective. Indeed, H itself is a subgroup of $N(H)$ that fixes M^H. Since H is a normal subgroup of $N(H)$, we therefore obtain a new group action by $N(H)/H$ on M^H.

4 Group actions and positive curvature: Planting the seed

Among simply connected, closed manifolds of dimension 2 and 3, the classification of those that admit positive curvature is complete. Only spheres arise. In higher dimensions, one finds other examples such as the projective spaces $\mathbb{CP}^n$, $\mathbb{HP}^n$, and the Cayley plane $\mathrm{Ca}\mathbb{P}^2$.

Other known examples include certain homogeneous spaces and so-called biquotient spaces, however there is only a finite list of families of manifolds that are known to admit a metric with positive sectional curvature. Aside from the rank one symmetric spaces listed above, the dimension of the known examples are 6, 7, 12, 13, and 24. That's it.

In this section, we wish to prove a beautiful theorem about isometric group actions on positively curved spaces:

Theorem 4.0.4 (Hsiang–Kleiner, [HK89])**.** *Let M^4 be a closed, simply connected manifold. If M admits a Riemannian metric with positive sectional curvature and an isometric circle action, then M is homeomorphic to $\mathbb{S}^4$ or $\mathbb{CP}^2$.*

This result has been improved, generalized, and used as motivation for a variety of results in higher dimensions. We make a few comments on related results in dimension four.

- Using classification results of Fintushel and Pao for smooth circle actions on 4-manifolds, the conclusion can be improved to a diffeomorphism classification. Recently, Grove and Wilking strengthened the conclusion to an equivariant diffeomorphism classification (see [**?**]). In particular, the action of the circle on M is equivalent to either a linear action on $\mathbb{S}^4$ or to a linear action on $\mathbb{S}^5$ that descends to an action on $\mathbb{CP}^2$.

- If one removes the assumption that $\pi_1(M) = 0$, the only additional example that arises is $\mathbb{RP}^2$. (This follows immediately from Synge's classical theorem and was included [HK89]. Note that $\mathbb{CP}^2$ does not admit a free $\mathbb{Z}_2$ action.)

- If one removes weakens the assumption on the metric so that it is only non-negatively curved, then the unpublished part of Kleiner's thesis (see also Searle–Yang's paper [**?**]) includes the proof that M is homeomorphic to $\mathbb{S}^4$, $\mathbb{CP}^2$, $\mathbb{S}^2 \times \mathbb{S}^2$, or one of the two possible connected sums of $\mathbb{CP}^2$ with itself. As with the positive curvature case, the work of Grove–Wilking mentioned above improves this conclusion to a classification up to equivariant diffeomorphism.

In addition to these results for 4-manifolds, the Hsiang–Kleiner theorem sparked a new research program on positively (and non-negatively) curved metrics with symmetry. Karsten Grove was instrumental in many of the early developments of the program, so it is now called the Grove research program. We will survey some portion of the results and techniques from the Grove program in the next section. In this section, we prove the Hsiang–Kleiner theorem.

4.1 The Hsiang–Kleiner theorem

The proof brings together many important ideas from Riemannian geometry and the theory of group actions. We will spend the rest of this section building up to and proving this result. Along the way, we will sometimes introduce relevant ideas in a broader context than was originally discovered.

First, the proof uses Freedman's classification of simply connected manifolds. Since M is a smooth manifold, the Kirby-Siebenmann invariant vanishes. Freedman's classification in this case means that the homeomorphism type of M is determined by its intersection form

$$H^2(M;\mathbb{Z}) \otimes H^2(M;\mathbb{Z}) \longrightarrow H^4(M;\mathbb{Z}).$$

If $b_2(M) = 0$, then Freedman's theorem implies that M is homeomorphic to $\mathbb{S}^4$, and if $b_2(M) = 1$, then the theorem implies that M is homeomorphic to $\mathbb{CP}^2$. Finally, since M is a closed, simply connected manifold, we can determine $b_2(M)$ from the Euler characteristic $\chi(M) = 2 + b_2(M)$. We therefore must show the following:

Claim: $\chi(M) \leq 3$.

