Heyi Zhu
Extra Credit, Homework 8
MATH 501
April 3rd, 2016

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Extra Credit

(a) Given the metric,

$$E = G = \frac{4}{1-(x^2+y^2)}, \quad F = 0.$$

$$E_x = G_x = \frac{8x}{(1-(x^2+y^2))^2}, \quad E_y = G_y = \frac{8y}{(1-(x^2+y^2))^2}$$

Thus $E_x/(2E) = G_x/(2G) = -(-G_x/(2E))$. i.e.

$$\Gamma^x_{xx} = \Gamma^y_{xy} = -\Gamma^x_{yy} = \frac{x}{1-(x^2+y^2)}$$

$E_y/(2E) = G_y/(2G) = -(-E_y/(2G))$ i.e.

$$\Gamma^x_{xy} = \Gamma^y_{yy} = -\Gamma^y_{xx} = \frac{y}{1-(x^2+y^2)}$$

Thus the equations for geodesics are:

$$(1-(x^2+y^2))x'' + x\cdot x'^2 + 2y\, x'\cdot y' - x\cdot y'^2 = 0 \qquad ①$$

$$(1-(x^2+y^2))y'' - y\cdot x'^2 + 2x\, x'\cdot y' + y\cdot y'^2 = 0 \qquad ②$$

Let $z = x+iy$. The above equations are equivalent to ①+i②

$$(1-(x^2+y^2))(x''+iy'') + (x-iy)x'^2 + (x-iy)2i x'\cdot y' + (x-iy)(-y'^2) = 0$$

$$\Leftrightarrow (1-|z|^2)z'' + \bar{z}(x'^2 + 2i x'y' - y'^2) = 0$$

$$\Leftrightarrow (1-|z|^2)z'' + \bar{z}(z')^2 = 0$$

□

(b) & (c) Suppose $c(t) = (1-a)e^{i\alpha t} + a e^{-i\beta t}$. [Assuming a real]

$$Re[c(t)] = (1-a)\cos(\alpha t) + a\cos(\beta t) =: x(t)$$

$$Im[c(t)] = (1-a)\sin(\alpha t) - a\sin(\beta t) =: y(t)$$

$$c'(t) = i\alpha(1-a)e^{i\alpha t} - i\beta a e^{-i\beta t}$$

$$Re[c'(t)] = -\alpha(1-a)\sin(\alpha t) - \beta a\sin(\beta t) =: x'(t)$$

$$Im[c'(t)] = \alpha(1-a)\cos(\alpha t) - \beta a\cos(\beta t) =: y'(t)$$

$$\Rightarrow x'^2 + y'^2 = \alpha^2(1-a)^2 + \beta^2 a^2 + 2\alpha\beta(1-a)a[\sin\alpha t\sin\beta t - \cos\alpha t\cos\beta t]$$

$$= \alpha^2(1-a)^2 + \beta^2 a^2 - 2\alpha\beta(1-a)a\cos(\alpha+\beta)t$$

$$x^2+y^2 = (1-a)^2 + a^2 + 2(1-a)a\cos(\alpha+\beta)t$$

$$\Rightarrow 1-(x^2+y^2) = 2a - 2a^2 - 2(1-a)a\cos(\alpha+\beta)t$$

$$\Rightarrow |c'(t)|^2 = \frac{4(x'^2+y'^2)}{1-(x^2+y^2)} = 2\,\frac{\alpha^2(1-a)^2 + \beta^2a^2 - 2\alpha\beta(1-a)a\cos(\alpha+\beta)t}{a-a^2-(1-a)a\cos(\alpha+\beta)t}$$

(cont)

Since $c(t)$ is geodesic, $\|c'(t)\|^2$ is constant. Since $\alpha\beta$ is a constant, we have that

$$\|c'(t)\|^2 = 4\alpha\beta, \quad \alpha^2(1-a)^2 + \beta^2 a^2 = 2\alpha\beta(a-a^2).$$

Thus $\alpha, \beta \neq 0$ and we can divide the second equation by β^2 on both sides and get

$$(1-a)^2\left(\frac{\alpha}{\beta}\right)^2 - 2\left(\frac{\alpha}{\beta}\right)(a-a^2) + a^2 = 0.$$

$$\Rightarrow \left((1-a)\frac{\alpha}{\beta} - a\right)^2 = 0$$

$$\Rightarrow \frac{\alpha}{\beta} = \frac{a}{1-a}.$$

Let $\alpha = ak$ and $\beta = (1-a)k$ for some constant real k.

