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A Unified Setting for Sequencing, Ranking, and Selection Algorithms for Combinatorial Objects

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1. INTRODUCTION

The recurrent construction of combinatorial objects proceeds in steps at each of which we modify the partially constructed object by certain elementary operations such as adjunction of a new element, relabeling, etc. The end result of this sequence of decisions is then available, and this subset, permutation, tree, etc. is called the output combinatorial object.

It is, of course, just a record of the totality of decisions which were made during its construction, and if it seems profitable to do so, we may regard that sequence of decisions as being itself the combinatorial object, whatever the final presentation may be. Then any natural setting which we construct for dealing with the sequence of decisions itself will ultimately result in procedures for processing the objects. The theory of directed graphs offers such a setting, and from it we are able to present algorithms which deal simultaneously with many families of objects.

The prototype of this situation is perhaps the binomial identity

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}. \tag{1}$$

As is well known, the terms on the right count those k-subsets of an n-set which do (resp., do not) contain the element n. To construct such a k-subset, then, is repeatedly to decide "include m" or "do not include m" ($m = n, n-1, \ldots, 1$), the first choice being made k times.

Consider a walk on the lattice points (n', k') such that $n' \geqslant k' \geqslant 0$ which begins at (n, k), and at each stage proceeds from (n', k') to either $(n'-1, k'-1)$ or to $(n'-1, k')$, and halts when it reaches the origin. By following such a walk, at each step making the decision indicated by the step, at the end we have constructed a particular k-subset of $\{1, 2, \ldots, n\}$. As the walk varies over all such

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walks, the subset will vary over all $\binom{n}{k}$ of the subsets. We might as well say that the walk *is* the subset, and that in presenting it in familiar form, such as {2, 5, 8, 9, 13}, we are merely making it socially acceptable.

In the same way we find that the objects in many well-known combinatorial families can be identified with the totality of certain walks on certain graphs, the initial point of the walk being thought of as the "order" of the object, and the walk itself as a recurrent recipe for its construction in familiar form. In this setting we describe algorithms for sequencing, ranking, and selection of objects, which therefore apply at once to k-subsets of an n-set, permutations of n letters with k cycles, k "runs," partitions of an n-set into k classes, partitions of an integer n into k parts, vector subspaces of dimension k of n-space over a finite field, and so forth.

In the next section we present the general framework in which the analysis takes place, and then a special case which accounts for an important subclass. Then we give the precise manner in which the six examples above may be obtained as special cases of the theory. We also give the general algorithms which, simultaneously for all objects considered,

(a) proceed from a given object to its immediate lexicographic successor in the set of objects of its order,

(b) compute the rank of a given object among those of its order,

(c) given the rank and order, construct the desired object,

(d) select, uniformly at random, an object of given order.

2. Combinatorial Families

Let G be a directed, nonempty, connected graph (multiple edges allowed) on a (usually infinite) set $V = V(G)$ of vertices, called "orders." We suppose that for each order $v \in V(G)$, the invalence $\rho_i(v)$ and its out-valence $\rho_0(v)$ are finite. Further suppose that for every vertex v there is no infinite path which begins at v, so that, in particular, G is acyclic. Finally, suppose that at each order v, the outgoing edges from v are consecutively numbered $0,..., \rho_0(v) - 1$, and that G contains a unique terminal vertex $\tau (\rho_0(\tau) = 0)$.

By a *combinatorial object of order* v we mean a path from v to the terminal vertex of G. Let $b(v)$ denote the number of objects in the family which are of order v. We have, evidently, the general recurrence relation

$$b(v) = \begin{cases} \sum_{w} g_{vw} b(w) & v \in V(G) - \tau \\ 1 & v = \tau, \end{cases} \tag{2}$$

where g_{vw} is the number of outgoing edges from v to w.