Second, we will prove this claim by analyzing the fixed point set $M^{\mathbf{S}^1}$ of the circle action. The relevance of this information comes from the following, old result:

Theorem 4.1.1 (Conner [Con57], Kobayashi [Kob58])**.** *For smooth actions of* $\mathbf{S}^1$ *on a closed manifold* M*,* $\chi(M) = \chi(M^{\mathbf{S}^1})$*.*

Proof. Cover M by open neighborhoods U and V, where U is a (possibly empty or disconnected) tubular neighborhood of $M^{\mathbf{S}^1}$ and V is the complement of $M^{\mathbf{S}^1}$. A consequence of the Mayer-Vietoris theorem is that

$$\chi(M) = \chi(U) + \chi(V) - \chi(U \cap V).$$

Since U deformation retracts onto $M^{\mathbf{S}^1}$, it suffices to show that $\chi(V) = \chi(U \cap V) = 0$.

One way to proceed is to to argue that V and $U \cap V$ have vanishing Euler characteristic since they admit a smooth, nowhere vanishing vector field. However some care has to be taken as these are not closed manifolds. An alternative is to apply the facts that the circle action on M has finitely many isotropy types and that cyclic groups of prime order always act semifreely (since every nontrivial element is a generator). Together, these facts imply that $M^{\mathbf{S}^1} = M^{\mathbb{Z}_p}$ for all subgroups $\mathbb{Z}_p \subseteq \mathbf{S}^1$ of sufficiently large prime order. In particular, $\mathbb{Z}_p$ acts freely on V and $U \cap V$. By covering space theory, this implies that

$$\chi(V) = p\chi(V/\mathbb{Z}_p) \equiv 0 \bmod p$$

and likewise for $\chi(U \cap V)$. As p was arbitrarily large, we conclude $\chi(V) = \chi(U \cap V) = 0$. □

4.2 Proof of the Hsiang–Kleiner theorem

Given the prerequisites of the previous section, we see that it suffices to prove that $\chi(M^{\mathbf{S}^1}) \leq 3$. By **??**, we see that the components of $M^{\mathbf{S}^1}$ are positively curved, closed, oriented manifolds of dimension 0 or 2. In other words, the components of $M^{\mathbf{S}^1}$ are isolated fixed points or two-dimensional spheres.

Suppose for a moment that a two-dimensional fixed-point component $N \subseteq M^{\mathbf{S}^1}$ exists. By Wilking's connectedness theorem, the inclusion $N \to M$ is c-connected with

$$c = 4 - 2(2) + 1 + 1 = 2.$$

In particular, $\mathbb{Z} \cong H_2(N; \mathbb{Z}) \to H_2(M; \mathbb{Z})$ is a surjection, so $b_2(M) \leq 1$, as required.

Alternatively, one may apply the following, which directly classifies the diffeomorphism type of M:

Theorem 4.2.1 (Grove–Searle: Diffeomorphism classification of fixed point homogenous circle actions)**.** *If $\mathbf{S}^1$ acts isometrically on a closed, positively curved Riemannian n-manifold M such that $\mathbf{S}^1$ acts transitively on the normal spheres of some component $N \subseteq M^{\mathbf{S}^1}$, then M is diffeomorphic to $\mathbb{S}^n$ or $\mathbb{CP}^{n/2}$.*

Note that the assumption on the circle action is equivalent to there existing a component of codimension two (or dimension two when $\dim(M) = 4$, as in our case).

