A GEOMETRY WHERE EVERYTHING IS BETTER THAN NICE

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Abstract. We present a geometry in the disk whose metric truth is curiously arithmetic.

1. The better-than-nice metric

Consider the metric

$$g = \frac{4}{1-r^2}(dx^2 + dy^2)$$

in the unit disk. Here $r^2 = x^2 + y^2$ as usual. Everything of interest can be computed explicitly, and with surprising results.

1.1. **Hypocycloids in the disk.** Consider the curve

$$c(t) = (1-a)e^{i\theta(t)} + ae^{-i\phi(t)}, \qquad 0 < a < 1,$$

thought of as a point on a circle of radius a turning at a rate $\dot\phi$ in the clockwise direction as the centre of the circle rotates on a circle of radius $1-a$ rotating at a rate $\dot\theta$ in the counterclockwise direction.

For the small circle of radius a to roll without slipping on the inside of the circle of radius 1 requires the point $c(t)$ to have velocity 0 when $|c(t)| = 1$, which is the relation

$$a\dot\phi = (1-a)\dot\theta.$$

Because g is rotationally invariant, without loss of generality we may take

$$\theta(t) = \frac{at}{2\sqrt{a(1-a)}}, \qquad \phi(t) = \frac{(1-a)t}{2\sqrt{a(1-a)}}.$$

Theorem 1.1. *The curve $c(t)$ is a geodesic for the metric g, parameterized by arclength.*

Proof. For our g, the equations $\ddot u^i + \Gamma^i_{jk}\dot u^j \dot u^k = 0$ determining geodesics $u(t) = (x(t), y(t))$ parameterized proportional to arclength are

$$(1 - x^2 - y^2)\ddot x + x(\dot x^2 - \dot y^2) + 2y\dot x\dot y = 0 \quad \text{and}$$
$$(1 - x^2 - y^2)\ddot y - y(\dot x^2 - \dot y^2) + 2x\dot x\dot y = 0,$$

which can be expressed in terms of $z = x + iy$, as the single equation

$$(1 - z\bar z)\ddot z + \bar z\dot z^2 = 0. \tag{1}$$

Date: March 16, 2016.

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Given the formula for $c(t)$ it is straightforward to verify that $c(t)$ satisfies (1) and moreover that $|\dot{c}| = \sqrt{1-|c|^2}/2$, from which it follows that $\|\dot{c}\|_g = 1$, meaning that the parameterization is by arclength. q.e.d.

THEOREM 1.2. *The closed geodesics (i.e. keep rolling the generating circle of the hypocycloid until it closes up) have length* $4\pi\sqrt{n}$*, and the number of geometrically distinct geodesics of length* $4\pi\sqrt{n}$ *is given by the arithmetic function* $\psi(n)$*.*

The function $\psi(n)$ counts the number of different ways that the integer n may be written as a product $n = pq$, with $p \le q$, $(p, q) = 1$. Values of this function are tabulated in sequence $A007875$ in the online encyclopedia of integer sequences [1].

Proof. Beginning at $c(0) = 1$, the geodesic $c(t)$ first returns to the boundary circle at

$$c\left(4\pi\sqrt{a(1-a)}\right) = e^{2\pi i(1-a)},$$

returning again at points of the form $e^{2\pi im(1-a)}$ $(m \in \mathbf{Z}_+)$. The corresponding succession of cycloidal geodesic arcs winds clockwise around the origin if $0 < a < 1/2$ and counterclockwise if $1/2 < a < 1$; when $a = 1/2$, $c(t)$ traverses back and forth along the x-axis. Thus $c(t)$ forms a once-covered closed geodesic precisely when $2\pi m(1-a) = q2\pi$ for some relatively prime pair of positive integers $q < m$, in which case the geodesic has length

$$4\pi m\sqrt{a(1-a)} = 4\pi\sqrt{(m-q)q}.$$

To count geometrically distinct closed geodesics we restrict to $0 < a \le 1/2$, in which case $p := m - q = ma \le m(1-a) = q$. Given any relatively prime positive integers $p \le q$ the geodesic of length $4\pi\sqrt{pq}$ occurs when $a = p/(p+q)$. q.e.d.

1.2. **Eigenfunctions and eigenvalues of the Laplacian.** Set the Laplacian Δ to be

$$\Delta = -g^{ab}\nabla_a\nabla_b$$

where ∇_a is the covariant derivative operator associated to the metric g via the Levi-Civita connection. Consider the eigenvalue problem

$$\Delta u = \lambda u$$

for functions u with the boundary value $u(r = 1) = 0$. This problem has a number of remarkable features.

THEOREM 1.3. *The eigenfunctions and eigenvalues satisfy*

(1) *The eigenvalues* λ_n *are precisely the positive integers* $n = 1, 2, 3, \ldots$.
(2) *The eigenfunctions are polynomials.*
(3) *The dimension of the eigenspace for eigenvalue n is the number of divisors of n. (The number of divisors function is denoted by* $\tau(n)$*.)*

Proof. Since the operator

$$-\Delta + \lambda = \frac{1-r^2}{4}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + \lambda$$

is analytic hypoelliptic, distributional solutions to the eigenvalue equation $\Delta u = \lambda u$ are necessarily real analytic, and representable near $(x, y) = (0, 0)$ by absolutely convergent Fourier series

$$u(r, \theta) = \sum_{n \in \mathbf{Z}} a_n(r) e^{in\theta}.$$

Observing that $(-\Delta + \lambda)(a_n(r)e^{in\theta})$ has the form $A_n(r)e^{in\theta}$ for some A_n, it follows that u is an eigenfunction of Δ only if each summand $a_n(r)e^{in\theta}$ is, so it suffices to consider just products of the form

$$u(r, \theta) = f(r)e^{in\theta}.$$

Expressing $-\Delta + \lambda$ in polar coordinates yields

$$(-\Delta + \lambda)(f(r)e^{in\theta}) = \frac{1 - r^2}{4}\left(f''(r) + \frac{1}{r} f'(r) + \left(\frac{4\lambda}{1 - r^2} - \frac{n^2}{r^2}\right) f(r)\right) e^{in\theta}.$$

