ASSIGNMENT # 8

MATH 660 , DIFFERENTIAL GEOMETRY

(1) Consider the hyperbolic metric $ds^2 = \frac{dx^2+dy^2}{y^2} = \frac{dzd\bar{z}}{\mathrm{Im}(z)^2}$ on the upper half plane $\mathbb{H}^2 = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\} \subset \mathbb{C}$ and recall that it has Gauss curvature -1.
   (a) Show that the fractional linear transformations $z \to \frac{az+b}{cz+d}$ with $a, b, c, d \in \mathbb{R}$, $ad - bc = 1$, are isometries of ds^2.
   (b) Compute the full isometry group of ds^2 and show it acts transitively on $\mathbb{H}^2$.
   (c) Show that the geodesics are precisely the circles perpendicular to the x-axis.
   (d) Show that ds^2 is complete.

(2) Show that $ds^2 = \frac{dx_1^2+dx_2^2+\cdots+dx_n^2}{x_n^2}$ has constant curvature -1 in $x_n > 0$ and that the geodesics are again circles perpendicular to $x_n = 0$ and hence the metric is complete. The challenge is to give a proof much simpler than the one in Do Carmo....

(3) Show that the metric $ds^2 = \frac{dx_1^2+\ldots+dx_n^2}{(1+\frac{c}{4}(x_1^2+\ldots+x_n^2))^2}$ has constant sectional curvature equal to c (on the domain where it is defined). Compute the geodesics and show the metric is complete .

(4) Show that the so called Cayley transform $C(z) = -i\frac{z+i}{z-i}$ takes the 2-dimensional disc model to the upper half space model and takes circles perpendicular to the boundary to circles perpendicular to the x-axis (and hence those are the geodesics as well).

(5) (a) Show that in normal coordinates we have the Taylor expansion

$$g_{ij} = 1 - \frac{1}{3}R_{ikjl}x_k x_l + \ldots$$

   (b) Assume that J is a Jacobi field along the geodesics γ with $\gamma'(0) = v$, $J(0) = 0$, $J'(0) = w$ and v, w orthonormal. Show that we have the Taylor expansion

$$|J(t)| = t - \frac{t^2}{6}\sec(v, w) + \ldots$$