A combinatorial determinant

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Abstract

A theorem of Mina evaluates the determinant of a matrix with entries $D^j(f(x)^i)$. We note the important special case where the matrix entries are evaluated at $x = 0$ and give a simple proof of it, as well as some special additivity properties that hold in this case, but not in general. Some applications are given also. We then give a short proof of the general case.

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An old theorem of Mina [1], which "deserves to be better known" [3], states that

$$\det\left\{\left(\frac{d^j}{dx^j}f(x)^i\right)_{i,j=0}^{n-1}\right\} = 1!2!\ldots(n-1)!f'(x)^{n(n-1)/2}. \tag{1}$$

A proof of Mina's theorem can be found, for instance, in [2]. We will first give a short proof of the special case in which both sides are evaluated at $x = 0$, with some applications, and then give a short proof of the general case. The special case shows an interesting structure owing to the fact that all matrices of that form can be simultaneously triangularized by multiplying them by a certain universal triangular matrix.

1 Coefficients of powers of power series

Theorem 1 *Let $f = 1 + a_1x + a_2x^2 + \ldots$ be a formal power series, and define a matrix c by*[1]

$$c_{i,j} = [x^j]f^i \qquad (i, j \geq 0). \tag{2}$$

Then

$$\det\left((c_{i,j})_{i,j=0}^{n}\right) = a_1^{n(n+1)/2} \qquad (n = 0, 1, 2, \ldots). \tag{3}$$

[1] "$[x^k]g$" means the coefficient of x^k in the series g.

To prove this, define a matrix b by

$$b_{i,j} = (-1)^{i+j}\binom{i}{j}, \qquad (i,j \geq 0). \tag{4}$$

Then we claim that bc is upper triangular with powers of a_1 on its diagonal. Indeed we have

$$\begin{aligned} \sum_j b_{i,j}c_{j,k} &= \sum_j (-1)^{i+j}\binom{i}{j}[x^k]f^j = (-1)^i[x^k]\sum_j(-1)^j\binom{i}{j}f^j \\ &= (-1)^i[x^k](1-f)^i = [x^k](a_1x + a_2x^2 + \ldots)^i = \begin{cases} 0, & \text{if } k < i; \\ a_1^i, & \text{if } k = i, \end{cases} \end{aligned}$$

as claimed. Since bc is this upper triangular matrix, and $\det b = 1$, the determinant of c is $\prod_{i=0}^n a_1^i = a_1^{n(n+1)/2}$. □

Ed Bender has noted that the hypothesis $a_0 = 1$ can be removed. Indeed if $a_0 \neq 0$, apply the result to f/a_0 and discover that the theorem is unchanged. If $a_0 = 0$ the result follows by continuity.

In order to gain an extra free parameter in the identities that are to follow, as well as to introduce the idea of the proof of Mina's theorem in general form, we'll restate Theorem 1 in terms of the zth power of f.

Theorem 2 *Let $f = 1 + a_1x + a_2x^2 + \ldots$ be a formal power series, let z be a complex number, and define a matrix c by*

$$c_{i,j} = [x^j]f^{zi} \qquad (i,j \geq 0). \tag{5}$$

Then

$$\det\left((c_{i,j})_{i,j=0}^n\right) = (za_1)^{n(n+1)/2} \qquad (n = 0,1,2,\ldots). \tag{6}$$

2 Some examples

1. This investigation began when I was looking at the infinite matrix whose (i,j) entry is the number of representations of the integer j as a sum of i squares of nonnegative integers $(i,j = 0,1,2,\ldots)$, and noticed that its determinant is 1. This matrix begins as

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & \ldots \\ 1 & 2 & 1 & 0 & 2 & 2 & 0 & 0 & \ldots \\ 1 & 3 & 3 & 1 & 3 & 6 & 3 & 0 & \ldots \\ 1 & 4 & 6 & 4 & 5 & 12 & 12 & 4 & \ldots \\ 1 & 5 & 10 & 10 & 10 & 21 & 30 & 20 & \ldots \\ 1 & 6 & 15 & 20 & 21 & 36 & 61 & 60 & \ldots \\ 1 & 7 & 21 & 35 & 42 & 63 & 112 & 141 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

This is an array of the type considered in (2) above, where $f = 1 + x + x^4 + x^9 + x^{16} + \ldots$. Hence by Theorem 1, the determinant of every upper-left $n \times n$ section of this infinite matrix is 1. The same will be true if "squares" is replaced by "cubes" or any higher power, or for that matter by any increasing sequence $\{0, 1, \ldots\}$ at all!

2. Take $f = 1 + x$ in Theorem 2 to discover that

$$\det\left(\binom{zi}{j}\right)_{i,j=0}^{n} = z^{\binom{n+1}{2}} \qquad (n = 0, 1, 2, \ldots).$$

3. With $f = (e^x - 1)/x$ in Theorem 2 we find a determinant that involves the Stirling numbers of the second kind,

$$\det\left(\frac{(zi)!}{(zi+j)!}\left\{ {zi+j \atop zi} \right\}\right)_{i,j=0}^{n} = \left(\frac{z}{2}\right)^{n(n+1)/2} \qquad (n = 0, 1, 2, \ldots).$$

4. With $f = \log(1 + x)/x$ in Theorem 2 we evaluate one that contains the Stirling numbers of the first kind,

$$\det\left(\frac{(zi)!}{(zi+j)!}\left[{zi+j \atop zi} \right]\right)_{i,j=0}^{n} = \left(\frac{z}{2}\right)^{n(n+1)/2} \qquad (n = 0, 1, 2, \ldots).$$

