16.7

Stokes Theorem

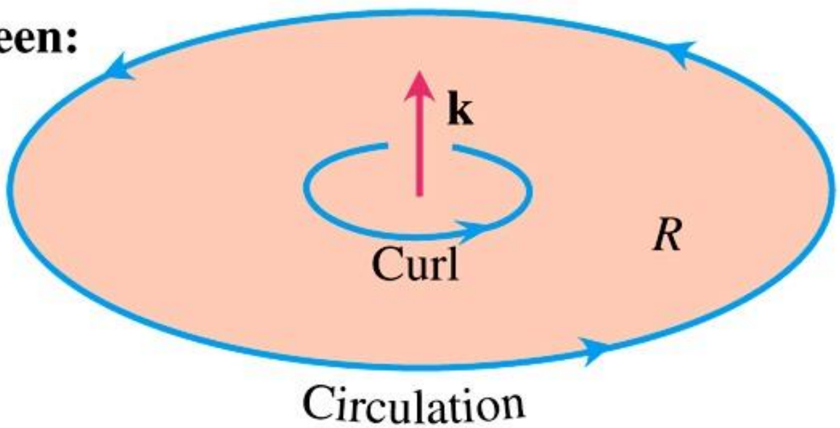
Review: **Stokes Theorem**

Let S be a smooth oriented surface in $\mathbf{R}^3$ with a smooth closed boundary C whose orientation is consistent with that of S. Assume that $\mathbf{F} = \langle f, g, h \rangle$ is a vector field whose components have continuous first partial derivatives on S. Then

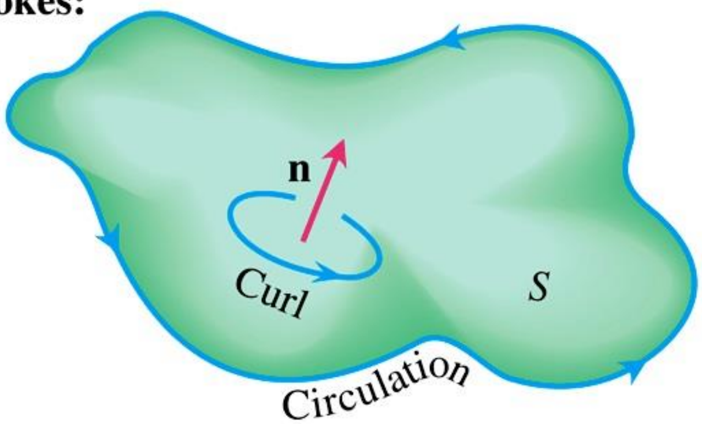
$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, dS,$$

where $\mathbf{n}$ is the unit vector normal to S determined by the orientation of S.

Green:



Stokes:

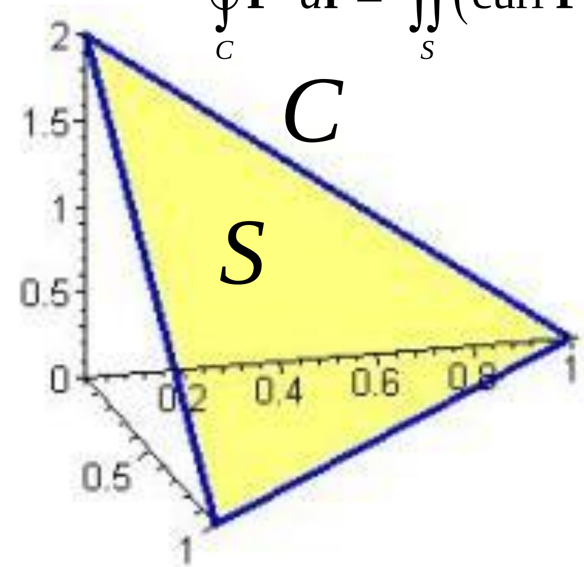


Recall Green's theorem:

$$\mathbf{F} = M\mathbf{i} + N\mathbf{j} \qquad \oint_C \mathbf{F} dr = \oint_C M dx + N dy = \iint_R \left(N_x - M_y \right) dA = \iint_R \left(\text{curl } \mathbf{F} \right) \cdot \mathbf{k} dA$$

Example: Evaluate the line integral $\oint_C \mathbf{F}\cdot d\mathbf{r}$ when $\mathbf{F}=\langle z^2, y^2, xy\rangle$, C is the triangle defined by $(1,0,0),(0,1,0)$, and $(0,0,2)$, and C is traversed counter clockwise as viewed from the origin.