An object of order v can be thought of concretely as its edge sequence

$$(e_1, e_2, \ldots, e_m), \tag{3}$$

in which e_1 is the number of the outbound edge from v, etc. Given the order v and the sequence of edges (*codeword*) (3) we can unambiguously follow the walk. In many important special cases we see that an object can be traced backward from its codeword and terminal vertex, without listing the vertex sequence also.

We lexicographically arrange the objects of a given order v by their codewords. The first object of order v has a codeword which is just a string of 0's, since as we arrive at each vertex w we exit from it via the first edge, numbered 0. The following algorithm is basically a standard backtracking routine.

ALGORITHM NEXT [Given an object ω of order v. Output its immediate lexicographic succesor]:

From the terminal vertex, back up along ω until reaching, for the first time, an edge e which is not the last outbound edge from its initial vertex. If no such edge exists, the input object ω was the "last," and the algorithm terminates. Otherwise, set $e \leftarrow e + 1$, and then complete the new walk from the final vertex of the new e by choosing, at each step, edge 0, until the terminal vertex is reached. ▌

Now for any edge e, let init(e), fin(e) denote, respectively, the initial and final vertices of e. Suppose we have available, from (2) or otherwise, a table of values of $b(v')$, for each $v' \overset{\alpha}{=} v$, in the partial order defined by G. Then define the edge function

$$f(e) = \sum_{\substack{\text{init}(e')=\text{init}(e) \\ e'<e}} b(\text{fin}(e')) \qquad (f(0) = 0), \tag{4}$$

on each edge e of G.

The meaning of $f(e)$ is that it is the change in the rank of an object ω' of order fin(e) if we extend it to an object of order init(e) by adjoining e. In other words, it is the number of objects of lower rank which we ignore by choosing the edge e for the extension. This function has the following properties:

(A) The rank of any object ω of order v is

$$r = \sum_{e \in \omega} f(e) \qquad (0 \leqslant r \leqslant b(v) - 1). \tag{5}$$

(B) Every integer r, $0 \leqslant r \leqslant b(v) - 1$, is uniquely representable in the form (5) for some path of order v (this generalizes a familiar property of binomial coefficients).

ALGORITHM RANK [Given an object of order v. Output its rank r in the lexicographically arranged list of all objects of order v]:

Set $r \leftarrow 0$. Walk forward on the path, on each edge e augmenting r by $f(e)$. Halt at the terminal vertex. ∎

ALGORITHM UNRANK [Given an order v and an integer r, $0 \leqslant r \leqslant b(v) - 1$. Output the object ω of rank r among those of order v]:

Begin at v with $r' = r$. Having arrived at w with r', exit along the highest e for which $f(e) < r'$ and set $r' \leftarrow r' - f(e)$. Continue from fin(e) with r'. Halt at the terminal vertex with 0. ∎

A number of particular ranking algorithms in the literature are special cases of the above.

We give two algorithms for uniformly random selection of an object of order v.

ALGORITHM RANDOM 1. Choose an integer r at random, $0 \leqslant r \leqslant b(v) - 1$. Set $\omega \leftarrow$ UNRANK (r). ∎

In the graph G there is a natural partial ordering of "reachability" on the vertices: $v \overset{\propto}{=} w$ if there is a path from w to v. Suppose that G can be *topologically sorted* with respect to this ordering, i.e., that the vertices of G can be numbered 1, 2, 3,..., so that $v \overset{\propto}{=} w \rightarrow$ number $(v) \leqslant$ number (w). Then we have the following algorithm, which does not use the topological sorting numbers at all, except in its proof of validity.

ALGORITHM RANDOM 2. Begin at v. Having arrived at w, choose the next vertex w' in the walk according to the probabilities

$$\mathrm{Prob}(w') = g_{ww'} b(w')/b(w) \qquad (w' \in G).$$

Choose an edge $w \rightarrow w'$ at random. Continue from w'. Halt at τ. ∎

To prove this, let $P(m)$ be the proposition that the vertex v whose number is m has all objects of its order chosen with equal probability $1/b(v)$. If $P(1)$, $P(2)$,..., $P(m-1)$, let ω be an object of order v (number $(v) = m$) which consists of a single step to v' followed by an object ω' of order v'. Then

$$\begin{aligned}\mathrm{Prob}(w) &= \mathrm{Prob}(v') \cdot \mathrm{Prob}(\text{edge}) \cdot \mathrm{Prob}(\omega') \\ &= \frac{g_{vv'} b(v')}{b(v)} \cdot \frac{1}{g_{vv'}} \cdot \frac{1}{b(v')} \\ &= \frac{1}{b(v)},\end{aligned}$$

as required.