We claim that $c(t)$ is a geodesic for any k.

$$(1-(x^2+y^2))x'' + xx'^2 + 2yx'y' - xy'^2$$

$$= -[(a-1)\alpha + a\beta][(a-1)(\alpha + a\alpha + a\beta)\cos(\alpha t) + a((1-a)\alpha - (a-2)\beta)\cos(\beta t)]$$

$$= -[(a-1)ak + a(1-a)k][\cdots] \quad \text{since } \alpha = ak, \ \beta = (1-a)k$$

$$= 0.$$

$$(1-(x^2+y^2))y'' - yx'^2 + 2xx'y' + yy'^2$$

$$= -[(a-1)\alpha + a\beta][(a-1)(\alpha + a\alpha + a\beta)\sin(\alpha t) + a((a-1)\alpha + (a-2)\beta)\sin(\beta t)]$$

$$= -[(a-1)ak + a(1-a)k][\cdots] \quad \text{since } \alpha = ak, \ \beta = (1-a)k$$

$$= 0.$$

Thus both ① and ② holds and $c(t)$ is a geodesic.

If we require $c(t)$ to be arclength-parametrized, then

$$\|c'(t)\|^2 = 4\alpha\beta = 1 \Rightarrow \alpha\beta = 1/4$$

$$\Rightarrow a(1-a)k^2 = 1/4.$$

$$\Rightarrow k = \frac{1}{2\sqrt{a(1-a)}}$$

$$\Rightarrow \alpha = \frac{a}{2\sqrt{a(1-a)}}, \quad \beta = \frac{1-a}{2\sqrt{a(1-a)}}$$

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Extra Credit.

(a) Suppose w.l.o.g. that $c(t) : [0, b] \to \mathbb{D}$. where $\mathbb{D} = \{(x,y) \mid x^2+y^2<1\}$
As before, let

$$k = \frac{1}{2\sqrt{a(1-a)}} \Rightarrow \alpha = ak, \ \beta = (1-a)k.$$

By previous work, $\|c'(t)\|^2 = 1 \Rightarrow \|c'(t)\| = 1$ and the length of $c(t)$ is

$$L = \int_{c(t)} \sqrt{\frac{4dx^2 + dy^2}{1-(x^2+y^2)}} = \int_0^b \|c'(t)\| dt = \int_0^b 1 \, d(t). = b$$

Suppose $c(b) = c(0)$. Then.

$$(1-a)e^{i\alpha b} + ae^{-i\beta b} = (1-a) + a = 1.$$

$$\Rightarrow (1-a)\cos(\alpha b) + a\cos(\beta b) + i[(1-a)\sin(\alpha b) - a\sin(\beta b)] = 1.$$

$$\Rightarrow (1-a)\cos(\alpha b) + a\cos(\beta b) = 1 \text{ and } (1-a)\sin(\alpha b) = a\sin(\beta b)$$

Thus

$$\begin{aligned}(1-a)^2 &= (1-a)^2\sin^2(\alpha b) + (1-a)^2\cos^2(\alpha b) \\ &= a^2\sin(\beta b) + (1-a\cos(\beta b))^2 \\ &= 1 + a^2 - 2a\cos(\beta b).\end{aligned}$$

$$\Rightarrow \cos(\beta b) = 1.$$

$$\Rightarrow (1-a)kb = \beta b = 2m\pi. \text{ for some } m \in \mathbb{Z}, m > 0.$$

Since $\cos(\beta b) = 1$,

$$(1-a)\cos(\alpha b) = 1 - a\cos(\beta b) = 1-a.$$

$$\Rightarrow \cos(\alpha b) = 1.$$

$$\Rightarrow akb = \alpha b = 2\ell\pi \text{ for some } \ell \in \mathbb{Z}, \ell > 0.$$