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_S (\text{curl } \mathbf{F})\cdot \mathbf{n}\, d\sigma$$



S: plane, we need to find the equation using a point and the normal vector to the plane

We can get the normal vector by taking the cross product of two vectors in the plane.

Vector from $(1,0,0)$ to $(0,1,0)$

$v_1=\langle 0-1, 1-0, 0-0\rangle = \langle -1,1,0\rangle$

Vector from $(1,0,0)$ to $(0,0,2)$

$v_2=\langle 0-1, 0-0, 2-0\rangle = \langle -1,0,2\rangle$

$$\mathbf{v}=v_1\times v_2=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ -1 & 1 & 0\\ -1 & 0 & 2\end{vmatrix}=\langle 2,2,1\rangle$$ hence $\mathbf{n}=\frac{1}{3}\langle 2,2,1\rangle$ wrong orientation! choose $\mathbf{n}=\frac{-1}{3}\langle 2,2,1\rangle$

$S: 2x+2y+z=d$ plug in any point to find d

$(1,0,0)\to 2(1)+2(0)+(0)=d=2$

$\Rightarrow 2x+2y+z=2$ so $\boxed{S: z=2-2x-2y}$

$$dS=\sqrt{1+(z_x)^2+(z_y)^2}\,dA$$

$$dS=\sqrt{1+(-2)^2+(-2)^2}\,dA$$

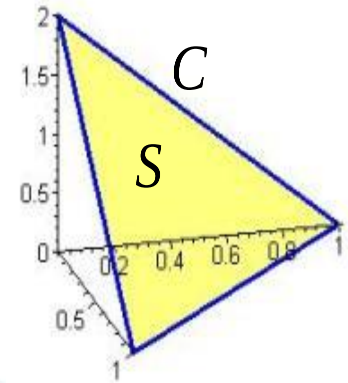
$$dS=3dA$$

Example (continued): Evaluate the line integral $\oint_C \mathbf{F}\cdot d\mathbf{r}$ when $\mathbf{F} = \langle z^2, y^2, xy\rangle$ and C is the triangle defined by $(1,0,0),(0,1,0)$, and $(0,0,2)$.

$$\mathbf{n} = \frac{-1}{3}\langle 2,2,1\rangle$$

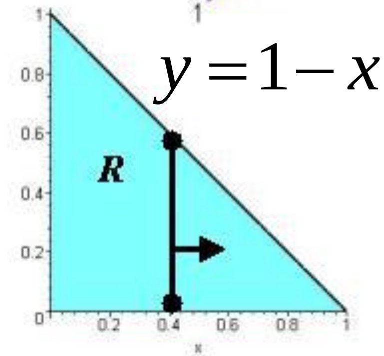
$$dS = 3dA$$

plane $S: z = 2-2x-2y$



$$\text{curl}\,\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z^2 & y^2 & xy \end{vmatrix} = \langle x, 2z-y, 0\rangle$$

$$\iint_S (\text{curl}\,\mathbf{F})\cdot\mathbf{n}\,d\sigma \;= -\iint_R (2x+4z-2y)\,dA$$



$$= -\int_0^1\int_0^{1-x}\left[2x+4(2-2x-2y)-2y\right]dy\,dx$$

$$= -\int_0^1\int_0^{1-x}\left[8-6x-10y\right]dy\,dx = -\int_0^1\left[(8-6x)y-5y^2\right]_0^{1-x}dx -$$

$$= -\int_0^1\left[(8-6x)(1-x)-5(1-x)^2\right]dx = -\int_0^1\left(x^2-4x+3\right)dx$$

$$= -\left(\frac{x^3}{3}-2x^2+3x\right)_0^1 = \frac{-1}{3}+2-3 = \boxed{-\frac{4}{3}}$$

Example: Evaluate the flux integral $\iint_S (\text{curl } \mathbf{F}) \cdot \mathbf{n} \, d\sigma$ where $\mathbf{F} = \langle 2z - y, x - z, y - x \rangle$ and S is the portion of the sphere $x^2 + y^2 + z^2 = 9$ with $z \geq y$ (a hemisphere!) and $\mathbf{n}$ points away from the origin.