3. Binomial Families

A special type of vertex set V and graph G holds many applications. We refer to these as families of binomial type.

Let ϕ, ψ be two nonnegative integer valued functions, defined on the lattice points (μ, ν) of the plane. A point (μ, ν) will belong to the vertex set of our graph G if

$$\phi(\mu + 1, \nu) + \psi(\mu + 1, \nu + 1) > 0. \tag{6}$$

From vertex (μ, ν) of V there go exactly $\phi(\mu, \nu)$ edges, numbered 0, 1,..., $\phi - 1$ to the point $(\mu - 1, \nu)$ and exactly $\psi(\mu, \nu)$ edges, numbered $\phi, \phi + 1$,..., $\phi + \psi - 1$ to $(\mu - 1, \nu - 1)$. We suppose that the vertex set V consists of *all* lattice points in a region $\mu \geqslant \nu \geqslant \nu^* \geqslant 0$, and no other vertices except possibly the terminal vertex τ. Finally we assume that

$$\phi(\mu, \nu) \uparrow_\nu \text{ on its interval of positivity.} \tag{7}$$

A typical graph G is shown in Fig. 1.

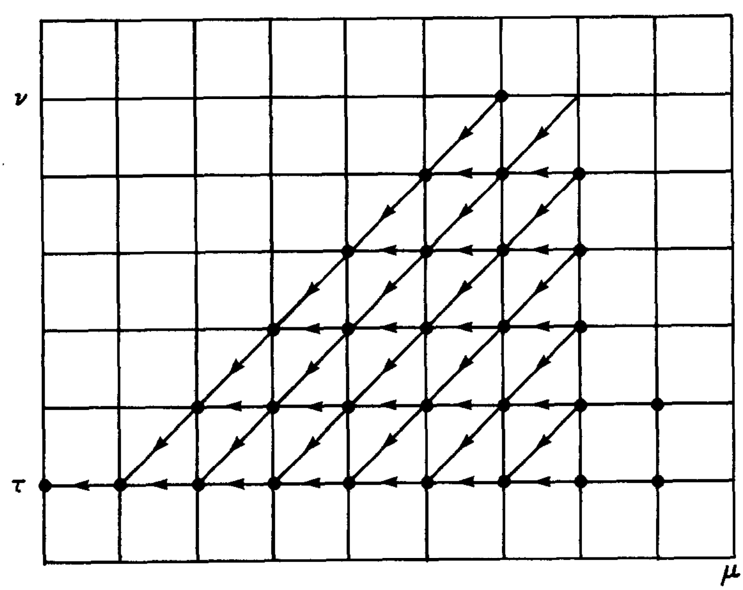

Figure 1

We shall find that on this kind of graph, the combinatorial objects of order (n, k) are among the various familiar families mentioned in the introduction. Before we turn to these examples, observe that a walk on such a grid can be traced backward given only its codeword, its terminal vertex, and the index h of the last horizontal edge e_h in the path.

Indeed, consider a point $(\mu, \nu) \in V$, a vertex in the walk, and let e be the (number of the) incoming edge at (μ, ν). If

$$0 \leqslant e \leqslant \phi(\mu + 1, \nu) - 1, \tag{8}$$

then we know that the previous vertex was $(\mu + 1, \nu)$, whereas if

$$\phi(\mu + 1, \nu + 1) \leqslant e \leqslant \phi(\mu + 1, \nu + 1) + \psi(\mu + 1, \nu + 1) - 1,$$

the predecessor was $(\mu + 1, \nu + 1)$. These two intervals are disjoint precisely when

$$\phi(\mu + 1, \nu) \leqslant \phi(\mu + 1, \nu + 1), \tag{9}$$

which is guaranteed by the monotonicity condition (7) except in one instance, when $\mu = \nu$, i.e., (μ, ν) is on the outside diagonal, and the right side of (9) vanishes. With the aid of the index h, however, we can climb up this first diagonal until we leave it, after which (9) holds.

