13.3

Arc Length

Review:

curve in space: $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$

Tangent vector: $\mathbf{r}'(t_0) = f'(t)\mathbf{i} + g'(t)\mathbf{j} + h'(t)\mathbf{k}$

Tangent line at $t = t_0$: $\mathbf{s} \to \mathbf{r}(t_0) + s\mathbf{r}'(t_0)$

Velocity: $\mathbf{v}(t) = \mathbf{r}'(t)$ speed = $|\mathbf{v}(t)|$

Acceleration $\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t)$

Projectile motion: $\mathbf{r}(t) = -\frac{1}{2}gt^2\mathbf{j} + \mathbf{v}_0 t + \mathbf{r}_0$ initial velocity $\mathbf{v}_0$, initial position $\mathbf{r}_0$

$g = 32 \text{ ft}/\text{sec}^2$ gravitational constant

Initial speed $|\mathbf{v}_0|$, initial height h, launching angle θ :

$$\mathbf{r}(t) = \langle \left(|\mathbf{v}_0|\cos\theta\right)t,\ h + \left(|\mathbf{v}_0|\sin\theta\right)t - \frac{1}{2}gt^2 \rangle$$

Recall the length of a curve in the plane:

curve as a graph $y = f\left(x\right)$: length from $t = a$ to $t = b$ $\quad = \int_a^b \sqrt{1 + [f'(\mathrm{x})]^2}\, dx$

curve in parametric form : $x = f\left(t\right)$ and $y = g\left(t\right)$: length $\quad = \int_a^b \sqrt{[f'(\mathrm{t})]^2 + [g'(\mathrm{t})]^2}\, dt$

A curve in 3-space : $\mathbf{r}\left(t\right) = \left\langle f\left(t\right), g\left(t\right), h\left(t\right)\right\rangle$

Length or Arc Length $= \int_a^b \sqrt{[f'(\mathrm{t})]^2 + [g'(\mathrm{t})]^2 + [h'(\mathrm{t})]^2}\, dt$

$$\text{Length } = \int_a^b |\mathbf{r}'(t)|\, dt$$

Example 1:

A line from **a** to **b**: $\mathbf{r}(t)=\mathbf{a}+t(\mathbf{b}-\mathbf{a}) \quad , \quad 0\le t\le 1$

$$\text{length} = \int_0^1 |\mathbf{r}'(t)|\,dt = \int_0^1 |\mathbf{b}-\mathbf{a}|\,dt = |\mathbf{b}-\mathbf{a}| = \text{distance from } \mathbf{a} \text{ to } \mathbf{b}$$

Example 2:

Compute the length (circumference) of circle: $(x-x_0)^2+(y-y_0)^2=r^2$

parametric description:

$$x=x_0+r\cos(t),\; y=y_0+r\sin(t) \quad 0\le t\le 2\pi$$

$$\text{length} = \int_0^{2\pi} |\mathbf{r}'(t)|\,dt = \int_0^{2\pi} \sqrt{r^2\sin^2(t)+r^2\cos^2(t)}\,dt = \int_0^{2\pi} r\,dt = 2\pi r$$

Problem : A drunken bee travels along the path $\mathbf{r}(t) = \langle \cos(2t), \sin(2\,\mathrm{t}), \mathrm{t} \rangle$ for 10 seconds. It then travels at constant speed in a straight line for 10 more seconds. How far did the bee travel?

velocity: $\mathbf{v}(t) = \mathbf{r}'(t) = \langle -2\sin(2t), 2\cos(2\,\mathrm{t}), 1 \rangle$ Speed = $|\mathbf{v}(t)| = \sqrt{9} = 3$

distance traveled along the helix: $\int_0^{10} |\mathbf{r}'(t)|\, dt = 30$

At time $t_0 = 10$, it travels along the tangent line

tangent line: $\mathbf{u}(s) = \mathbf{r}(t_0) + s\mathbf{v}(t_0)$

length = $\int_{10}^{20} |\mathbf{u}'(\mathrm{s})|\, ds = \int_{10}^{20} |\mathbf{v}(t_0)|\, ds = 3 \cdot (20 - 10) = 30$