The boundary C of S is the circle obtained by intersecting the sphere with the plane $z = y$

This circle is not so easy to parametrize, so instead we write C as the boundary of a disc D in the plane $y = z$.

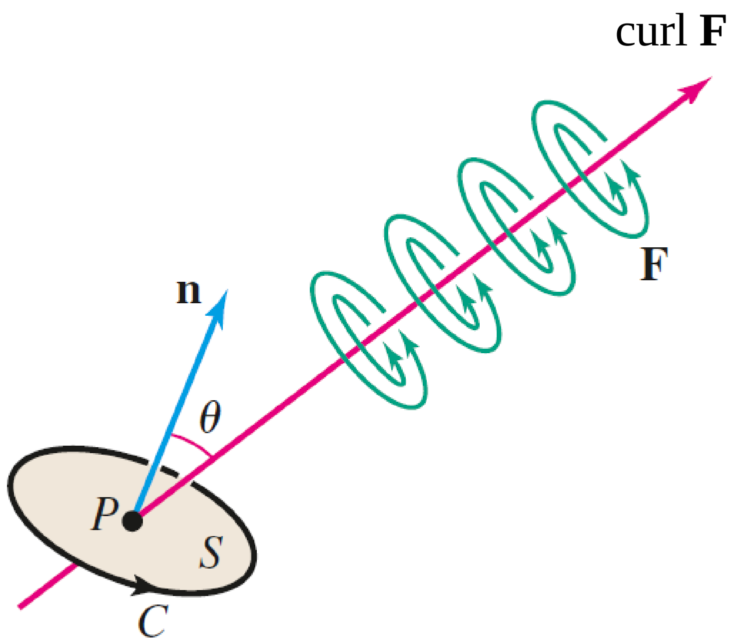
Using Stokes theorem twice, we get $\iint_S (\text{curl } \mathbf{F}) \cdot \mathbf{n} \, d\sigma = \oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_D (\text{curl } \mathbf{F}) \cdot \mathbf{n}_2 \, d\sigma$

But now $\mathbf{n}_2$ is the normal to the disc D, i.e. to the plane $y = z$: $\mathbf{n}_2 = \frac{1}{\sqrt{2}} \langle 0, -1, +1 \rangle$ (check orientation!)

$$\text{curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2z - y & x - z & y - x \end{vmatrix} = 2\mathbf{i} + 3\mathbf{j} + 2\mathbf{k} \qquad \text{curl } \mathbf{F} \cdot \mathbf{n}_2 = \frac{-1}{\sqrt{2}}$$

$$\iint_S (\text{curl } \mathbf{F}) \cdot \mathbf{n}_2 \, d\sigma = \iint_D \frac{-1}{\sqrt{2}} d\sigma = \frac{-1}{\sqrt{2}} area(D) = \frac{-1}{\sqrt{2}} \pi 3^2 = \boxed{\frac{-9}{\sqrt{2}} \pi}$$

Application of Stokes $\iint_S (\text{curl } \mathbf{F}) \cdot \mathbf{n} \, dS = \oint_C \mathbf{F} \cdot d\mathbf{r}$



Fix a point P and let $\mathbf{n}$ be a vector at P

Let S be a disc of radius r in a plane orthogonal to $\mathbf{n}$ and centered at P

For small r, the left hand side is approximately

$$(\text{curl } \mathbf{F})_P \cdot \mathbf{n} \times \iint_S 1 \cdot d\sigma \quad = (\text{curl } \mathbf{F})_P \cdot \mathbf{n} \times area(\text{S}) = (\text{curl } \mathbf{F})_P \cdot \mathbf{n} \times \left(\pi r^2\right)$$

Thus $(\text{curl } \mathbf{F})_P \cdot \mathbf{n} \approx \dfrac{1}{\pi r^2} \oint_C \mathbf{F} \cdot d\mathbf{r}$

