Math 601 Spring 2015

Midterm Solutions

(1) Show that X is simply connected iff for any 2 points $p, q \in X$, any two paths from p to q are homotopic to each other.

Solution: $\Rightarrow$ Let α, β be 2 paths from p to q. Then $\alpha \star \bar{\beta} \simeq 1_p$ and hence $\alpha \simeq \alpha \star (\bar{\beta} \star \beta) \simeq (\alpha \star \bar{\beta}) \star \beta \simeq 1_p \star \beta \simeq \beta$.
$\Leftarrow$ If $\alpha \in \pi_1(X, p)$, then $\alpha \simeq 1_p$ since they have the same endpoints.

(2) Compute the fundamental group and homology groups of $\mathbb{R}^n$ with a line taken out. You can assume $n \geq 3$, but you need to present a complete proof of your claim.

Solution: We can assume the line is $x_n = 0$. Then $\mathbb{R}^n - L$ deformation retracts to $\mathbb{R}^{n-1} - \{0\}$ via the homotopy $F(x,t) = (x_1, \cdots, x_{n-1}, tx_n)$, which deformation retracts to $\mathbb{S}^{n-2}$ via the homotopy $G(x,t) = tx + (1-t)x/||x||$. Thus $\pi_1 = \mathbb{Z}$ if $n = 3$ and 0 other wise. Similarly, $H_k = \mathbb{Z}$ for $k = 0, n-2$ and 0 otherwise.

(3) Sow that every continuous map $f\colon \mathbb{S}^2 \to \mathbb{S}^1$ is homotopic to a constant map.

Solution: Since $\pi_1(\mathbb{S}^2) = 0$, there exists a lift $\tilde{f}\colon \mathbb{S}^2 \to \mathbb{R}$ under the cover $p\colon \mathbb{R} \to \mathbb{S}^1$, i.e. $p \circ \tilde{f} = f$. Since $\mathbb{R}$ is contractible, there is a null homotopy $F\colon \mathbb{S}^2 \times I \to \mathbb{R}$, and then $p \circ F$ is a null homotopy of f.

(4) Let $p_i\colon Y_i \to X_i$, i=1,2 be covering spaces with Y_i simply connected. Show that if X_1 and X_2 are homotopy equivalent, then so are Y_1 and Y_2.

Solution: Let $f\colon X_1 \to X_2$ and $g\colon X_2 \to X_1$ be such that $f \circ g \simeq \mathrm{Id}$ and $g \circ f \simeq \mathrm{Id}$. Since $\pi_1(Y_i) = 0$, there exists a lift $\tilde{f}$ of the composition $f \circ p_1$ under the cover p_2 and a lift of $\tilde{g}$ of the composition $g \circ p_2$ under the cover p_1. Then $\tilde{g} \circ \tilde{f}$ is a lift of $g \circ f \circ p_1$ under the cover p_1. The homotopy lifting property then implies that $\tilde{g} \circ \tilde{f} \simeq \mathrm{Id}$, and similarly $\tilde{f} \circ \tilde{g} \simeq \mathrm{Id}$.

(5) Let X be the Δ complex obtained by starting with two simplices $[v_0, v_1, v_2]$ and $[w_0, w_1, w_2]$ and identifying corresponding vertices, i.e. $v_i \sim w_i$. Compute its homology.

Solution: We have the chain complex $0 \to C_2 \simeq \mathbb{Z}^2 \to C_1 \simeq \mathbb{Z}^6 \to C_0 \simeq \mathbb{Z}^3$. Let $\alpha = [v_0, v_1, v_2]$, $\beta = [w_0, w_1, w_2]$ be a basis of C_2, and $a = [v_1, v_2], b = [v_0, v_2], c = [v_0, v_1], \bar{a} = [w_1, w_2], \bar{b} = [w_0, w_2], \bar{c} = [w_0, w_1]$ a basis of C_1, and $v_0 = w_0, v_1 = w_1, v_2 = w_2$ a basis of C_0. Since $\partial_2(\alpha) = a - b + c$ and $\partial_2(\beta) = \bar{a} - \bar{b} + \bar{c}$, the boundary map ∂_2 is clearly injective, and hence $H_2(X) = 0$.
$\partial_1(a) = v_2 - v_1 = \delta_1(\bar{a})$ and similarly for b, c. Thus v_0, v_1, v_2 are homologous, and $H_0(X) = \mathbb{Z}$. The kernel $\ker \partial_1$ has a $\mathbb{Z}$ basis $\{a - \bar{a}, b - \bar{b}, a - b + c, \bar{a} - \bar{b} + \bar{c}\}$ and hence $H_2(X) = \mathbb{Z} \oplus \mathbb{Z}$ with basis $[a - \bar{a}], [b - \bar{b}]$.