Total distance traveled: 60 ft

Arc length function describes distance from a beginning point $t = a$:

$$s(t) = \int_a^t \left|\mathbf{r}'(u)\right| du \qquad \text{Notice: } \frac{ds}{dt} = |\mathbf{r}'(t)|$$

$\mathbf{r}(t)$ gives position on the curve as a function of time

its more natural to look at the odometer or the mile markers to tell where you are after you have traveled a certain distance s

we would like to have t as a function of s so that we could reparametrize in terms of s.

Using the arc length function $s = s(t)$ we can find t as a function of s

$t = t(s)$ giving us $\mathbf{r}(t(s))$ or just $\mathbf{r}(s)$

Example:

Parametrize the following curve by arc length: $\mathbf{r}(t) = 3\sin(\mathrm{t}^2)\mathbf{i} + 3\cos(t^2)\mathbf{k}$

$$\mathbf{r}'(t) = 6t\cos(t^2)\mathbf{i} + 6t\sin(t^2)\mathbf{k}$$

$$|\mathbf{r}'(t)| = \sqrt{36t^2\cos^2(t^2) + 36t^2\sin(t^2)} \quad = \sqrt{36t^2} = 6t$$

$$s = s(t) = \int_0^t 6u\,du \quad = 3u^2\Big|_{u=0}^{u=t} = 3t^2$$

Solve for t: $\mathrm{t} = \sqrt{\frac{s}{3}}$

Plug into $\mathbf{r}(t)$: $\mathbf{r}(t(s)) = \mathbf{r}(\sqrt{\mathrm{s}/3}) \quad = 3\sin(\frac{s}{3})\mathbf{i} + 3\cos(\frac{s}{3})\mathbf{k}$

Arc length parametrization: $\mathbf{r}(s) = 3\sin(\frac{s}{3})\mathbf{i} + 3\cos(\frac{s}{3})\mathbf{k}$

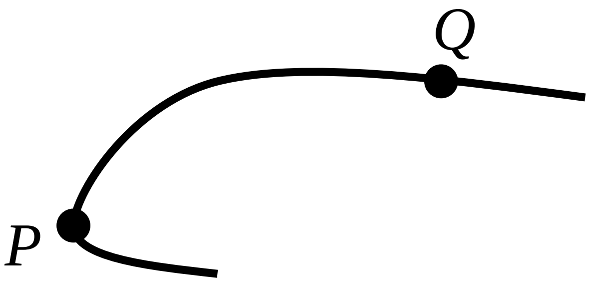
Notice: $|\mathbf{r}'(s)| = 1$ length from $s = a$ to $s = b$: $\int_a^b |\mathbf{r}'(s)|\,ds = \int_a^b 1\,ds = b - a$

t is an arc length parameter if $|\mathbf{r}'(t)| = 1$!

13.4

Curvature

The **curvature** of a curve $\mathbf{r}(t)$ measures the amount the tangent vector bends.



curvature at P > curvature at Q

Tangent vector $\mathbf{r}'(t)$ Unit tangent vector $\mathbf{T} = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$ $(= \mathbf{r}'(s)\ !\ )$

Curvature - the magnitude of the rate of change of the unit tangent vector **T** with respect to the arc length parameter *s*.