From now one, we assume that M^T has no two-dimensional component. It suffices to show that there are at most three isolated fixed points. We first need the following:

Lemma 4.2.2. *Assume $\mathbf{S}^1$ acts on the unit sphere $\mathbb{S}^3 \subseteq \mathbb{C}^2$ by $z \cdot (z_1, z_2) = (z^k z_1, z^l z_2)$ for some nonzero, relatively prime integers k and l. If $x_1, x_2, x_3 \in \mathbb{S}^3/\mathbf{S}^1$, then*

$$\sum_{1 \leq i < j \leq 3} d_{\mathbb{S}^3/\mathbf{S}^1}(x_i, x_j) \leq \pi.$$

Note that we will apply this lemma to the the isotropy representation $\mathbf{S}^1 \to \mathbf{SO}(T_p M)$ at an isolated fixed point $p \in M^{\mathbf{S}^1}$. The condition that k and l are nonzero holds since p is an isolated fixed point, and the relatively prime condition holds since the $\mathbf{S}^1$ action on M is effective.

Proof. We define "coordinates" on $U = \{(z_1, z_2) \in \mathbb{S}^3 \mid z_1 \neq 0, z_2 \neq 0\} = \mathbb{S}^3 \setminus (\mathbf{S}^1 \times 0 \cup 0 \times \mathbf{S}^1)$ using the map $(0, \pi/2) \times \mathbf{S}^1 \times \mathbf{S}^1 \to U$ given by

$$\phi : (t, \theta_1, \theta_2) \mapsto (\cos(t) e^{i\theta_1}, \sin(t) e^{i\theta_2}).$$

(I am using Petersen's notation from this book.) The circle action restricts to a free action on U and is given in coordinates by

$$e^{is} \cdot \phi(t, \theta_1, \theta_2) = \phi(t, \theta_1 + ks, \theta_2 + ls).$$

The quotient $\pi : U \to U/\mathbf{S}^1$ is defined by

$$(\cos(t)e^{i\theta_1}, \sin(t)e^{i\theta_2}) \mapsto [\cos(t)e^{i\theta_1}, \sin(t)e^{i\theta_2}] = [\cos(t), \sin(t)e^{i(k\theta_2 - l\theta_1)/k}].$$

We define similar "coordinates" $\Phi : (0, \pi/2) \times \mathbf{S}^1 \to \pi(U)$ on the quotient of U by

$$\Phi : (r, \theta) \mapsto [\cos r, \sin r e^{i\theta/k}].$$

Since the circle action on U is smooth, the orbital distance function on $\mathbb{S}^3/\mathbf{S}^1$ is induced by the submersion metric. We wish to calculate this metric in coordinates. First, note that the round metric on $U \subseteq \mathbb{S}^3$ is given by

$$g = dr^2 + \cos^2 r d\theta_1^2 + \sin^2 r d\theta_2^2.$$

The action field of the circle action is given in coordinates by

$$X^* = k\frac{\partial}{\partial\theta_1} + l\frac{\partial}{\partial\theta_2}.$$

We can extend this to an orthogonal basis using

$$\partial_r = \frac{\partial}{\partial r} \quad \text{and} \quad Y = -l\sin^2(r)\frac{\partial}{\partial\theta_1} + k\cos^2(r)\frac{\partial}{\partial\theta_2}.$$

The metric $\bar{g}$ on $\mathbb{S}^3/\mathbf{S}^1$ such that $\mathbb{S}^3 \to \mathbb{S}^3/\mathbf{S}^1$ is a Riemannian submersion satisfies

$$|\pi_*(\partial_r)|_{\bar{g}} = |\partial_r|_g, \quad |\pi_*(Y)|_{\bar{g}} = |Y|_g, \quad \text{and} \quad \bar{g}(\pi_*(\partial_r), \pi_*(Y)) = g(\partial_r, Y) = 0.$$

Since $\pi_*(\partial_r) = \partial_r$, we conclude from this that the metric on the quotient takes the form

$$\bar{g} = dr^2 + f(r)^2 d\theta^2$$

for some f (see also the original paper). To complete the calculation of $f(r)$, we calculate

$$\pi_*(Y) = -l\sin^2(r)\pi_*(\partial_{\theta_1}) + k\cos^2(r)\pi_*(\partial_{\theta_2}) = (l^2\sin^2(r) + k^2\cos^2(r))\partial_\theta$$

and

$$|Y|_g = (-l\sin^2(r))^2|\partial\theta_1|^2 + (k\cos^2(r))^2|\partial_{\theta_2}|^2 = (k^2\cos^2 r + l^2\sin^2 r)^2\cos^2 r\sin^2 r.$$