4. Examples

(1) Subsets: Take

$$\begin{aligned} \phi(\mu, \nu) &= 1 \quad \text{if} \quad \mu \geqslant \nu + 1, \quad \nu \geqslant 0 \\ &= 0 \quad \text{else} \end{aligned} \tag{10a}$$

$$\begin{aligned} \psi(\mu, \nu) &= 1 \quad \text{if} \quad \mu \geqslant \nu \geqslant 1 \\ &= 0 \quad \text{else.} \end{aligned} \tag{10b}$$

The objects of order (n, k) in the family can be identified with the k-subsets of $\{1,..., n\}$ by following the walk and adjoining j on every diagonal edge e_j.

(2) Partitions of sets: Now choose

$$\begin{aligned} \phi(\mu, \nu) &= \nu \quad \text{if} \quad \mu \geqslant \nu + 1, \quad \nu \geqslant 1 \\ &= 0 \quad \text{else} \end{aligned} \tag{11a}$$

$$\begin{aligned} \psi(\mu, \nu) &= 1 \quad \text{if} \quad \mu \geqslant \nu \geqslant 2 \\ &= 1 \quad \text{if} \quad \mu = \nu = 1 \\ &= 0 \quad \text{else.} \end{aligned} \tag{11b}$$

This family can be identified with set partitions. An object of order (n, k), i.e., a walk from (n, k) to $(0, 0)$, is a partition with k classes.

To construct the set partition in familiar form from an object ω of order

(n, k), we can begin at $(0, 0)$ with the empty partition and trace the path backwards. A step to (μ, ν) from $(\mu - 1, \nu)$ along edge j means "insert μ into the jth class of the partition so far constructed," while an edge from $(\mu - 1, \nu - 1)$ means "adjoin μ as a singleton class to the partition." For instance, the object $(3, 0, 2, 0, 0)$ of order (5.3) is the partition $(124)(3)(5)$ in familiar form.

If $S(n, k)$ is the number of objects of order (n, k), the identity (2) is the familiar recurrence relation for the Stirling numbers of the second kind.

(3) Permutations with given cycles: Let

$$\begin{aligned} \phi(\mu, \nu) &= \mu - 1 \qquad \text{if} \quad \mu \geqslant \nu + 1, \quad \nu \geqslant 0 \\ &= 0 \qquad\qquad \text{else}, \end{aligned} \tag{12}$$

with ψ as in (11b).

This family can be identified with permutations, in fact, an object of order (n, k) "is" a permutation of n letters with k cycles. If an object of order (n, k) is given then we construct the corresponding permutation, in cycle form, as follows. Begin at the terminal vertex of ω with the empty permutation. If we encounter an edge

$$(\mu - 1, \nu - 1) \xleftarrow{\mu-1} (\mu, \nu)$$

as we trace ω backward, then we adjoin the letter μ as a singleton cycle (fixed point) to the permutation so far constructed.

Some standardization is necessary to describe the horizontal edges. Suppose we write each cycle with its smallest letter first, and order the cycles in ascending order of their smallest elements.

If, at a certain stage, $\mu - 1$ elements have already been inserted into our permutation, think of these as being arranged as beads around a number of necklaces, with $\mu - 1$ spaces between consecutive beads. An edge

$$(\mu - 1, \nu) \xleftarrow{j} (\mu, \nu) \qquad (0 \leqslant j \leqslant \mu - 2)$$

asks us to insert the letter μ into the jth one of these spaces. For example, the object of order $(5, 3)$ with codeword $(2, 3, 0, 0, 0)$ describes the permutation $(13)(25)(4)$, in familiar cycle form.