$$\kappa = \left|\frac{d\mathbf{T}}{ds}\right| = \left|\mathbf{T}'(s)\right|$$

$$\kappa = \left|\frac{d\mathbf{T}}{ds}\right| = \left|\frac{d}{ds}\underbrace{\left(\frac{d\mathbf{r}}{ds}\right)}_{\mathbf{T}}\right| = \left|\mathbf{r}''(s)\right|$$

$$\kappa = \left|\mathbf{r}''(s)\right|$$ if s is the arc length parameter

Examples:

a) The curvature of a straight line:

$$\mathbf{r}(t) = \mathbf{r}_0 + \mathbf{t}\,\mathbf{v} \qquad \mathbf{r}'(t) = \mathbf{v}$$

what is the arc length parametrization of the line:

$$\mathbf{r}(\mathrm{s}) = \mathbf{r}_0 + \mathbf{s}\frac{\mathbf{v}}{|\mathbf{v}|} \qquad \text{since then } |\mathbf{r}'(\mathrm{s})| = \left|\frac{\mathbf{v}}{|\mathbf{v}|}\right| = 1$$

$$\kappa = \left|\frac{d\mathbf{T}}{ds}\right| = \left|\mathbf{r}''(s)\right| = 0$$

(if not arc lenght: $\mathbf{r}(t) = \mathbf{r}_0 + t^2\,\mathbf{v}$, then $\mathbf{r}''(t) = 2t\,\mathbf{v}$)

b) The curvature of a circle of radius *r*:

$$\mathbf{r}(t) = \langle r\cos(t), r\sin(t)\rangle \qquad \mathbf{r}'(t) = \langle -r\sin(t), r\cos(t)\rangle$$

$$\left|\mathbf{r}'(t)\right| = r \qquad \text{hence } s = \int_0^t \left|\mathbf{r}'(u)\right| du = rt \text{ or } t = r/s$$

arc length parametrization: $\mathbf{r}(\mathrm{s}) = \langle r\cos(\frac{s}{r}), r\sin(\frac{s}{r})\rangle$ $\qquad \mathbf{T} = \mathbf{r}'(\mathrm{s}) = \langle -\sin(\frac{s}{r}), \cos(\frac{s}{r})\rangle$

$$\mathbf{T}' = \frac{d\mathbf{T}}{ds} = \langle -\frac{1}{r}\cos(\frac{s}{r}), -\frac{1}{r}\sin(\frac{s}{r})\rangle \qquad \kappa = \left|\frac{d\mathbf{T}}{ds}\right| = \sqrt{\frac{1}{r^2}} = \frac{1}{r}$$

small radius means large curvature!

It is usually too difficult to parametrize a curve by arc length.

Express curvature in terms of a general parameter t :

Recall $s(t) = \int_a^t |\mathbf{r}'(u)|\, du$ hence: $\dfrac{ds}{dt} = |\mathbf{r}'(t)|$

$$\kappa = \left|\frac{d\mathbf{T}}{ds}\right| = \left|\frac{dt}{ds}\frac{d\mathbf{T}}{dt}\right| = \left|\frac{\frac{d\mathbf{T}}{dt}}{\frac{ds}{dt}}\right| = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|}$$

$$\boxed{\kappa = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|}}$$

Example: Compute the curvature of an ellipse: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

$\mathbf{r}(t) = \langle a\cos(t), \mathrm{b}\sin(t)\rangle$ $\quad\mathbf{r}'(t) = \langle -a\sin(t), \mathrm{b}\cos(t)\rangle$ $\quad|\mathbf{r}'(t)| = \sqrt{a^2\sin^2(t) + b^2\cos^2(t)}$

$$\mathrm{T}(t) = \frac{\langle -a\sin(t), \mathrm{b}\cos(t)\rangle}{\sqrt{a^2\sin^2(t) + b^2\cos^2(t)}} \qquad \mathrm{T}'(t) = \frac{\langle -ab^2\cos(t), -\mathrm{b}a^2\sin(t)\rangle}{(a^2\sin^2(t) + b^2\cos^2(t))^{3/2}}$$

(a lengthy computation…)

$$|\mathrm{T}'(t)| = \frac{ab}{a^2\sin^2(t) + b^2\cos^2(t)} \qquad \kappa = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \frac{ab}{(a^2\sin^2(t) + b^2\cos^2(t))^{3/2}}$$

Sometimes it is easier to use another formula:

$$\kappa = \frac{\left|\mathbf{r}'(t)\times\mathbf{r}''(t)\right|}{\left|\mathbf{r}'(t)\right|^3}$$

Why is this true?