Thus

$$(1-a)/a = m/\ell \Rightarrow am = (1-a)\ell \Rightarrow a = \frac{\ell}{m+\ell}.$$

$$\Rightarrow 1-a = \frac{m}{m+\ell}, \quad k = \frac{1}{2\sqrt{(1-a)a}} = \frac{m+\ell}{2\sqrt{m\ell}} \text{ since } m, \ell > 0.$$

Thus

$$\begin{aligned}2\ell\pi &= akb \\ &= \frac{\ell}{2\sqrt{m\ell}} b\end{aligned}$$

$$\Rightarrow b = 4\sqrt{m\ell}\,\pi \text{ for } m, \ell \in \mathbb{Z}. \ m, \ell > 0.$$

$$\Rightarrow L = b = 4\pi\sqrt{n} \text{ for } n \in \mathbb{Z}, n > 0.$$

$\square$

(b) For any $n \in \mathbb{Z}$, $n > 0$, fixed. For each un-ordered pair $p, q \in \mathbb{Z}$, $0 < p \leq q$, $(p, q) = 1$ with $n = pq$, let either

$$a = \frac{p}{p+q}, \quad (1-a) = \frac{q}{p+q} \quad \text{irreducible} \quad ①$$

or

$$(1-a) = \frac{p}{p+q}, \quad a = \frac{q}{p+q} \quad \text{irreducible} \quad ②$$

In either case,

$$k = \frac{1}{2\sqrt{(1-a)a}} = \frac{p+q}{2\sqrt{pq}}, \quad b = 4\pi\sqrt{pq} = 4\pi\sqrt{n}$$

Notice that

$$c(p, q; t) = \frac{q}{p+q} e^{i\frac{1}{2}\sqrt{\frac{p}{q}}t} + \frac{p}{p+q} e^{-i\frac{1}{2}\sqrt{\frac{q}{p}}t}$$

$$c(q, p; t) = \frac{p}{p+q} e^{i\frac{1}{2}\sqrt{\frac{q}{p}}t} + \frac{q}{p+q} e^{-i\frac{1}{2}\sqrt{\frac{p}{q}}t}$$

Thus $c(p, q; t) = c(q, p; -t)$. Since $c(t)$ and $c(-t)$ share the same trace [image], each un-ordered pair p, q with $p, q \in \mathbb{Z}$, $0 < p \leq q$, $(p, q) = 1$, $n = pq$ corresponds to a closed geodesic which can be parametrized as $c(p, q; t)$ on $[0, 4\pi\sqrt{n}]$. Thus the number of closed geodesics of length $4\pi\sqrt{n}$ for some $n \in \mathbb{Z}$, $n > 0$ is $\psi(n)$.

In fact, if the prime decomposition of n is

$$n = p_1^{e_1} p_2^{e_2} \cdots p_{w(n)}^{e_{w(n)}}, \quad w(n) = \text{number of prime factors of } n.$$

where p_i's are prime integers and $e_i \geq 1$. Then

$$\psi(n) = \frac{1}{2} \sum_{i=0}^{w(n)} \binom{w(n)}{i} = \frac{1}{2} \cdot 2^{w(n)} = 2^{w(n)-1}.$$

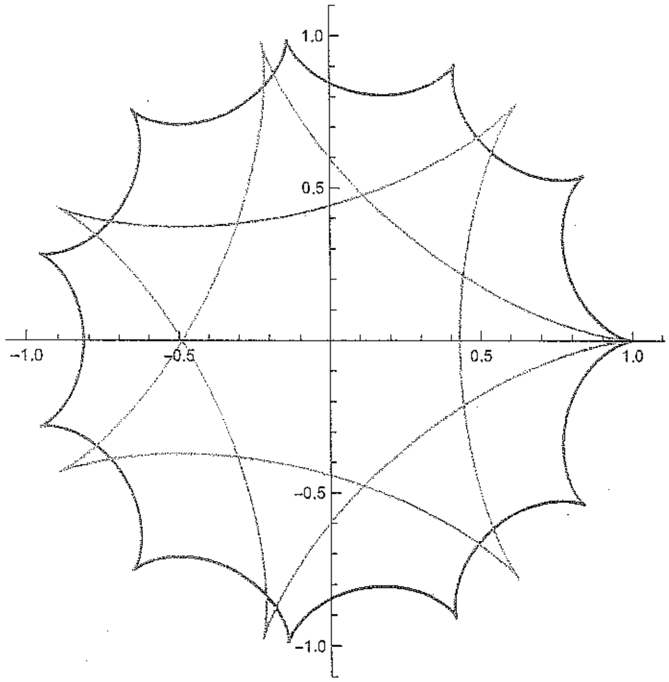
$\square$.

