The variance of the Stirling cycle numbers

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Abstract

We show that the probability that two permutations of n letters have the same number of cycles is

$$\sim \frac{1}{2\sqrt{\pi \log n}}.$$

Our purpose here is to prove the following

Theorem 1 *Let two permutations of n letters be chosen independently uniformly at random. The probability that they have the same number of cycles is*

$$\sim \frac{1}{2\sqrt{\pi \log n}} \qquad (n \to \infty).$$

This question was raised by Miklós Bóna.

The Stirling cycle numbers $\left[{n \atop k}\right]$ are defined by

$$\prod_{j=0}^{n-1}(x+j) = \sum_{k=1}^{n} \left[{n \atop k}\right] x^k, \tag{1}$$

and, as is well known, $\left[{n \atop k}\right]$ is the number of permutations of n letters that have exactly k cycles. It follows that

$$\sum_k \left[{n \atop k}\right] = n!,$$

of course, but what can be said about $f(n) = \sum_k \left[{n \atop k}\right]^2$?

Since for any polynomial $g(z) = \sum_{k=0}^{n-1} a_k z^k$ with real or complex coefficients we have

$$\sum_{k=0}^{n-1} |a_k|^2 = \frac{1}{2\pi} \int_0^{2\pi} |g(e^{i\theta})|^2 d\theta,$$

we have in particular that

$$\begin{aligned} f(n) &= \sum_k \left[{n \atop k}\right]^2 \\ &= \frac{1}{2\pi} \int_0^{2\pi} \left| \prod_{j=0}^{n-1} (e^{i\theta} + j) \right|^2 d\theta \\ &= \frac{(n-1)!^2}{2\pi} \int_0^{2\pi} \left| \prod_{j=0}^{n-1} \left(1 + \frac{e^{i\theta}}{j}\right) \right|^2 d\theta \\ &= \frac{(n-1)!^2}{2\pi} \int_0^{2\pi} e^{2H_{n-1}\cos\theta} \left| \prod_{j=0}^{n-1} \left(1 + \frac{e^{i\theta}}{j}\right) e^{-e^{i\theta}/j} \right|^2 d\theta, \end{aligned}$$

where $H_{n-1} = \sum_{m=1}^{n-1} 1/m$ is the $(n-1)$st harmonic number.

Now the familiar Gamma function of analysis is given by

$$\frac{e^{-\gamma z}}{z\Gamma(z)} = \prod_{r=1}^{\infty} \left(1 + \frac{z}{r}\right) e^{-z/r},$$

and consequently

$$\begin{aligned} f(n) &= \frac{(n-1)!^2}{2\pi} \int_0^{2\pi} e^{2H_{n-1}\cos\theta} \left\{ \left| \frac{e^{-\gamma e^{i\theta}}}{e^{i\theta}\Gamma(e^{i\theta})} \right|^2 + o(1) \right\} d\theta \\ &= (1 + o(1)) \frac{(n-1)!^2}{2\pi} \int_0^{2\pi} \frac{e^{2\cos\theta \log n}}{|\Gamma(e^{i\theta})|^2} d\theta \\ &= (1 + o(1)) \frac{n!^2}{2\pi} \int_0^{2\pi} \frac{e^{2(\cos\theta - 1)\log n}}{|\Gamma(e^{i\theta})|^2} d\theta \end{aligned} \tag{2}$$

The method of Laplace for integrals (e.g., [1]) now shows that the behavior of our integral

$$I(n) = \int_0^{2\pi} \frac{e^{2(\cos\theta - 1)\log n}}{|\Gamma(e^{i\theta})|^2} d\theta$$

for large n is given by

$$I(n) \sim \sqrt{\frac{\pi}{\log n}}.$$

The final result is that

$$f(n) \sim \frac{n!^2}{2\sqrt{\pi \log n}}. \tag{3}$$

which completes the proof of the theorem.

References

[1] N.G. de Bruijn, Asymptotic Methods in Analysis, North-Holland, 1958.