(6) Take the torus $\mathbb{S}^1 \times \mathbb{S}^1$ and identify points in the circle $\mathbb{S}^1 \times \{x_0\}$ that differ by a $2\pi/m$ rotation and identify points in the circle $\{x_0\} \times \mathbb{S}^1$ that differ by a $2\pi/n$ rotation. Make it into a CW complex and compute its homology.

Solution: The CW complex has one 2-cell e^2 and two 1-cells $e_1^1 = \{(x_0, e^{it}) \mid 0 \leq t \leq 2\pi/n\}$ and $e_2^1 = \{(e^{it}, x_0) \mid 0 \leq t \leq 2\pi/m\}$ and one 0-cell. We have boundary maps $\partial(e^2) = ne_1^1 + me_2^1 - ne_1^1 - me_2^1 = 0$ and $\partial(e_1^1) = \partial(e_2^1) = 0$ and hence $H_0(X) = H_2(X) = \mathbb{Z}$ and $H_1(X) = \mathbb{Z} \oplus \mathbb{Z}$.

(7) Let A be a 3x3 matrix with real positive entries. Show that A has a real positive eigenvalue.

Hint: You may want to consider the set

$$A := \{v = (x_1, x_2, x_3) \in \mathbb{R}^3 \mid \sum x_i^2 = 1,\ x_i \geq 0\}.$$

Solution: Let $f\colon A \to A$ bedefined by $f(v) = Av/||Av||$ which is well defined since A has positive entries. By the Brower fixed point theorem, there exists a $v \in A$ with $f(v) = v$ and hence $Av = ||Av||v$.

(8) (Extra credit if you have time left) Consider a 3 simplex $[v_0, v_1, v_2, v_3]$ with opposite edges identified (orientation preserving). What is the homology of this space.

Solution: C_3 is spanned by $\alpha = [v_0, v_1, v_2, v_3]$ and C_2 by

$$a = [v_1, v_2, v_3], b = [v_0, v_2, v_3], c = [v_0, v_1, v_3], d = [v_0, v_1, v_2]$$

and C_1 by

$$u = [v_0, v_1] = [v_2, v_3], v = [v_0, v_2] = [v_1, v_3], w = [v_0, v_3] = [v_1, v_2].$$

All vertices are identified to one point $[v_0]$ and hence $H_0(X) = \mathbb{Z}$. We have $\partial_3(\alpha) = a - b + c - d$ and

$$\partial_2(a) = u - v + w = \partial_2(d), \quad \partial_2(b) = u + v - w = \partial_2(c), \quad \partial_1 = 0.$$

Thus $H_3(X) = 0$ and

$$H_2(X) = \ker(\partial_2)/\operatorname{Im}(\partial_3) =< a - d, b - c > /(a - d = b - c) \simeq \mathbb{Z}.$$

Finally,

$$\begin{aligned} H_1(X) &= \ker(\partial_1)/\operatorname{Im}(\partial_2) =< u, v, w > /(u - v + w = 0, u + v - w = 0v) \\ &=< u, v - w, w > /(2u = 0, 2(v - w) = 0) = \mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2. \end{aligned}$$

Correction (thanks to Zhen Zeng! Excercise: Find my mistake.....)

$$\begin{aligned} H_1(X) &= \ker(\partial_1)/\operatorname{Im}(\partial_2) =< u, v, w > /(u - v + w = 0, u + v - w = 0v) \\ &=< u, u - v + w, w > /(2u = 0, u - v + w = 0) = \mathbb{Z}_2 \oplus \mathbb{Z}. \end{aligned}$$