(c) See the attachment, which includes the plots of the geodesics when

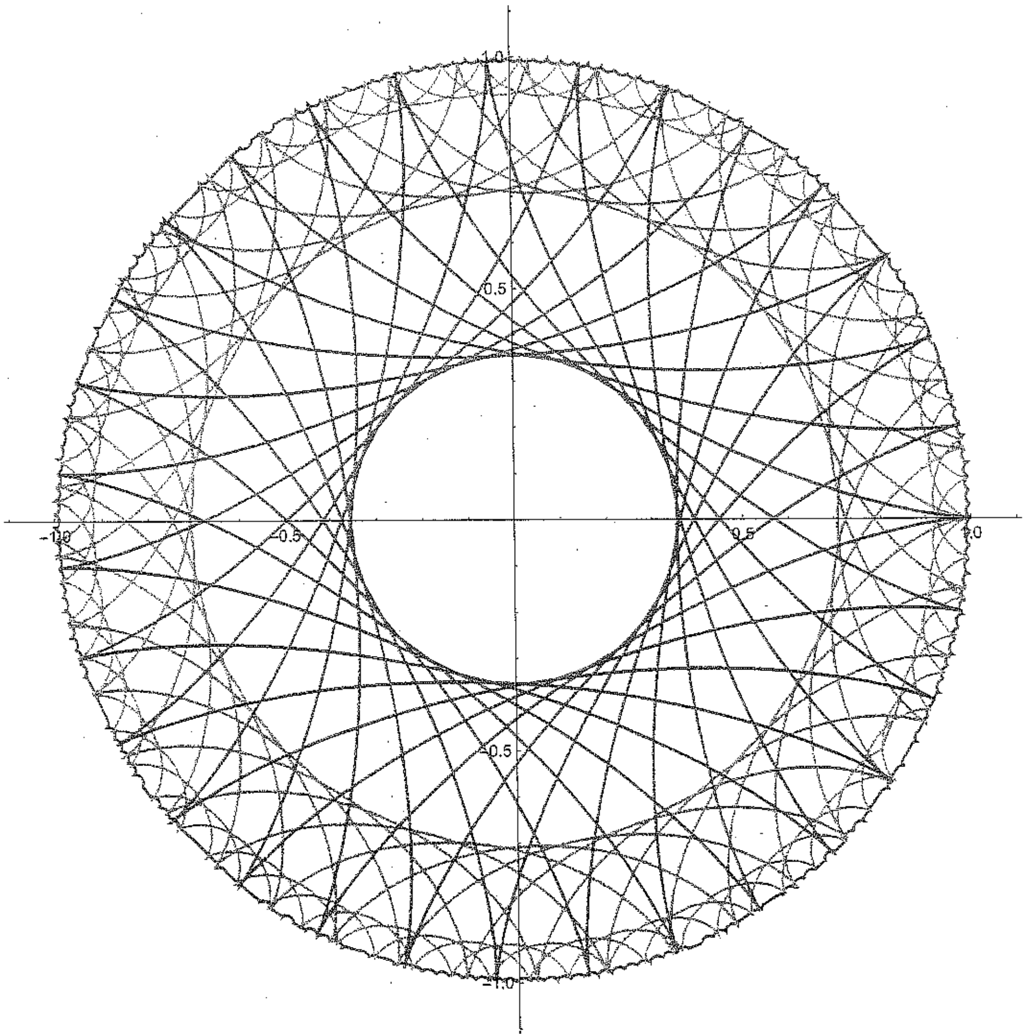
n	1	2	3	4	10	11492	210	2310
$w(n)$	1	1	1	1	2	3	4	5
$\psi(n)$	1	1	1	1	2	4	8	16