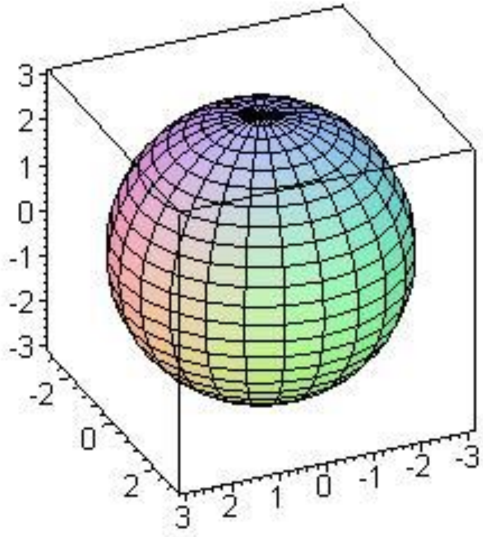
as r goes to 0 we get:

$$\boxed{(\text{curl } \mathbf{F})_P \cdot \mathbf{n} = \lim_{r \to 0} \frac{1}{\pi r^2} \oint_C \mathbf{F} \cdot d\mathbf{r}}$$

16.8

Divergence Theorem

Divergence Theorem



- D: a closed and bounded region in 3-space
- S: the piecewise smooth boundary of D
- $\mathbf{n}$: the unit normal to S, pointing outward
- $\mathbf{F}$: $\mathbf{F} = \langle P, Q, R \rangle$ is a vector field with P, Q, R, and all first partial derivatives continuous in the region in D

$$\iint_S (\mathbf{F}\cdot\mathbf{n})\ d\sigma = \iiint_D div\ \mathbf{F}\ dV$$

total outward flux through the surface S = integral of local flux over the interior

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Compare with flux version of Green's theorem for $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$

$$\oint_C \mathbf{F}\cdot\mathbf{n}ds = \oint_C -Ndx + Mdy \quad = \iint_R \left(M_x + N_y\right)dA \quad \text{or} \quad \oint_C \mathbf{F}\cdot\mathbf{n}ds = \iint_R div(\mathbf{F})d\sigma$$

Example: Let S be the surface of the cube D: $0 \le x \le 1, 0 \le y \le 1, 0 \le z \le 1$

and $\mathbf{F} = \left(e^x + z\right)\mathbf{i} + \left(y^2 - x\right)\mathbf{j} + \left(-xe^y\right)\mathbf{k}$.

Compute the outward flux $\iint_S \left(\mathbf{F} \cdot \mathbf{n}\right)\, d\sigma$

$$\iint_S \left(\mathbf{F} \cdot \mathbf{n}\right)\, d\sigma = \iiint_D div\ \mathbf{F}\ dV$$

$$= \iiint_D \left(e^x + 2y\right)\, dV$$

$$= \int_0^1 \int_0^1 \int_0^1 \left(e^x + 2y\right) dxdydz$$

$$= \int_0^1 \int_0^1 \int_0^1 e^x dxdydz + \int_0^1 \int_0^1 \int_0^1 2y dxdydz$$

$$= e^x \Big|_0^1 + y^2 \Big|_0^1 = e - 1 + 1 = e$$

Example: Use the divergence theorem to find the outward flux $\iint_S (\mathbf{F}\cdot\mathbf{n})\, d\sigma$ of the vector field $\mathbf{F} = x^3\mathbf{i} + y^3\mathbf{j} + z^3\mathbf{k}$ with D the region bounded by the sphere $x^2 + y^2 + z^2 = a^2$.

$$\iint_S (\mathbf{F}\cdot\mathbf{n})\, d\sigma = \iiint_D div\ \mathbf{F}\ dV$$

$$= \iiint_D (3x^2 + 3y^2 + 3z^2)\, dV$$

$$= \int_0^{2\pi}\int_0^{\pi}\int_0^{a} (3\rho^2)\rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi}\left[\tfrac{3}{5}\rho^5\right]_0^a \sin\phi\, d\phi\, d\theta$$

$$= \tfrac{3}{5}a^5\int_0^{2\pi}\int_0^{\pi}\sin\phi\, d\phi\, d\theta$$

$$= \tfrac{3}{5}a^5\int_0^{2\pi}\left[-\cos\phi\right]_0^{\pi} d\theta \quad = \tfrac{6}{5}a^5\int_0^{2\pi} d\theta \quad = \boxed{\frac{12\pi}{5}a^5}$$