Recall: $\mathbf{T} = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$ and $|\mathbf{r}'(t)| = \dfrac{ds}{dt}$ $\quad \kappa = \dfrac{\left|\mathbf{T}'(t)\right|}{\left|\mathbf{r}'(t)\right|}$

Hence: $\mathbf{r}'(t) = |\mathbf{r}'(t)|\,\mathbf{T} = \dfrac{ds}{dt}\mathbf{T}$ and $\mathbf{r}''(t) = \dfrac{d}{dt}\left(\dfrac{ds}{dt}\mathbf{T}\right) = \dfrac{d^2s}{dt^2}\mathbf{T} + \dfrac{ds}{dt}\mathbf{T}'$

$$\mathbf{r}'(t)\times\mathbf{r}''(t) = \frac{ds}{dt}\mathbf{T} \times \left(\frac{d^2s}{dt^2}\mathbf{T} + \frac{ds}{dt}\mathbf{T}'\right) = \frac{ds}{dt}\mathbf{T} \times \left(\frac{ds}{dt}\mathbf{T}'\right) = \left(\frac{ds}{dt}\right)^2 \mathbf{T} \times \mathbf{T}'$$

but $\mathbf{T}\cdot\mathbf{T} = 1$ implies $\mathbf{T}'\cdot\mathbf{T} + \mathbf{T}\cdot\mathbf{T}' = 2\mathbf{T}\cdot\mathbf{T}' = 0$ i.e., $\mathbf{T}$ is perpendicular to $\mathbf{T}'$

This implies $|\mathbf{T}\times\mathbf{T}'| = |\mathbf{T}|\,|\mathbf{T}'|\sin(\theta) = |\mathbf{T}'| = \kappa\,|\mathbf{r}'|$

Put together, we get: $|\mathbf{r}'(t)\times\mathbf{r}''(t)| = \left(\dfrac{ds}{dt}\right)^2 |\mathbf{T} \times \mathbf{T}'| = \left(\dfrac{ds}{dt}\right)^2 \kappa\,|\mathbf{r}'| = \kappa\,|\mathbf{r}'|^3$

Example: Ellipse revisited:

$$\mathbf{r}(t) = \langle a\cos(t), \mathrm{b}\sin(t)\rangle \quad \text{or} \quad \mathbf{r}(t) = \langle a\cos(t), \mathrm{b}\sin(t), 0\rangle$$

$$\mathbf{r}'(t) = \langle -a\sin(t), \mathrm{b}\cos(t), 0\rangle$$

$$\mathbf{r}''(t) = \langle -a\cos(t), -\mathrm{b}\sin(t), 0\rangle$$

$$\mathbf{r}'(t)\times\mathbf{r}''(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -a\sin(t) & b\cos(t) & 0 \\ -a\cos(t) & -b\sin(t) & 0 \end{vmatrix} = \quad (ab\sin^2(t) + ab\cos^2(t))\mathbf{k} = ab\mathbf{k}$$

$$\kappa = \frac{|\mathbf{r}'(t)\times\mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} \quad = \frac{ab}{(a^2\sin^2(t) + b^2 cos^2(t))^{3/2}}$$

Example: Curvature of the (elliptical) helix:

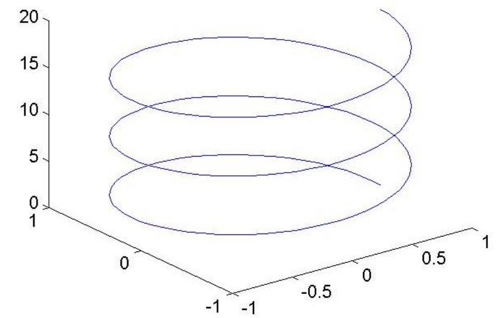
$$\mathbf{r}(t) = \langle a\cos(t), \mathrm{b}\sin(\mathrm{t}), c\,\mathrm{t}\rangle \qquad \mathbf{r}'(t) = \langle -a\sin(t), \mathrm{b}\cos(t), \mathrm{c}\rangle$$

$$\mathbf{r}''(t) = \langle -a\cos(t), -\mathrm{b}\sin(\mathrm{t}), 0\rangle$$

$$\mathbf{r}'(t)\times\mathbf{r}''(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -a\sin(t) & b\cos(t) & c \\ -a\cos(t) & -b\sin(t) & 0 \end{vmatrix} = (0 + bc\sin(t))\mathbf{i} + (-ac\cos(t) - 0)\mathbf{j} + (ab\sin^2(t) + ab\cos^2(t))\mathbf{k}$$

$$= bc\sin(t)\mathbf{i} - ac\cos(t)\mathbf{j} + ab\mathbf{k}$$

$$|\mathbf{r}'(t)\times\mathbf{r}''(t)| = \sqrt{b^2c^2\sin^2(t) + a^2c^2\cos^2(t) + a^2b^2}$$



$$|\mathbf{r}'(t)| = \sqrt{a^2\sin^2(t) + \mathrm{b}^2\cos^2(t) + \mathrm{c}^2}$$

$$\kappa = \frac{|\mathbf{r}'(t)\times\mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} = \frac{\sqrt{b^2c^2\sin^2(t) + a^2c^2\cos^2(t) + a^2b^2}}{\left(a^2\sin^2(t) + \mathrm{b}^2\cos^2(t) + \mathrm{c}^2\right)^{3/2}}$$

Circular helix: $b = a, \quad \kappa = \frac{\sqrt{a^2c^2 + a^2a^2}}{\left(a^2 + \mathrm{c}^2\right)^{3/2}} = \frac{a\sqrt{c^2 + a^2}}{\left(a^2 + \mathrm{c}^2\right)^{3/2}}$

$$\boxed{\kappa = \frac{a}{a^2 + \mathrm{c}^2}}$$ constant!

Appplication: Curvature of a plane curve:

$$\kappa = \frac{\left|\mathbf{r}'(t)\times\mathbf{r}''(t)\right|}{\left|\mathbf{r}'(t)\right|^3}$$

$$\mathbf{r}(t) = \langle x(t), \mathrm{y(t)}, 0\rangle \qquad \mathbf{r}'(t) = \langle x'(t), \mathrm{y'(t)}, 0\rangle$$

$$\mathbf{r}'(t)\times\mathbf{r}''(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x' & y' & 0 \\ x'' & y'' & 0 \end{vmatrix} = (x'y'' - x''y')\mathbf{k}$$

$$\kappa = \frac{|x'y'' - x''y'|}{\left((x')^2 + (y')^2\right)^{3/2}}$$

Summary:

Length of a curve $= \int_a^b |\mathbf{r}'(t)|\, dt$

Arc length function $s(t) = \int_a^t \left|\mathbf{r}'(u)\right| du$ $\qquad \dfrac{ds}{dt} = |\mathbf{r}'(t)|$

Arc length parametrization $\mathbf{r}(s)$ with $|\mathbf{r}'(s)| = 1$

Unit tangent vector $\mathbf{T} = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \mathbf{r}'(\mathrm{s})$

Curvature: $\kappa = \left|\dfrac{d\mathbf{T}}{ds}\right| = \left|\mathbf{r}''(s)\right| = \dfrac{\left|\mathbf{T}'(t)\right|}{\left|\mathbf{r}'(t)\right|} = \dfrac{\left|\mathbf{r}'(t) \times \mathbf{r}''(t)\right|}{\left|\mathbf{r}'(t)\right|^3}$