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The Dark Side of the Moebius Strip

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The Moebius strip has now been around for over a century. Everybody knows how to make one, as a real-world object as well as abstractly. And yet, as a geometrical configuration in three-dimensional Euclidean space, there is still no satisfactory model for it. Furthermore, most of the attempts in this direction have remained relatively unknown. The following is an account of what I found after some search, and some research, about these matters. In particular: a smooth flat model, as had been also found by Sadowsky [1930, no. 4]; two smooth flat algebraic models, one defined by Wunderlich [1962], and one found independently by myself [1988]; some results concerning the shortest Moebius strip, by Barr [1964], who also reports results of Gardner; some by myself, first reported here; a formulation of a variational problem—the Moebius strip of least elastic energy—by Sadowsky [1930, no. 5]; and, finally, my own conjecture, as it emerges from all the facts listed here plus some observations of real-world Moebius strips.

1. What exactly is a Moebius strip? On one hand, it is often defined as the topological space attained by starting with a (closed) rectangle, endowed with the "usual" topology, and identifying two opposite edges point by point with each other, so that each vertex gets identified with the one diagonally across. This "abstract Moebius strip" serves in topology as the canonical example of a nonorientable manifold.

On the other hand, there is a physical model of the abstract strip, and it is usually denoted by the same term. Its inventor Moebius [1865] (also spelled "Möbius", but "Mobius" is not an acceptable spelling), described it as follows: a paper rectangle that is sufficiently long and narrow is bent and twisted so that its two shorter edges can be glued together in the required manner.

Between the abstract topological space and its physical model lie some concepts of intermediate degrees of abstraction. For example, besides a physical model for the topological strip, one can ask for a geometrical model in three-dimensional Euclidean space. Such a model will be provided by a subset of 3-space, that is homeomorphic to the topological strip. A simple approach to constructing such an "embedding" of the strip in 3-space consists of imitating the physical twisting that takes place when the paper strip is produced. Fix a line in three space, and pick a segment that is coplanar with the line, perpendicular to it and disjoint from it. Rotate the plane, and the segment with it, around the fixed line. At the same time, in the rotating plane, rotate the segment around its midpoint. If the second rotation is carried out at half the angular velocity of the first one, the segment will have completed half a turn around its center when it has gone once around the line, and thus will meet itself with the required twist. If the fixed line is chosen as the z-axis of a coordinate system, the initial position of the segment is chosen to be $[R-1, R-1]$ on the x-axis, for some $R > 1$, and the angles by which the two rotations have progressed are t and $\frac{1}{2}t$, respectively, then the equations

$$x(s,t) = \left(R + s\cos\tfrac{1}{2}t\right)\cos t \qquad y(s,t) = \left(R + s\cos\tfrac{1}{2}t\right)\sin t,$$
$$z(s,t) = s\sin\tfrac{1}{2}t,$$

for $-1 \leqslant s \leqslant 1$, $0 \leqslant t \leqslant 2\pi$, define the embedding. The fact that for all s the functions at $(s, 2\pi)$ agree with their values at $(-s, 0)$ reflects the identifying of the opposite sides of the rectangle. Besides being homeomorphic to the topological Moebius strip, this model has the advantage of using only analytic functions of s and t. In fact, it is not only analytic—it is algebraic: it is possible to eliminate s and t from the expressions for the coordinates, and to represent the strip as part of an algebraic surface, that is, as the set of points where a polynomial in the coordinates takes on the value 0.

