Math 600 Fall 2019

Final, December 10, 2019

I could not find any interesting problems from the time before we had any theorems (apart from Frobenius). So the problems are all about deRham cohomology and applications. I hope you enjoy them....

Each problem is worth 10 points (partial credit will be given). You can use Lee's book, your class notes and your homework as a reference. You can only use theorem's in Lee's book that we proved in class. You are not allowed to use the internet, or work with other students. Be as rigorous as possible! Write legibly and well organized! The exam should be turned in

Don'y stress out about the exam... Finish whatever you can, including ideas even if not competely rigourous. I decided to give you an extra day, so the exam is due Thursday at 5pm. I won't be here, so you have to send it via email (scan it in the math department if written by hand).

(1) Show that $H^1_{DR}(S^1) \simeq \mathbb{R}$ with your bare hands. The only theorem you are allowed to use is Stokes theorem.

(2) The deRham cohomology with compact support, denoted by $H_C(M, \mathbb{R})$, (on a non-compact manifold) is defined as closed forms with compact support modulo exact forms with compact support. Show that $H^1_C(\mathbb{R}) \simeq \mathbb{R}$ with your bare hands.

Remark: This generalizes to $H^n_C(\mathbb{R}^n) \simeq \mathbb{R}$, but is more difficult to prove. The proof of Poincare duality (without using the Hodge decomposition theorem) relies on this fact.

(3) Assume that M is compact. Show that the homomorphism:

$$L\colon H^1_{DR}(M^n) \to \mathrm{Hom}(\pi_1(M), \mathbb{R}) \quad : [\omega] \to \left\{ [\gamma] \to \int_\gamma \omega \right\}$$

is onto. You can assume, since you already proved it, that L is well defined and injective.

(4) Prove the Cauchy integral formula using Stokes theorem. Let $D \subset \mathbb{C}$ be a connected and simply connected domain with smooth boundary ∂D, contained in a larger domain D'. Assume that $f\colon D' \to \mathbb{C}$ is holomorphic. Show that

$$f(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(\zeta)}{\zeta - z} d\,\zeta$$

Hint: Relate the property of being holomorphic with the existence of closed 1-forms.

(5) You can assume in the following that $\pi_1(\mathbb{R}^2 \backslash \{0\}) \simeq \mathbb{Z}$.

(a) Let $\omega = \dfrac{xdy - ydx}{x^2 + y^2} \in \Omega^1(\mathbb{R}^2 \backslash \{0\})$. Recall that we showed that ω is closed but not exact. Show that if γ is a closed curve with $\{0\} \notin \mathrm{Im}(\gamma)$, then $\dfrac{1}{2\pi}\int_\gamma \omega \in \mathbb{Z}$. This is called the winding number of γ around the origin, denoted by $w(\gamma)$.

(b) Show that if $\gamma_1,\ \gamma_2$ are two closed curves with $\{0\} \notin \mathrm{Im}(\gamma_i)$ then $w(\gamma_1)$ is homotopic to $w(\gamma_2)$ in $\mathbb{R}^2 \backslash \{0\}$ if and only if $w(\gamma_1) = w(\gamma_2)$.

(c) Generalize Cauchy's theorem to

$$\frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z} d\zeta = w(\gamma) f(z)$$

if γ is a closed curve with $\{0\} \notin \mathrm{Im}(\gamma)$