A more elegant decoding algorithm follows the path forward from (n, k), starting from the codeword $e_1, e_2, \ldots, e_n$ and an initially blank array $a_1, \ldots, a_n$. At a generic jth step of the walk, if the number of cycles not yet started is equal to the number of letters remaining in the codeword, fill all remaining blanks in the array a with $-1, -2, -3, \ldots$, and exit. Otherwise, if $e_j = n - j$, insert $-(n + 1 - j)$ into the $(1 + e_j)$th blank space, while if $e_j < n - j$, put $n + 1 - j$ into the $(2 + e_j)$th blank space. At termination, the array $a_1, \ldots, a_n$ holds the familiar cycle form of the desired permutation, and the leftmost element of each cycle is flagged with a negative sign.

Insertion into the jth blank space in a linear array can be accomplished in $O(\log j)$ operations by use of standard binary tree techniques.

(4) Vector subspaces over a finite field: Now take

$$\begin{aligned} \phi(\mu, \nu) &= q^{\nu} \quad \text{if} \quad \mu \geq \nu + 1, \quad \nu \geq 0 \\ &= 0 \quad \text{else} \end{aligned} \tag{14a}$$

$$\begin{aligned} \psi(\mu, \nu) &= 1 \quad \text{if} \quad \mu \geq \nu \geq 1 \\ &= 0 \quad \text{else.} \end{aligned} \tag{14b}$$

A walk from (n, k) to $(0, 0)$ "is" a vector subspace V_k of dimension k of n-space V_n over a groundfield of q elements. The number of such subspaces is given by the Gaussian coefficient

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q^n - 1)(q^{n-1} - 1) \cdots (q^{n-k+1} - 1)}{(q^k - 1)(q^{k-1} - 1) \cdots (q - 1)}, \tag{15}$$

and their recurrence

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q + q^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_q, \tag{16}$$

is the present case of the general recurrence (2). There are q^k edges from (n, k) to $(n - 1, k)$, numbered $0, 1, \ldots, q^k - 1$, and there is 1 edge from (n, k) to $(n - 1, k - 1)$, numbered q^k. The combinatorial interpretation of (16) appears in [1]. Briefly, a subspace is given by the $k \times n$ matrix of its basis vectors in reduced echelon form. An edge in G of the form

$$(n - 1, k) \overset{j}{\leftarrow} (n, k) \qquad (0 \leq j \leq q^k - 1) \tag{17}$$

is an instruction to adjoin to the basis matrix so far constructed a new column of k elements chosen from the field, namely, the jth such k-tuple, whereas an edge

$$(n - 1, k - 1) \overset{q^k}{\longleftarrow} (n, k), \tag{18}$$

tells us to border the basis matrix by a column from the identity matrix and a row of zeros (see [1] for details). As usual, then, a walk ω is a complete blueprint for the familiar form of the object.

(5) Permutations with given runs: Here we take the functions

$$\begin{aligned} \phi(\mu, \nu) &= \nu \quad && \text{if} \quad \mu \geq \nu + 1, \quad \nu \geq 1 \\ &= 1 && \text{if} \quad \mu = \nu = 1 \\ &= 0 && \text{else} \end{aligned} \tag{19a}$$

$$\begin{aligned} \psi(\mu, \nu) &= \mu - \nu + 1 \quad && \text{if} \quad \mu \geq \nu, \quad \nu \geq 1 \\ &= 0 && \text{else.} \end{aligned} \tag{19b}$$

If $\langle {n \atop k} \rangle$ is the number of objects of order (n, k) in the family, then the $\langle {n \atop k} \rangle$ are the Eulerian numbers, and (2) is their recurrence relation.