There is, however, also a drawback to this model. The nature of this drawback will become clear after taking a closer look at the physical model, the paper strip. It was produced by bending and glueing, but no stretching was used: any curve drawn on the paper rectangle would have became a space curve of the same length. A mathematical model that reflects this property of the paper strip would be an *isometric* embedding. The following considerations show that the model defined above fails on that score. The center line of the rectangle is the line $s = 0$, and its image under the embedding is the circle of radius R around the origin in the x-y plane. Its length is $2\pi R$, and an appropriate rescaling would have matched this value with the length of the rectangle in the (s,t)-plane. However, the other lines of the form $s =$ *constant* go into curves of greater length. To see this, find the velocity at which a point on the strip moves when s is fixed, and t is regarded as "time." The velocity can be split into two components, one in the plane through the point and the z-axis, and one perpendicular to it. Their magnitudes are $R + s\cos\frac{1}{2}t$ and $\frac{1}{2}s$, respectively. The total velocity is, therefore, strictly greater than the first part, whenever s is not 0. Since integration of the first part from $t = 0$ to $t = 2\pi$ yields just $2\pi R$ (note that the cosine term contributes nothing to the definite integral), the space-curve corresponding to nonzero constant s is strictly longer than the circle of radius R. The model is therefore not a valid representation of any Moebius strip that can be made out of a paper rectangle. Could it fit a strip made out of a differently shaped piece of paper? Again the answer is no, but in order to see this, a local consequence of the absence of stretching will have to be considered.

A differentiable surface can be approximated in the neighborhood of any one of its points by a plane, the tangent plane at the point. If the surface is twice differentiable, it can be approximated to a higher degree of approximation by a quadric. Among twice differentiable surfaces, those that are obtained by bending, but not stretching, paper, are characterized by the property that the approximating quadrics will be planes or parabolic cylinders. Surfaces with this property, that holds if and only if the matrix of the second derivatives of z with respect to x and y is singular, are called "flat surfaces." It is one of the classical results of differential geometry, that among twice differentiable surfaces, a piece of plane can be mapped isometrically only onto flat ones.

To the point ($s = 0$, $t = 0$) of the rectangle there corresponds the point ($x = R$, $y = 0$, $z = 0$) in space. In a neighborhood of this point, the Taylor expansion shows that $z = \frac{1}{2}(x - R)y/R$ is true modulo third and higher powers of the distance from between (x, y) and $(R, 0)$. The strip is, therefore, locally approximated by a hyperboloid at that point. Therefore, the surface is not flat, and consequently, the model fails to represent any paper model of the Moebius strip.

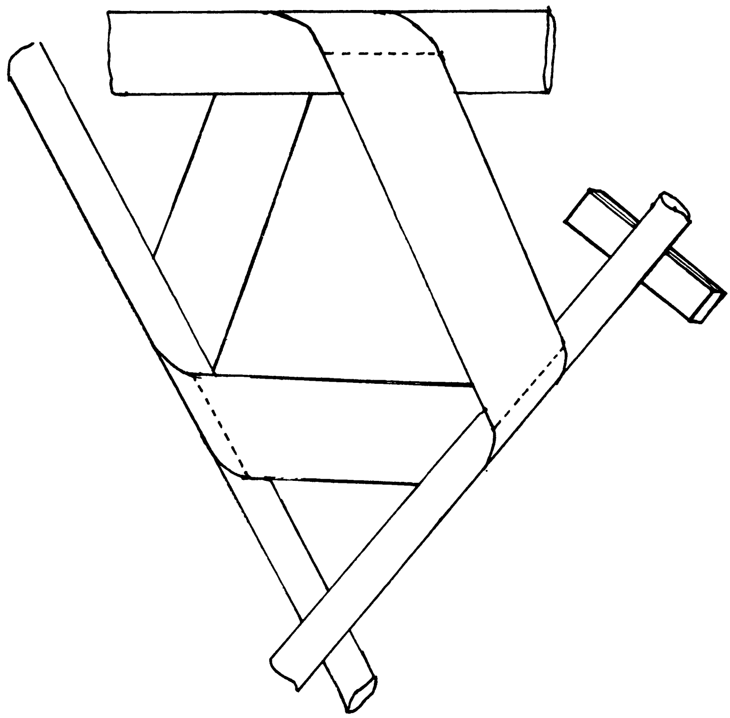
2. The simplest flat surfaces are planes and cylinders. A Moebius strip consisting of three planar parts and three cylindrical parts, smoothly joined together, was found by Sadowsky [1930, no. 1]. I had no access to his papers, and before I read about them in Wunderlich [1962], I found such a model myself. This model is best described in physical terms. Consider a long cylinder of radius r, and a long paper rectangle of width w. Imagine the rectangle lying on a table, and the cylinder lying on the rectangle at an angle, that is, so that the axis of the cylinder forms an acute angle of, say, α with the long sides of the rectangle. Clearly one can get hold of one end of the rectangle, and pulling it up, wrap the rectangle tightly half a turn around the cylinder, and then let it continue at an angle of $\pi - \alpha$ to the axis, on the plane that is parallel to the table, a distance of $2r$ above it.