Best picture among these

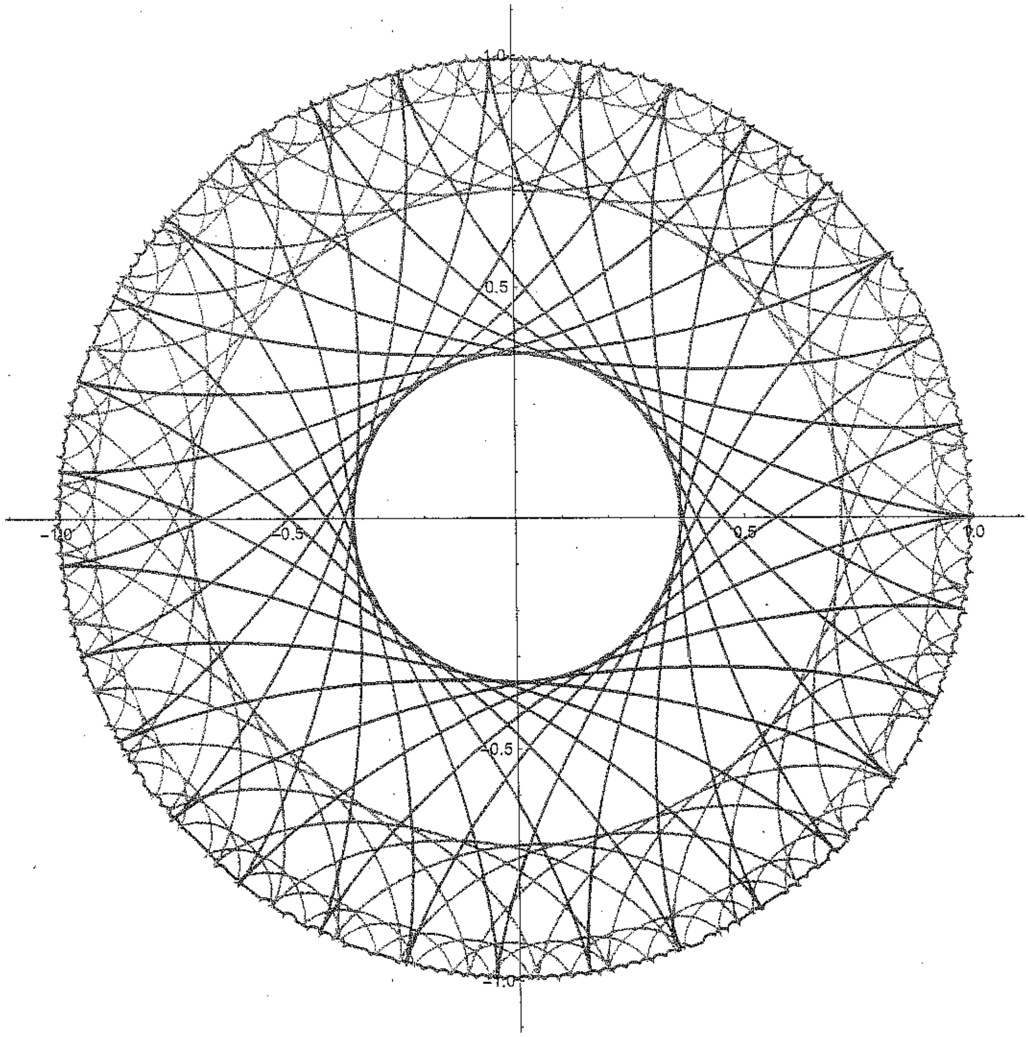
```
(* n=10 *)

ParametricPlot[{ClosedGeo[1, 10, t], ClosedGeo[2, 5, t]}, {t, 0, 4 √10 π}]
```



