ASSIGNMENT # 5

MATH 660 , DIFFERENTIAL GEOMETRY

(1) Show that for a compact hypersurface in $M^n \subset \mathbb{R}^{n+1}$, there exists a point p where $\sec(\sigma) > 0$ for all 2-planes $\sigma \subset T_pM$. What can you say if the codimension is bigger than one?

(2) Show that if $M^n \subset \mathbb{R}^{n+1}$ is a compact submanifold with positive sectional curvature, then M is diffeomorphic to $\mathbb{S}^n$.

(3) Show that the second Bianchi identity implies $2d(Scal) = div(\mathrm{Ric})$. You should find out how to define the divergence of any tensor on a Riemannian manifold.

(4) Show that on a 3-dimensional manifold the Ricci curvature determines the sectional curvature. Show that and Einstein metric has constant sectional curvature.

(5) Show that for a hypersurface in $\mathbb{R}^n$, positive sectional curvature implies that the curvature operator $\hat{R}$: $\Lambda^2 T_pM \to \Lambda^2 T_pM$ is positive definite.

Hint: Show that $\hat{R}$ is diagonal in a basis of principal curvatures.
The converse is far from true in general, which can hence be interpreted as an obstruction to find a local isometric embedding as a hypersurface for a manifold with positive curvature. Indeed a theorem of Böhm-Wilking states that $\hat{R} > 0$ implies M is diffeomorphic to $\mathbb{S}^n(1)/\Gamma$ with $\Gamma \subset O(n+1)$.