Each one of the long sides of the rectangle will become a space curve consisting of a piece of a helix, and two straight-line pieces, each tangent to the helical piece at one of its end. By rolling out the cylinder onto a plane, the helical piece would become the hypothenuse of a right triangle with one side of length πr, and an angle of α facing it. Its length is therefore $\pi r/\sin \alpha$. Now project the picture down onto the table. Each of the two space curves becomes a pair of segments forming an angle of $\pi - 2\alpha$, whose missing tip is replaced by a sine curve. Note that the missing tip consists of two sides of an isosceles triangle with an angle of $\pi - 2\alpha$ facing a base of length $\pi r \cot \alpha$, and its length is $\pi r/\sin \alpha$, the same as the helical part of the space curve. Since the segment that was on the higher plane does not change its length when it is projected onto the table, and the other segment stays put, the whole space curve retains its length when it is first projected onto the table, and then its rounded tip is replaced by continuing the segments until they meet.

There is a "physical" consequence of this fact: when a (long) rectangle is folded once on the table, a cylinder, say of radius r, can be inserted into the fold, so that the rectangle wraps tightly around the cylinder for half a turn, and then continues in the horizontal plane that lies $2r$ above the table, directly over its previous position on the table.

A flat smooth Moebius strip will now be built from three of these configurations. Choose three angles α, β and γ, that sum to π; then $\pi - 2\alpha$, $\pi - 2\beta$ and $\pi - 2\gamma$ also add up to π. For the corresponding radii choose r, r and $2r$ respectively. Start with a rectangle lying on the table. Put a cylinder of radius r on it, so that its axis forms an angle of α with the long sides of the rectangle, and wrap the rectangle half a turn around it. The rectangle continues now on a plane that lies $2r$ above the table. Now repeat the procedure with the second cylinder, sufficiently far from the first one so as not to interfere with it (in "reality" a slab of thickness $2r$ will have to be inserted under the part of the rectangle that is r above the table, to keep it there). After this operation the rectangle continues on a plane

$4r$ above the table. In the third step, where a cylinder of double radius is used, wrap downwards, causing the strip to return to the plane of the table, where it can be made to meet the first part that never left the table.



The surface that was constructed here is easily seen to have a continuously varying tangent plane: it is a smooth surface. Since each half-turn wrapping flips it around once, and it meets itself after three of them, it is nonorientable, and represents the Moebius strip. On the debit side, it is not analytic at the "seams" where cylinders meet planes. In fact, the second derivatives that are needed for defining the second-order Taylor series approximation are not defined at these points; still, the surface is "flat" in the extended sense: it is locally isometric to a piece of a plane.

The construction can be modified so that higher derivatives will exist. All that is required, is that the circular cross-sections of the cylinders will be replaced by cross sections of the form

$$\frac{y^{2n}}{s^{2n}} + \frac{z^{2n}}{r^{2n}} = 0,$$

where s is chosen so that the (generalized) helix will have the right length of $\pi r/\sin \alpha$, and the surface will have $2n - 1$ derivatives. By using more complicated cross sections, derivatives of all orders may be made to exist, but this will also not yield analyticity; as long as there are planar portions of the model, it cannot be analytic.