Using the fact that $|Y|_g = |\pi_*(Y)|_{\bar{g}}$, we conclude

$$\bar{g} = dr^2 + \frac{\cos^2 r\sin^2 r}{k^2\cos^2 r + l^2\sin^2 r}d\theta.$$

Using this, we define a distance-decreasing homeomorphism $F : X_{1,1} \to X_{k,l}$ by declaring that the restriction $U_{1,1} \to U_{k,l}$ is the identity map in the $\Phi_{k,l}$ coordinates.

Observe that $F : U_{1,1} \to U_{k,l}$ is a diffeomorphism. Let $x, y \in U_{1,1}$. Using the fact that $U_{1,1}$ is all of $X_{1,1}$ except for two points, we can choose a sequence of smooth curves c_i in $U_{1,1}$ from x to y such that their lengths $L(c_i) \to d(x, y)$ as $i \to \infty$. Since we have the inequality of metrics $\bar{g}_{1,1} \geq \bar{g}_{k,l}$, the integral formula for arclength implies

$$L(c_i) \geq L(F \circ c_i) \geq d(F(x), F(y))$$

for all i. Taking i to infinity, we conclude that F is distance-decreasing on $U_{1,1}$. Since $U_{1,1} \subseteq X_{1,1}$ is dense, this proves that $F : X_{1,1} \to X_{k,l}$ is distance decreasing.

An immediate corollary is that, given $x_1, x_2, x_3 \in X_{k,l}$, we can choose $y_1, y_2, y_3 \in X_{1,1}$ with $F(y_i) = x_i$ and conclude that

$$\sum d(x_i, x_j) = \sum d(F(y_i), F(y_j)) \leq \sum d(y_i, y_j),$$

and hence that it suffices to prove the lemma for the case $(k, l) = (1, 1)$. However, the metric $\bar{g}_{1,1}$ on $U_{1,1} \subseteq X_{1,1}$ is

$$\bar{g}_{1,1} = dr^2 + \cos^2 r \sin^2 r d\theta^2 = dr^2 + \frac{\sin^2(2r)}{4} d\theta^2,$$

which extends smoothly to all of $X_{1,1}$ (see the discussion of surfaces of revolution in the first section of these notes). Moreover, it has curvature

$$-(\sin(2r)/2)''/(\sin(2r)/2) = 4$$

on U and likewise on $X_{1,1} \setminus U_{1,1}$, hence $X_{1,1}$ is the round sphere of radius $1/2$. It is an fun, undergraduate level exercise to verify that, on the round sphere of radius $1/2$, the the maximal sum of distances between three points is π. This concludes the proof. □

Lemma 4.2.3. *There are at most three isolated fixed points.*

Proof. Suppose $p_1, p_2, p_3, p_4 \in M^{\mathbf{S}^1}$ are (distinct) isolated fixed points of the $\mathbf{S}^1$ action. For $1 \leq i < j \leq 4$, let C_{ij} denote the set of minimal-length, unit-speed geodesics from p_i to p_j. Given $\gamma \in C_{ij}$ and $\delta \in C_{ik}$ with $j < k$ and $i \notin \{j, k\}$, we can measure the angle between $\gamma'(0)$ and $\delta'(0)$. We denote the minimum such angle by $\alpha_{i,jk}$. Note that the minimum exists since γ and δ range over a compact set. There are twelve such (minimum) angles, and we calculate their sum in two ways.