Therefore $u(r, \theta) = f(r)e^{in\theta}$ is an eigenfunction of Δ only if f satisfies

$$r^2(1 - r^2) f'' + r(1 - r^2) f' + (4\lambda r^2 - n^2(1 - r^2)) f = 0.$$

Assume for definiteness that $n \geq 0$ and write $f(r) = r^n g(r^2)$, so that g satisfies the hypergeometric equation

$$r(1 - r)g'' + (c - (a + b + 1)r)g' - ab\, g = 0,$$

where

$$a = \left(n + \sqrt{n^2 + 4\lambda}\right)/2, \quad b = \left(n - \sqrt{n^2 + 4\lambda}\right)/2, \text{ and } c = n + 1.$$

This has a unique non-singular solution (up to scalar multiplication), the hypergeometric function

$$g(r) = {}_2F_1(a, b; n + 1; r) \quad \text{where} \quad g(1) = \frac{n!}{\Gamma(1 + a)\Gamma(1 + b)},$$

which is zero at $r = 1$ if and only if $b = -m$ for some integer $m \geq 1$. This implies $\lambda = m(m + n)$ is a positive integer, proving part (1) of the theorem. If $n > 0$ there are two corresponding eigenfunctions $u(r, \theta) = r^n g(r^2)e^{\pm in\theta}$, making a total of $\tau(\lambda)$ eigenfunctions for each positive integer eigenvalue λ. That these are linearly independent (part (3)), and that the eigenfuctions are polynomials (part (2)) can be verified using an explicit formula for the eigenfunctions, as follows.

Using the formulation $\Delta = -(1 - z\bar{z})\frac{\partial^2}{\partial \bar{z} \partial z}$, where $z = x + iy$, one can check directly that the Rodrigues-type formula

$$u^{(p,q)}(z) = \frac{(-1)^p}{q(p + q - 1)!}(1 - z\bar{z})\frac{\partial^{p+q}}{\partial \bar{z}^p \partial z^q}(1 - z\bar{z})^{p+q-1} \tag{2}$$

represents eigenfunctions corresponding to $\lambda = pq$, i.e., $\Delta u^{(p,q)} = pq\, u^{(p,q)}$. q.e.d.

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1.3. **Two corollaries.**

Corollary 1.4. *The spectral function is precisely the square of the Riemann zeta function*

$$\sum_n \frac{1}{(\lambda_n)^s} = \sum_n \frac{\tau(n)}{n^s} = (\zeta(s))^2.$$

Corollary 1.5. *Comparison with the standard vibrating membrane yields the harmonic drumhead, as the eigenfrequencies are all commensurate.*

1.4. **Acoustics and combinatorics.** Supplemented by monomials $u^{(p,0)}(z) := z^p$ for $p \geq 0$, the eigenfunctions of Δ are the special functions suited to acoustic imaging of layered media. Let $G(t)$ denote the normal incidence plane wave impulse response at the boundary of an n-layered medium having layer thicknesses $L = (L_1, \dots, L_n)$ (in two-way travel time) and reflection coefficients at layer interfaces $R = (R_1, \dots, R_n) \in (-1, 1)^n$.

Theorem 1.6.

$$G(t) = \sum_{k \in \{1\} \times \mathbf{Z}_+^{n-1}} \left(\prod_{j=1}^{n} u^{(k_j, k_{j+1})}(R_j) \right) \delta(t - \langle L, k \rangle).$$

Here $u^{(0,q)} \equiv 0$ if $q \geq 1$ and $k_j = 0$ if $j > n$.

Proof. Expanding the binomial $(1 - z\bar{z})^{p+q-1}$ in the formula (2), and then applying the derivative $\partial^{p+q} / \partial \bar{z}^p \partial z^q$, yields

$$u^{(p,q)}(z) = \frac{(-1)^{q+\nu+1}}{q} (1 - z\bar{z}) z^{m+\nu-q+1} \bar{z}^{m+\nu-p+1} \sum_{j=0}^{\nu} (-1)^j \frac{(j+\nu+m+1)!}{j!(j+m)!(\nu-j)!} (z\bar{z})^j,$$

where $m = |p - q|$ and $\nu = \min\{p, q\} - 1$. Switching to polar form $z = re^{i\theta}$, it follows that

$$u^{(p,q)}(re^{i\theta}) = e^{i(p-q)\theta} \frac{(-1)^{q+\nu+1}}{q} (1 - r^2) r^m \sum_{j=0}^{\nu} (-1)^j \frac{(j+\nu+m+1)!}{j!(j+m)!(\nu-j)!} r^{2j}.$$

For $\theta = 0, \pi$, the latter conicide with the functions $f^{(p,q)}$ occurring in [2, Thm. 2.4, 4.3]. q.e.d.

The tensor products $\prod_{j=1}^{n} u^{(k_j, k_{j+1})}(R_j)$ have a combinatorial interpretation in that they count weighted Dyck paths having $2k_j$ edges at height j. See [2].

References

[1] *The On-Line Encyclopedia of Integer Sequences*, published electronically at https://oeis.org, 2016, Sequence A007875.

[2] P. C. Gibson. The combinatorics of scattering in layered media. *SIAM J. Appl. Math.*, 74(4):919–938, 2014.

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Mar 16 2016 09:34:35 EDT
Version 2 - Submitted to PROC