16.8

Divergence Theorem
Review: Divergence Theorem

- **D**: a closed and bounded region in 3-space
- **S**: the piecewise smooth boundary of **D**
- **n**: the unit normal to **S**, pointing outward
- **F**: \( \mathbf{F} = \langle P, Q, R \rangle \) is a vector field with \( P, Q, R \), and all first partial derivatives continuous in the region in **D**

\[
\iint_S (\mathbf{F} \cdot \mathbf{n}) \, d\sigma = \iiint_D \text{div} \, \mathbf{F} \, dV
\]

total outward flux through the surface **S** = integral of local flux over the interior
**Example:** Compute the outward flux \( \iint_S (\mathbf{F} \cdot \mathbf{n}) \, d\sigma \)

of the vector field \( \mathbf{F} = (yz - x)\mathbf{i} + (x - y)\mathbf{j} + (1 + z^2)\mathbf{k} \)

through \( S \), which is the surface of the ellipsoid

\[
2x^2 + 2y^2 + z^2 = 8
\]

lying above the plane \( z = 0 \)

\[
\iint_S (\mathbf{F} \cdot \mathbf{n}) \, d\sigma = \iiint_D \text{div} \, \mathbf{F} \, dV
\]

The surface \( S \) is not closed, so cannot use divergence theorem

Add a second surface \( S' \) (any one will do) so that

\( S \cup S' \) is a closed surface with interior \( D \)

simplest choice: a disc \( x^2 + y^2 \leq 4 \) in the x-y plane

\[
\iint_S (\mathbf{F} \cdot \mathbf{n}) \, d\sigma + \iint_{S'} (\mathbf{F} \cdot \mathbf{n}') \, d\sigma = \iiint_D \text{div}(\mathbf{F}) \, dV
\]

hence:

\[
\iint_S (\mathbf{F} \cdot \mathbf{n}) \, d\sigma = \iiint_D \text{div}(\mathbf{F}) \, dV - \iint_{S'} (\mathbf{F} \cdot \mathbf{n}') \, d\sigma
\]
\[ \iiint_S (\mathbf{F} \cdot \mathbf{n}) \, d\sigma = \iiint_D \text{div}(\mathbf{F}) \, dV - \iiint_{S'} (\mathbf{F} \cdot \mathbf{n}') \, d\sigma \]

\[ \iiint_D \text{div}(\mathbf{F}) \, dV = \iiint_D 2(z-1) \, dV = 2 \iiint_D zdV - 2 \text{vol}(D) \quad \text{vol}(D) = \frac{4}{3} \pi \cdot 2 \cdot 2 \cdot \sqrt{8} = \frac{32\pi}{3} \sqrt{2} \]

\[ \iiint_D 2z \, dV = \int_{0}^{2} \int_{0}^{2 \sqrt{8-2x^2-2y^2}} \int_{0}^{2} 2zr \, dz \, dr \, d\theta = \int_{0}^{2} \int_{0}^{2 \pi} \left(8-2x^2-2y^2\right) r \, dr \, d\theta \]

\[ = \int_{0}^{2} \int_{0}^{2 \pi} \left(8-2r^2\right) r \, dr \, d\theta = 2\pi \left(8r - \frac{2}{3} r^3\right) \bigg|_{0}^{2} = \frac{64\pi}{3} \]

\[ \iiint_D 2(z-1) \, dV = \iiint_D 2z \, dV - 2 \text{vol}(D) = \frac{64\pi}{3} - \frac{64\pi}{3} \sqrt{2} = \frac{64\pi}{3} (1 - \sqrt{2}) \]

\[ \iiint_{S'} (\mathbf{F} \cdot \mathbf{n}') \, d\sigma = \iiint_{S'} (\mathbf{F} \cdot (-\mathbf{k})) \, d\sigma = \iiint_{S'} -(1+z^2) \, d\sigma = -\iiint_{S'} 1 \, d\sigma = -4\pi \]

\[ \iiint_{S} (\mathbf{F} \cdot \mathbf{n}) \, d\sigma = \iiint_D \text{div}(\mathbf{F}) \, dV - \iiint_{S'} (\mathbf{F} \cdot \mathbf{n}') \, d\sigma = 4\pi + \frac{64\pi}{3} (1 - \sqrt{2}) \]
Divergence theorem with holes

Suppose the vector field \( \mathbf{F} \) satisfies the conditions of the Divergence Theorem on a region \( D \) bounded by two smooth oriented surfaces \( S_1 \) and \( S_2 \), where \( S_1 \) lies within \( S_2 \). Let \( S \) be the entire boundary of \( D \) (\( S = S_1 \cup S_2 \)) and let \( \mathbf{n}_1 \) and \( \mathbf{n}_2 \) be the outward unit normal vectors for \( S_1 \) and \( S_2 \), respectively. Then

\[
\iiint_D \nabla \cdot \mathbf{F} \, dV = \iint_S \mathbf{F} \cdot \mathbf{n} \, dS = \iint_{S_2} \mathbf{F} \cdot \mathbf{n}_2 \, dS - \iint_{S_1} \mathbf{F} \cdot \mathbf{n}_1 \, dS.
\]

\( \mathbf{n}_1 \) is outward normal to \( S_1 \) and points into \( D \).
Outward normal to \( S \) on \( S_1 \) is \( -\mathbf{n}_1 \).

If \( \text{div} \, \mathbf{F} = 0 \), between \( S_1 \) and \( S_2 \), then

\[
\iint_{S_1} (\mathbf{F} \cdot \mathbf{n}_1) \, d\sigma = \iint_{S_2} (\mathbf{F} \cdot \mathbf{n}_2) \, d\sigma
\]

Can interchange surfaces!
New integral might be simpler.
Application: Recall electrical field (Coulomb's Law, 1785):

The electrical field of a charge \( Q \) centered at the origin:

\[
E = \frac{Q}{|r|^2} \frac{r}{|r|}
\]

force on a second particle of charge \( q \) is: \( F = \pm qE \)

Important property: \( \text{curl } E = 0 \) since conservative, and \( \text{div } E = 0 \) as well (away from origin).