3. Let us now have a closer look at the dimensions of the rectangle out of which the strip was made. The cylindrical parts of the surface will appear on the rectangle as three disjoint strips, two of width πr and one of width $2\pi r$, that form, with the long sides of the rectangle, angles of alternating signs, and of magnitudes α, β and γ, respectively. Let the width w of the rectangle be chosen as the unit of length. Clearly, the length of the rectangle must exceed the sum of the cotangents of α, β and γ. The average of the angles is $\pi/3$, and for positive acute angles the function cotangent is convex. By the Jensen Inequality, the average of the cotangents is at least the cotangent of the average, or $\cot(\pi/3)$, which is $1/\sqrt{3}$. The length of the rectangle therefore exceeds $\sqrt{3}$.

Conversely, starting with any rectangle whose length exceeds $\sqrt{3}$ times its width, one can, for $\alpha = \beta = \gamma = \pi/3$, and sufficiently small r, carry out the construction described above, and obtain a smooth flat Moebius strip. The question of how short a Moebius strip can be, is not completely solved by these considerations: for rectangles with *length/width* $\leqslant \sqrt{3}$ only this particular construction fails, and so a length to width ratio of more than $\sqrt{3}$ has been shown to suffice, but not to be necessary. In fact, in a certain sense there are arbitrarily short Moebius strips, if the assumption of a smooth embedding is relaxed, so as to admit "planar folded Moebius strips." These are again best described in physical terms. While the models that correspond to what can be made out of paper, without stretching and tearing, are flat, in the sense of preserving the lengths of curves, they do not have to be smooth, or even one-to-one. If the latter condition is violated by the surface cutting through itself, the model cannot represent a paper surface. Consider, however, a sheet of paper folded flat along a line. It is reasonably approximated by a mapping of a rectangle onto the plane, and this mapping is at some points two-to-one. The folded paper may then be bent, yielding a two-to-one mapping onto nonplanar surfaces. The extension to finitely many folds is obvious. Martin Gardener has constructed folded Moebius strips in which that side of the rectangle that gets identified with its opposite is arbitrarily long compared to the two other sides. The bound $\sqrt{3}$ is thus replaced by 0. After the description of Gardner's model, the bound $\sqrt{3}$ will make a come-back, in a slightly different role. Start with a rectangle of dimensions W and L. Pick a positive odd integer n, and divide the rectangle into n parallel strips of dimensions W/n by L. Folding along the dividing lines back and forth, accordion-fashion, an n-fold covered strip of length L and width W/n is obtained.

No matter how large W is, compared to L, for sufficiently large n, this strip will be narrow enough to permit its being bent and glued into a Moebius strip. Note that the fact that n is odd permits the different layers of the folded strip to be glued to each other individually, in a manner that is equivalent to the identification of sides required by the configuration of the Moebius strip.

4. Compare Gardner's folded model with another folded model, namely, the limiting configuration obtained from the three-plane-three-cylinder model, when the parameter r approaches 0. The latter, that can be constructed by folding a rectangle three times, is, by definition, the limit of a family of smooth flat models. The value of a parameter like *length/width* for such a model is the limit of its values for the flat smooth models that approximate it. Inclusion of such folded models won't change the least upper bound of a parameter. This is not the case for Gardner's model: it cannot be approximated by flat smooth surfaces. More generally: *Any paper surface in the making of which a folding-line is bent or folded, is a finite distance away, in an appropriate sense of distance, from all smooth flat surfaces*. This follows from the fact that every smooth flat surface is a ruled surface. For every point, one of the lines that pass through the point on the paper remains a straight line as the paper is bent to make the required surface. This condition is violated when the paper is folded, and then refolded, with the second fold cutting through the first one: let the acute angle between the two fold-lines be α. Then a lines of slope β through the point of intersection of the two folding-lines will be broken at an angle of $2\max(\alpha, \beta)$, which is at least twice the angle between the folding-lines. If, in making the surface, the folding line is bent, the straight lines through any of its points will also be bent by an angle bounded from below. Thus Gardner's model does not disprove the conjecture that *the length of a smooth flat Moebius strip exceeds $\sqrt{3}$ times its width*. The model that Barr describes in Chapter 3 of his book [1964], where a paper square is folded along one diagonal, and then along the other, also does not imply the violation of this bound, since its making involves the folding of a fold. If there is a counterexample to the conjecture, it won't be found by examining flat folded models. Indeed, *the length of a flat folded Moebius strip without refolded folds is at least $\sqrt{3}$ times its width. The bound is attained by the configuration consisting of three equilateral triangles on top of each other, the first connected to the second along one side, the second to the third along another side, and the third to the first along the third side.*