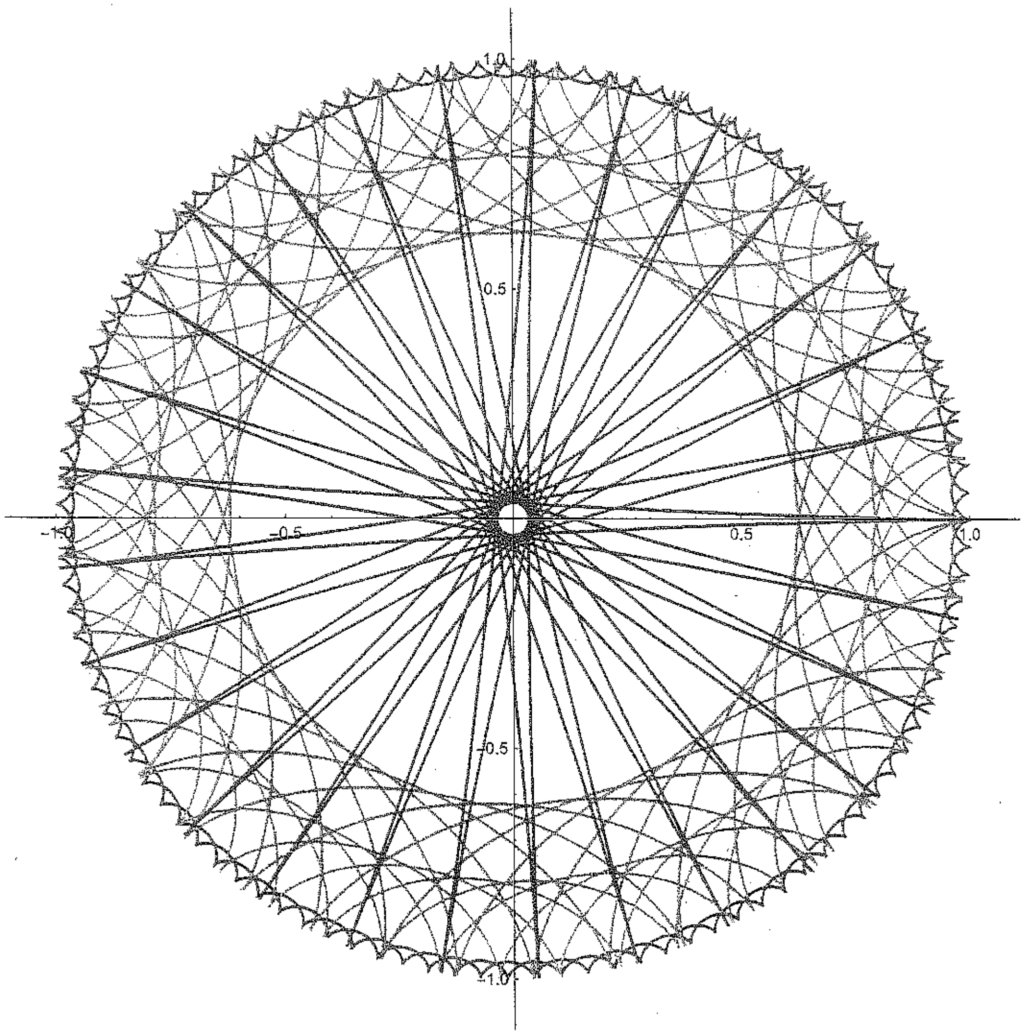
```
ParametricPlot[{ClosedGeo[1, 210, t], ClosedGeo[3, 70, t],
  ClosedGeo[6, 35, t], ClosedGeo[10, 21, t]}, {t, 0, 4 √210 π}]
```