A run is a permutation is a maximal ascending consecutive subsequence. An object ω of order (n, k) is a permutation of n letters with exactly k runs, according to the following recipe: Begin at the point (0, 1) and walk backward. Interpret an edge

$$(\mu - 1, \nu) \xleftarrow{j} (\mu, \nu) \qquad (0 \leqslant j \leqslant \nu - 1)$$

as an instruction to insert μ at the end of the jth existing run, whereas an edge

$$(\mu - 1, \nu - 1) \xleftarrow{j} (\mu, \nu)$$

calls for insertion of μ into the jth of the places which are interior to existing runs.

(6) Partitions of integers

If $p(n, k)$ is the number of partitions of n whose largest part is k, it is well known that (see, e.g. [2, p. 70])

$$p(n, k) = p(n - 1, k - 1) + p(n - k, k). \tag{20}$$

The first term on the right represents those partitions of n whose largest part is k and whose second largest part is less than k, since such partitions can be obtained from one of $n - 1$ whose largest part is $k - 1$ by adding 1 to the largest part. The partitions of n whose largest two parts are both k come from partitions of $n - k$ of largest part k by replicating the largest part.

A partition, then, is a series of decisions "add 1 to the largest part" or "adjoin another copy of the largest part." The vertex set of the graph is the set $\mu \geqslant \nu \geqslant 0$. There are $\phi(n, k)$ edges from (n, k) to $(n - k, k)$ and $\psi(n, k)$ edges from (n, k) to $(n - 1, k - 1)$ where

$$\begin{aligned} \phi(n, k) &= 1 \qquad \text{if} \quad n \geqslant 2k \geqslant 0 \\ &= 0 \qquad \text{else} \end{aligned} \tag{21a}$$

$$\begin{aligned} \psi(n, k) &= 1 \qquad \text{if} \quad n \geqslant k \geqslant 2 \\ &= 1 \qquad \text{if} \quad n = k = 1 \\ &= 0 \qquad \text{else.} \end{aligned} \tag{21b}$$

This is not quite a binomial family, but can be easily accommodated by an algorithm whose only inputs would be the functions ϕ and ψ, which would rank, unrank, sequence, and randomly select from, all of the above families.

The form which the general ranking algorithm takes in the case of partitions of an integer is quite interesting. Let

$$n = \mu_1 d_1 + \cdots + \mu_l d_l$$

be a partition of n whose distinct parts are $k = d_1 > d_2 > \cdots > d_l$. The rank of this partition in the list of all partitions of n whose largest part is k is

$$r = p(n_0, k) + p(n_1, k - 1) + p(n_2, k - 2) + \cdots +$$

in which the first $d_1 - d_2$ of the n_i are all equal to $n - \mu_1 d_1$, the next $d_2 - d_3$ of the n_i are all equal to $n - \mu_1 d_1 - \mu_2 d_2$, etc.

For instance, the rank of

$$\pi\colon 17 = 5 + 5 + 2 + 2 + 2 + 1$$

is

$$\begin{aligned} r(\pi) &= p(7, 5) + p(7, 4) + p(7, 3) + p(1, 2) \\ &= 9 \end{aligned}$$

in the list of partitions of 17 whose largest part is 5.

6. Young Tableaux

We conclude with an example where the vertex set is not of binomial type but is instead the set of all partitions of integers. There is an edge $\pi' \to \pi''$ if π'' is obtained from π' by deleting a corner dot from its Ferrers diagram. A walk from π to the empty partition is a Young tableau of shape (= order) π. To recover the familiar form, at a step $\pi' \to \pi''$ of the walk ω, suppose that π' is a partition of m. Then insert m into the space in the shape π which corresponds to the dot which is deleted on this step. Thus the walk

$$3322 \to 3222 \to 3221 \to 3211 \to 2211 \to 2111 \to 211 \to 111 \to 11 \to 1 \to \phi$$

is the tableau

1	4	7
2	6	10
3	8	
5	9	

Note added in proof. Part II of this paper will appear in the Proceedings of the Conference on Combinatorial Algorithms, held at Qualicum Beach, B.C., 1976, North-Holland Press.

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