This is proved by first eliminating all models in which the center-line of the rectangle becomes, after folding, a polygon of more than three sides. There remain only the models where it becomes a triangle, with the three fold-lines bisecting its three exterior angles. On the rectangle, the folding lines cut the center lines at angles α, $-\beta$ and γ. The cotangents must add up at least to the length over the width, and the proof concludes like above. Note, however, that here the bound is attained, because there is no extra room needed for any cylinders.

5. Three decades after the work of Sadowsky, the step from flat smooth models to flat analytic, and even algebraic models was accomplished by Wunderlich [1962]. Chicone [1984], who was evidently not aware of Wunderlich's article, studied the problem, and came up with a family of flat analytic surfaces in 3-space, homeomorphic to the Moebius strip, but he failed to check whether they contain parts isometric to a strip obtained from a rectangle. To understand what is missing in Chicone's model, consider the embedding of a cylindrical ("untwisted") band in the surface of a cone. It is easy to find a homeomorphic embedding, but it cannot be trimmed down to an isometric image of a cylindrical band, since there are no closed geodesics on a cone. Having seen Chicone's manuscript, but also unaware of the work of Wunderlich, I constructed a different flat algebraic model, also isometric to a rectangle [1988].

Wunderlich's model and my own are both based on the same observation: the bisecting line of the rectangle that is parallel to the long sides, becomes a (closed) geodesic on the surface. If a curve is a geodesic on a flat smooth surface, and the curve has no straight parts, then the surface is the *rectifying developable* of the curve. This is a flat surface, which forms the envelope of all the rectifying planes of the curve. The normal to the surface at any point of the curve will be the principal normal to the curve at that point. Therefore, a simple, closed curve without straight parts, whose principal normal changes its direction by 180 degrees when the curve is traversed once, will yield the required model: its rectifying developable will contain a band around the curve, isometric to a Moebius strip made from a rectangle; and if the band is narrow enough it will not intersect itself. A simple way to make sure that the principal normal will flip around, is to make the curve symmetric under a rotation by 180° around an axis that meets the curve at two points, one where the principal normal is parallel to the axis, and one where it is perpendicular to it. The second point must be a point of zero curvature of the curve. Wunderlich chose a parametrization in which this point is approached when the parameter t of the curve approaches $\pm\infty$. The coordinates of the curve are given in his model as rational functions (that is, quotients of polynomials) of the parameter. In my model, the point of zero curvature corresponds to $t = 0$, and the fact that the curve is closed is interpreted as the curve having a periodic parametrization. This suggested trying for a parametrization by trigonometric polynomials. The simplest curve that has the required properties was found to be

$$x = \sin t, \qquad y = (1 - \cos t)^3, \qquad z = \sin t(1 - \cos t),$$

which is indeed an algebraic curve; it is the intersection of the surfaces

$$z^3 = x^3 y, \qquad \text{and} \quad 8y = x^6 + 6x^2 y + y^2.$$

As was pointed out to me by Gregory Brumfiel, the Seidenberg-Tarski Theorem [1954] can be used to show that not only the curve, but also the strip obtained in this manner is part of an algebraic surface.

6. Let us now take another step in the direction from the abstract to the real world, and include some physical properties of paper in our considerations. If the substance of paper is assumed to be elastic, with a finite value for the constant in Hook's law, it would be possible to stretch it. Paper that cannot be stretched, yet can be bent if a finite force is applied, is well approximated by an infinitely thin plate with an infinite Hook's constant, so that bending any piece of it into part of a cylinder of radius r requires an amount of energy proportional to the area of the piece divided by r^2. Since every small part of a flat surface is locally approximable by a cylinder with radius equal to the finite one among the two radii of curvature (the other one must be infinite), the total energy expended in making the surface out of paper is the integral of r^{-2} over the surface. Left to itself, the paper will take on the shape of minimum energy among all shapes in 3-space that are compatible with the metric structure of the surface.

