Math 600 Fall 2019

Homework 6, due Thursday October 17

.

(1) Show that our two definitions of orientability of a manifold M^n are equivalent:
  (a) There exists an atlas (U_α, x_α) such that $D(x_\alpha \circ x_\beta^{-1})$ is orientation preserving (i.e. has positive determinant).
  (b) There exists a choice of orientations on T_pM for all p, i.e. a choice of (equivalence classes) of basis, which represents one of the possible orientations on this vector space, such that it varies smoothly (or continuously) with p. I.e. for each $p \in M$, there exists a neighborhood U of p and smooth (continuous is sufficient) vector fields $Y_1, \cdots Y_n$ on U such that for all $p \in U$, the basis $Y_1(p), \cdots Y_n(p)$ represents the orientation class you chose on T_pM.

(2) Let Γ be a group that acts properly discontinously on a manifold M^n with quotient M/Γ. Recall we already showed that M/Γ is a manifold.
Using the definition of orientability in (1) (b) :
  (a) Show that if M is orientable and Γ acts orientation preserving on M, then M/Γ is orientable as well.
  (b) If M/Γ is orientable, show that there exists an orientation on M such that Γ acts orientation preserving on M.
  (c) Make precise that $\mathbb{RP}^n$ is orientable iff n is odd. I.e. carefully define the orientation of the sphere, and show the restriction of the antipodal map to the sphere is orientation preserving iff n is odd. Then also make precise again that all lens space are orientable.

(3) Let F and G be two vector bundles over M and define the direct sum $E = F \oplus G$, where the fiber E_p over $p \in M$ is the direct sum of the vector spaces F_p and G_p.
  (a) Show that E is in fact a vector bundle.
  (b) Define the orientation of a vector bundle, and show that if both F and G are orientable, then E is orientable as well.
  (c) Show that if both E and F are orientable, then G is orientable as well.

(4) Show that a one dimensional vector bundle (i.e., the fiber over $p \in M^n$ is one dimensional) is trivial iff it is orientable.

(5) Let $f\colon M \to B$ be a submersion.
  (a) Show that $TM = f^*(TB) \oplus V$ where $f^*(TB)$ is the "pull back" vector bundle over M whose fiber over $p \in M$ is $T_{f(p)}B$, and V is the "vertical" bundle over M whose fiber over $p \in M$ is the tangent space of the fiber through p, i.e. $T_p(f^{-1}(f(p)))$.
  (b) Show that V is isomorphic to the vector bundle $\ker D(f)$ and $f^*(TB)$ to the normal bundle, if we endow M with a Riemannian metric.
  (c) Show that the fibers of f are orientable if M and B are. In particular, the Brieskorn varieties from the last assignment are orientable.

(6) (Extra credit 1)
Find a base B, and two vector bundles F and G over B, such that both F and G are non-orientable, but $E = F \oplus G$ is non- orientable as well.

(7) (Extra credit 2)

For those who know what the fundamental group $\pi_1(M)$ is. If you do not, you can look up the definition, which is not difficult.

(a) Construct a map $O\colon \pi_1(M) \to \mathbb{Z}_2$.
(b) Make this definition rigorous and show it is well defined and a homomorphism.
(c) Show that O is trivial iff M is orientable.

Comment: Next semester you will learn that a homomorphism $\alpha\colon \pi_1(M) \to \mathbb{Z}_2$ is an element of $H^1(M, \mathbb{Z}_2)$, which is called the first Stiefel Whitney class.