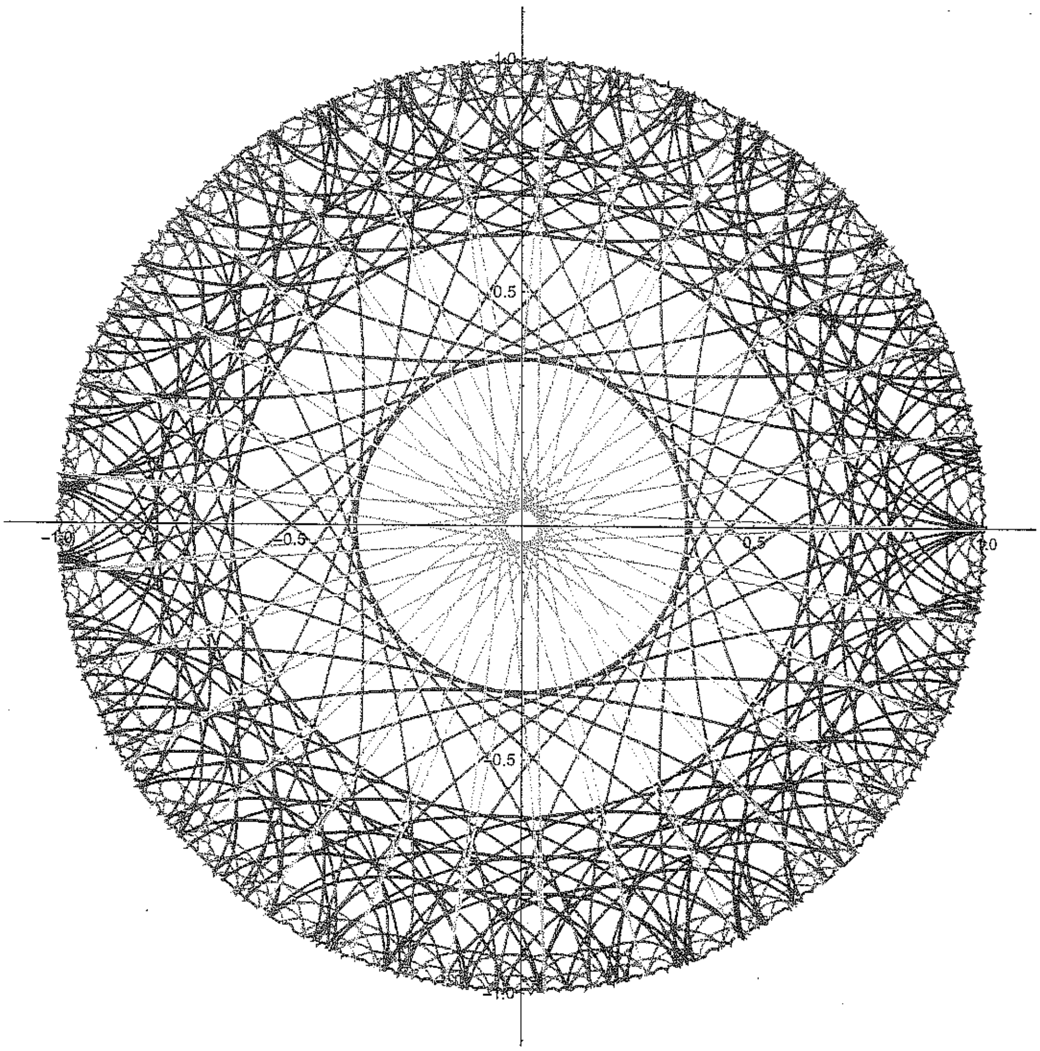
```
ParametricPlot[{ClosedGeo[1, 210, t], ClosedGeo[3, 70, t],
  ClosedGeo[6, 35, t], ClosedGeo[10, 21, t]}, {t, 0, 4 √210 π}]
```



```
ParametricPlot[{ClosedGeo[2, 105, t], ClosedGeo[5, 42, t],
  ClosedGeo[7, 30, t], ClosedGeo[14, 15, t]}, {t, 0, 4 √210 π}]
```



```
(* n=210 *)

FactorInteger[210]

{{2, 1}, {3, 1}, {5, 1}, {7, 1}}

PrimeNu[210]

4

ParametricPlot[{ClosedGeo[1, 210, t], ClosedGeo[2, 105, t], ClosedGeo[3, 70, t],
  ClosedGeo[5, 42, t], ClosedGeo[6, 35, t], ClosedGeo[7, 30, t],
  ClosedGeo[10, 21, t], ClosedGeo[14, 15, t]}, {t, 0, 4 √210 π}]
```



```
ParametricPlot[{ClosedGeo[5, 462, t], ClosedGeo[11, 210, t],
  ClosedGeo[22, 105, t], ClosedGeo[42, 55, t]}, {t, 0, 4 √2310 π}]
```