As an example of this phenomenon, consider a cylindrical band, made out of a paper rectangle *without* twisting. The integral of $1/r$ over any closed geodesic on the band is forced to remain 2π. Under this constraint, the minimum of the integral of the square of $1/r$ is attained when r is constant, which holds when the surface is a circular cylinder. Indeed, this is the shape the surface takes on "in the real world," and if gently deformed, it returns to this shape. Paper Moebius strips also appear to have a clear stable shape, if the paper is not too soft, and the width of the strip is not too small. Sadowsky [1930, 5] approached the search for a analytical Moebius strip from this angle. He did not find an explicit form for the shape of minimum energy, but he simplified the expression for the energy: he reduced it to a one-dimensional integral, with respect to arc length s along the curve whose rectifying developable is the strip. The integrand can be expressed in terms of the curvature $k(s)$ of the curve, its torsion $\tau(s)$ and the function

$$\psi(s) = \frac{1}{\tau\dfrac{dk}{ds} - k\dfrac{d\tau}{ds}}$$

as follows:

$$\int (k^2 + \tau^2)^2 \psi \log \frac{2k^2\psi + w}{2k^2\psi - w}\, ds,$$

where w is the width of the strip, and s ranges over its length.

Neither Sadowsky nor Wunderlich solved this variational problem—in all likelihood it is still open. On the basis of observation of physical models, Wunderlich states that "the shape appears to depend only slightly on the width w." This may be true for small values of w; in fact, Sadowsky showed that in the limit, for the "infinitely narrow band," the energy is approximately proportional to the integral of $(k^2 + \tau^2)^2/k^2$, and it is plausible that the solution to the minimum problem for finite w approaches the (also unknown) solution of this simpler problem when w goes to zero. Note, however, that for a fixed curve, when w grows, and approaches the minimum of $2k^2\psi$ along the curve, the energy becomes infinite. The dependence of the solution on w can, therefore, be expected to become much more pronounced as w becomes larger. My own observations of physical Moebius strips confirm this, and furthermore, lead to the following conjecture:

For *fixed length* L, *and width* w *less than* $L/\sqrt{3}$, *there exists a unique minimum-energy smooth flat Moebius strip in* 3-*space. As* w *approaches* $L/\sqrt{3}$ *from below, the strip approaches the configuration of three equilateral triangles on top of each other, each two connected along one side. If* w *exceeds* $L/\sqrt{3}$, *there is no smooth Moebius strip in* 3-*space, isometric to the corresponding rectangle.*

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REFERENCES

1. Stephen Barr, Experiments in Topology, Thomas Y. Crowell Company, New York 1964.
2. C. Chicone, Untitled manuscript, Univ. Missouri, 1984.
3. F. A. Möbius, Über die Bestimmung des Inhaltes eines Polyeders, *Ber. Verh. Ges. Wiss.* 17 (1865) 31–68. Gesammelte Werke, Bd. II, Leipzig.
4. M. Sadowsky, Ein elementarer Beweis für die Existenz eines abwickelbaren Möbiusschen Bandes und Zurückführung des geometrischen Problems auf ein Variationsproblem, *Sitzungsber. Preuss. Akad. Wiss.*, 22 (1930) 412–415.
5. ______, Theorie der elastisch biegsamen undehnbaren Bänder mit Anwendung auf das Möbiussche Band, *Verh. 3. Kongr. Techn. Mechanik*, Stockholm 1930, II, pp. 444–451.
6. G. Schwarz, A pretender to the title "Canonical Moebius Strip," *Pacific J. Math.*, 143 (1990) 195–200.
7. A. Seidenberg, A new decision method for elementary algebra, *Ann. Math.*, 60 (1954), 365–374.
8. W. Wunderlich, Über ein abwickelbares Möbiusband, *Monatshefte Math.*, 66 (1962), 276–289.