First, we group the angles into four groups, each corresponding to a triangle with vertices p_i, p_j, and p_k for some $1 \leq i < j < k \leq 4$. We claim that the sum

$$\alpha_{i,jk} + \alpha_{j,ik} + \alpha_{k,ij}$$

of the angles in each triangle is greater than π. To do this, we apply Toponogov's comparison theorem. Since M is compact and positively curved, its curvature is bounded below by some $\kappa > 0$. Choose vertices $\tilde{p_i}$, $\tilde{p_j}$, and $\tilde{p_k}$ of a triangle on the round sphere of curvature κ such that $d(p_i, p_j) = d(\tilde{p_i}, \tilde{p_j})$ and likewise for the other two side lengths. Now $\alpha_{i,jk}$ is realized as the angle between the initial vectors of some $\gamma \in C_{ij}$ and $\delta \in C_{ik}$, so Toponogov's theorem implies that $\alpha_{i,jk} \geq \tilde{\alpha_{i,jk}}$, where $\tilde{\alpha_{i,jk}}$ is the angle at $\tilde{p_i}$ in the comparison triangle. Repeating this argument (choosing a new hinge at each of the vertices p_j and p_k), we conclude that the cyclic sum of the $\alpha_{i,jk}$ is at least that of the $\tilde{\alpha_{i,jk}}$. But the latter sum is greater than π, so the proof of the claim is complete. Given the claim, it follows that the sum of all of the angles satisfies

$$\sum_{1 \leq i < j < k \leq 4} (\alpha_{i,jk} + \alpha_{j,ik} + \alpha_{k,ij}) > 4\pi.$$

On the other hand, we may regroup this sum according to the base points of the angles. The above inequality therefore implies

$$\sum_{1 \leq i \leq 4} \sum_{\substack{1 \leq j < k \leq 4 \\ i \notin \{j,k\}}} \alpha_{i,jk} > 4\pi.$$

We claim, however, that the interior sum is at most π for each $1 \leq i \leq 4$. We prove this for the $i = 4$ term, and observe that the proof at other points follows from shuffling indices.

Fix any $\gamma_j \in C_{4j}$ for $1 \leq j \leq 4$, and denote their initial (unit) vectors by $v_j \in T_{p_4}M$. Note that the angle between v_j and v_k is equal to the distance $d_{\mathbb{S}^3}(v_j, v_k)$ between v_j and v_k on the unit sphere in $T_{p_4}M$. By definition of $\alpha_{4,jk}$, therefore, $d_{\mathbb{S}^3}(v_j, v_k) = \alpha_{4,jk}$ for all $1 \leq j < k \leq 4$. Moreover, since $\mathbf{S}^1$ acts isometrically on M, the $\mathbf{S}^1$-orbits of v_j and v_k also have distance at least $\alpha_{4,jk}$ from each other. The images of v_1, v_2, and v_3 under the quotient map $\mathbb{S}^3 \to \mathbb{S}^3/\mathbf{S}^1$ satisfy the property that the sum of their distances is at least the sum of the $\alpha_{4,jk}$, which is greater than π. However this directly contradicts Lemma 4.2.2, so the proof of the lemma is complete. □

This concludes the proof of the Hsiang-Kleiner theorem.

4.3 Exercises

Exercise 4.3.1 (Frankel's theorem)**.** Let M be a closed Riemannian manifold, and let N_1 and N_2 be closed submanifolds of M. Prove that N_1 and N_2 intersect in each of the following situations:

1. M has positive sectional curvature, N_1 and N_2 are totally geodesic, and $\dim(N_1) + \dim(N_2) \geq \dim(M)$.

2. M has positive sectional curvature, and N_1 and N_2 are minimal hypersurfaces.

3. M has positive Ricci curvature, and N_1 and N_2 are totally geodesic hypersurfaces.

Exercise 4.3.2 (Frankel's theorem with symmetry)**.** Suppose M is a closed Riemannian manifold with positive sectional curvature. If a circle acts isometrically (and effectively) on M, and if N_1 and N_2 are distinct components of the fixed point set, then

$$\dim(N_1) + \dim(N_2) \leq \dim(M) - 2.$$

(Observe that this already follows from Frankel's theorem together with **??** in case where $\dim(M)$ is even.)

Exercise 4.3.3 (q-extent)**.** One might ask, in general, how large the average distance that q points on a sphere can be. This question is answered in [**?**]. For related results on quotients of spheres and their applications to positive or non-negative curvature, see [**?**, **?**, **?**, GGS12].

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