5. The preceding two examples generalize as follows. In an exponential family, let $c(n, k)$ be the number of objects of order n that have exactly k components. Then

$$\det\left(\frac{i!}{(i+j)!}c(i+j, i)\right)_{i,j=0}^{n} = \left(\frac{c(2,1)}{2}\right)^{n(n+1)/2} \qquad (n = 0, 1, 2, \ldots).$$

6. Now let $f = (1 - \sqrt{1 - 4x})/(2x) = 1 + x + \ldots$ in Theorem 2. If we use the fact that

$$\left(\frac{1 - \sqrt{1 - 4x}}{2x}\right)^k = \sum_{n \geq 0} \frac{k(2n + k - 1)!}{n!(k + n)!} x^n,$$

then the resulting determinantal identity can be put in the form

$$\det\left(\frac{(2j + zk - 1)!}{(zk + j)!}\right)_{j,k=1}^{n} = z^{\binom{n}{2}}\, 1!\, 2!\, 3! \ldots (n - 1)! \qquad (n = 1, 2, \ldots).$$

3 The additive structure of this case

This case of Mina's theorem, in which we evaluate at $x = 0$, has an interesting structure because, if the series involved are normalized to constant term 1, then the matrices are all triangularized by the same matrix b of (4).

Usually, determinants don't relate well to addition of matrices. That is, if $\det U$, $\det V$ can be evaluated in simple closed form, there is no reason to suppose that the same is true of $\det(U+V)$. But the determinants that we are now studying act nicely under matrix addition. Thus, if $u_{i,j} = [x^i]f^j$, and $v_{i,j} = [x^i]g^j$, then $b(u+v)$ is triangular, with diagonal entries $f'(0)^i + g'(0)^i$ ($i = 0, 1, 2, \ldots$), and so

$$\det(u_{i,j} + v_{i,j})_{i,j=0}^n = \prod_{i=0}^n (f'(0)^i + g'(0)^i).$$

For example,

$$\det\left(\binom{ri}{j} + \binom{si}{j}\right)_{i,j=0}^n = \prod_{i=0}^n (r^i + s^i),$$

and more generally if S is any set of numbers, and c is a function on S, then

$$\det\left(\sum_{r\in S} c(r)\binom{ri}{j}\right)_{i,j=0}^n = \prod_{j=0}^n \left(\sum_{r\in S} c(r) r^j\right).$$

An interesting special case is obtained by taking S to be a set of m equally spaced points in an interval, say $(0,1)$, taking the $c(r)$'s all equal to $1/m$, and taking the limit as $m \to \infty$. The result is that

$$\det\left(\int_0^1 f(x)\binom{xi}{j}dx\right)_{i,j=0}^n = \prod_{j=0}^n \mu_j(f),$$

where the $\mu_i(f) = \int_0^1 x^i f(x)dx$ are the moments of f. For example,

$$\det\left(\int_0^1 \binom{xi}{j}dx\right)_{i,j=0}^n = \frac{1}{(n+1)!}.$$

Let's state this additive structure as the following theorem.

Theorem 3 *Let $\{f_i(x)\}_{i=0}^m$ be formal power series, and suppose that $f_i(0) = 1$ for all i. Then we have*

$$\det\left\{\left([x^j]\left\{K_1 f_1(x)^i + K_2 f_2(x)^i + \ldots + K_m f_m(x)^i\right\}\right)_{i,j=0}^n\right\}$$
$$= \prod_{i=0}^n \left\{K_1 f_1'(0)^i + K_2 f_2'(0)^i + \ldots + K_m f_m'(0)^i\right\}.$$

4 Proof of Mina's theorem

Now we prove Mina's theorem by proving the following small generalization of it.

$$\det\left\{\left(\frac{d^j}{dx^j}f(x)^{x_i}\right)_{i,j=0}^{n}\right\} = f(x)^{\sum_0^n (x_i - i)}\left(f'(x)\right)^{n(n+1)/2}\prod_{0\le i<j\le n}(x_j - x_i). \tag{7}$$

This will follow easily from the following observation.

Lemma 1 *Let* $\{p_j(x)\}$ $(j = 0, 1, 2, ...)$ *be any sequence of polynomials such that the* j*th one is of degree* j*, for each* $j \ge 0$*. Then the determinant of the matrix* $Q = \{p_j(x_i)\}_{i,j=0}^n$ *is equal to the product of the highest coefficients of* $p_0, p_1, \ldots, p_n$ *times the discriminant of the* x_i*'s.*

Indeed, if A is the lower triangular matrix of coefficients of the p_j's and V is the Vandermonde matrix of the x_i's then $Q = AV$. □

Now let $\alpha_j = D^j(f(x)^z)$, where $D = d/dx$. We claim that, *for each* $j = 0, 1, \ldots,$ α_j *is of the form* f^{z-j} *times a polynomial in* z *of exact degree* j*, whose highest coefficient is* $(f')^j$. In view of Lemma 1, this claim will prove the theorem. However this claim is trivial to prove by induction on j. □

Theorem 2 is the special case of (7) in which each $x_i = iz$ $(i = 1, \ldots, n)$, and $x = 0$.

5 Remarks

1. The family of matrices of the form (2) was previously investigated by Doron Zeilberger [4], from the point of view of constant term identities.

2. My thanks go to Brendan McKay and Dennis Stanton for some e-mail exchanges that were helpful in clarifying the ideas here.

3. By comparing (1) and (7) we see that the absence of $f(x)$ on the right side of the former is because of the "lucky coincidence" that in the former, $x_0 + x_1 + \ldots + x_n = n(n+1)/2$.

References

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