(Recall this follows from inverse square law in 3-space)

Gauss's Law (1835): \[
\iint_S (E \cdot n) \, d\sigma = 4\pi Q
\]

for any surface \( S \) which contains the origin in its interior

Proof: Let \( S' \) be a sphere centered at the origin of a small radius \( a \) so that \( S' \) lies inside \( S \).

Since \( \text{div } E = 0 \), away from the origin, we have \[
\iint_S (E \cdot n) \, d\sigma = \iint_{S'} (E \cdot n') \, d\sigma
\]

\( n' = \frac{r}{|r|} \) is the normal to \( S' \)

\[
\iint_{S'} (E \cdot n') \, d\sigma = \iint_{S'} \left( Q \frac{r}{|r|^3} \cdot \frac{r}{|r|} \right) \, d\sigma = Q \iint_{S'} \left( \frac{r \cdot r}{a^4} \right) \, d\sigma = Q \iint_{S} \left( \frac{1}{a^2} \right) \, d\sigma
\]

\[
= Q \frac{1}{a^2} \cdot \text{area}(S) = Q \frac{1}{a^2} \cdot 4\pi a^2 = 4\pi Q
\]
Consequence of Gauss's Law: \[ \iint_S (\mathbf{E} \cdot \mathbf{n}) d\sigma = 4\pi Q \]

Assume there is a charge distribution in the region \( D \) inside \( S \) of charge density \( \delta \)

Gauss's law generalizes to: \[ \iint_S (\mathbf{E} \cdot \mathbf{n}) d\sigma = 4\pi \iiint_D \delta dV \]

Total outward flux across \( S \) = total charge in \( D \)

by divergence theorem: \[ \iint_S (\mathbf{E} \cdot \mathbf{n}) d\sigma = \iiint_D \text{div} \mathbf{E} dV \]

thus \[ \iiint_D \text{div} \mathbf{E} dV = 4\pi \iiint_D \delta dV \]

since this is true for all regions \( D \), this can only hold if

\[ \text{div} \mathbf{E} = 4\pi \delta \]

this is one of Maxwell's equations (1861)

others: \[ \text{curl}(\mathbf{E}) = -\frac{\partial \mathbf{B}}{\partial t}, \quad \text{div}(\mathbf{B}) = 0, \]

where \( \mathbf{B} \) is the magnetic field
Geometric interpretation of divergence as an application of Divergence Theorem\[ \iiint_{D} \text{div} \mathbf{F} \, dV = \iint_{S} (\mathbf{F} \cdot \mathbf{n}) \, d\sigma \]

Fix a point \( P \) and let \( B_r(p) \) be a small ball of radius \( r \) centered at \( P \) with boundary a sphere \( S_r \).

For small \( r \), the left hand side is approximately\[ \iiint_{B} \text{div} \mathbf{F} \, dV \approx (\text{div} \mathbf{F})_p \times \iint_{B} 1 \cdot dV = (\text{div} \mathbf{F})_p \times \text{vol}(B_r) \]

Thus \( (\text{div} \mathbf{F})_p \approx \frac{1}{\text{vol}(B_r)} \iiint_{B} \text{div} \mathbf{F} \, dV = \frac{1}{\text{vol}(B_r)} \iint_{S_r} (\mathbf{F} \cdot \mathbf{n}) \, d\sigma \) by Divergence theorem.

as \( r \) goes to 0 we get:\[ (\text{div} \mathbf{F})_p = \lim_{r \to 0} \frac{1}{\text{vol}(B_r)} \iint_{S_r} (\mathbf{F} \cdot \mathbf{n}) \, d\sigma \]

\( (\text{div} \mathbf{F})_p \) measures outward flux near \( p \) per unit volume per unit time.
Fundamental Theorem of Calculus
\[ \int_a^b f'(x) \, dx = f(b) - f(a) \]

Fundamental Theorem of Line Integrals
\[ \int_C \nabla f \cdot dr = f(B) - f(A) \]

Green’s Theorem (Circulation form)
\[ \iint_R (g_x - f_y) \, dA = \oint_C f \, dx + g \, dy \]

Stokes’ Theorem
\[ \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, dS = \oint_C \mathbf{F} \cdot dr \]

Divergence Theorem
\[ \iiint_D \nabla \cdot \mathbf{F} \, dV = \iint_S \mathbf{F} \cdot \mathbf{n} \, dS \]
Example: Let $S$ be the portion of the surface $z = 1 - \sqrt{x^2 + y^2}$ above the $xy$ plane, with normal vector pointing downward. Given the vector field $\mathbf{F} = yi + \sin(z^2)j + \cos(x^2)k$, evaluate the surface integral $\iint_S (\text{curl } \mathbf{F}) \cdot \mathbf{n} \, d\sigma$.

Example: Let $S$ be the portion of the surface $z = x^2 + y^2$ over the disc $x^2 + y^2 = 1$ in the $xy$ plane, with unit normal pointing downward. Compute the flux of $\mathbf{F}$ through $S$ if $\mathbf{F} = (e^y + x)i + (e^{\sin z} + \sin x)j + (-z + xy)k$.

Example: $\mathbf{F}$ is a vector field with $\text{curl } \mathbf{F} = i + 2j + 3k$. What is the work done by $\mathbf{F}$ along the square path from $(1,-1,3)$ to $(2,-1,3)$ to $(2,0,3)$ to $(1,0,3)$ and back to $(1,-1,3)$.