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THE POSSIBILITY OF TSCHEBYCHEFF QUADRATURE ON INFINITE INTERVALS

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Introduction.—Let $d\psi(x)$ be a positive measure, of total mass one, on the interval (a, b) of the real axis, and let N be a given positive integer. We say that Tschebycheff quadrature is possible for N if there are numbers $x_1, x_2, \ldots, x_N$ lying interior to (a, b) such that

$$\int_a^b f(x)d\psi(x) = 1/\mathrm{N} \sum_{\nu=1}^{N} f(x_\nu) \tag{1}$$

for every polynomial $f(x)$ of degree $\leqq N$. If Tschebycheff quadrature is possible on a sequence $\{n_j\}_1^\infty$ of values of N tending to infinity, we will say that the measure $d\psi(x)$ has property T on (a, b), and, in any event, the sequence $\{n_j\}$, finite or infinite, of values of N for which Tschebycheff quadrature is possible will be called the T-sequence for $d\psi(x)$.

Example 1: $d\psi(x) = \pi^{-1}(1 - x^2)^{-1/2}dx$ has property T on $(-1, 1)$ since, in this case, (1) is identical with Gauss-Jacobi quadrature. The T-sequence is simply $n_j = j$ $(j = 1, 2, \ldots)$.

Example 2: $d\psi(x) = dx$ does not have property T on $(0, 1)$. This was proved by S. Bernstein,[1] who later[2] showed that the T-sequence in this case is $\{1, 2, 3, 4, 5, 6, 7, 9\}$.

Example 3: For $d\psi(x) = [\Gamma(\alpha)]^{-1}x^{\alpha-1}e^{-x}dx$ on $(0, \infty)$, nothing is known except[3] that the T-sequence contains $N = 1, 2$ and does not contain N for $3 \leqq N \leqq 10$, if $\alpha = 1$.

Example 4: For $d\psi(x) = \pi^{-1/2}e^{-x^2}dx$ on $(-\infty, \infty)$, nothing is known except that the T-sequence contains $N = 1, 2, 3$ and does not contain N for $3 < N \leqq 10$.

The results of the present paper are first a simple necessary condition for a measure to have property T, second, an application of this condition to show that the T-sequence of a measure on an infinite interval is, roughly, very "sparse," third, an application of the same condition to settle the cases of Examples 3 and 4, showing in each case that property T is not present and exactly determining the T-sequences involved, and finally, some remarks about Bernstein's method.

It is the author's conjecture that if a measure has property T then it has zero mass outside of some finite interval, but as will be seen, our methods are not quite strong enough to prove this.

Jensen's Inequality.—Jensen's inequality[4] asserts that if $\xi_1, \xi_2, \ldots, \xi_n$ are non-negative, and if we define

$$\sigma_r = \left\{\sum_{\nu=1}^{n} \xi_\nu^r\right\}^{1/r} \qquad (r = 1, 2, \ldots)$$

then $s < t$ implies $\sigma_s \geqq \sigma_t$. Taking $f(x) = x^r$ in (1),

$$\mu_r = \int_a^b x^r d\psi(x) = 1/N \sum_{\nu=1}^{N} x_\nu^r \qquad (r = 1, 2, \ldots, N). \tag{2}$$

Suppose $a \geqq 0$. If Tschebycheff quadrature is possible for N, then $x_\nu \geqq a \geqq 0$ $(\nu = 1, \ldots, N)$, and therefore

$$\sigma_r = (N\mu_r)^{1/r} \qquad (r = 1, 2, \ldots, N) \tag{3}$$

is decreasing. Even if $a < 0$, however, we can apply Jensen's inequality to

$$\tau_r = \left\{\sum_{\nu=1}^{n} (\xi_\nu^2)^r\right\}^{1/r} \tag{4}$$

and deduce that the sequence

$$\tau_r = (N\mu_{2r})^{1/r} \qquad (r = 1, 2, \ldots, [N/2]) \tag{5}$$

decreases.

THEOREM 1. *Let* $\{n_j\}_1^\infty$ *be the T-sequence of a measure with moments* $\mu_r (r = 0, 1, \ldots)$. *Then for each fixed* $j (j = 1, 2, \ldots)$ *the sequence*

$$\tau_r = (n_j\mu_{2r})^{1/r} \qquad (r = 1, 2, \ldots, [n_j/2]) \tag{6}$$

decreases. If $a \geqq 0$, *then actually,*

$$\sigma_r = (n_j\mu_r)^{1/r} \qquad (r = 1, 2, \ldots, n_j) \tag{7}$$

decreases.

We now investigate the density of T-sequences. Let $\{n_j\}_1^\infty$ be the T-sequence of a measure $d\psi(x)$ on $(-\infty, \infty)$. Let $\{2p_j\}_1^\infty$, $\{2q_j + 1\}_1^\infty$ be, respectively, the subsequences of even and odd integers in $\{n_j\}$. Then, for each j, the sequence

$$\tau_r = (2p_j\mu_{2r})^{1/r} \qquad (r = 1, 2, \ldots, p_j)$$

decreases. Hence, in particular,

$$\tau_{p_{j-1}} \geqq \tau_{p_j}$$

or $(2p_j\mu_{2p_{j-1}})^{1/p_{j-1}} \geqq (2p_j\mu_{2p_j})^{1/p_j}$.

Temporarily writing $\lambda_j = \mu_{2p_j}$, we have

$$\lambda_j \leqq (2p_j)^{(p_j/p_{j-1}-1)}\lambda_{j-1}^{p_j/p_{j-1}}. \tag{8}$$

It is easy to see recursively that

$$\lambda_j \leqq \gamma_j(\lambda_1)^{p_j/p_1} \tag{9}$$

where the γ_j satisfy

$$\left.\begin{aligned} \gamma_{k+1} &= (2p_{k+1})^{(p_{k+1}/p_k - 1)}\gamma_k^{p_{k+1}/p_k} \\ \gamma_1 &= 1 \qquad\qquad (k = 2, 3, \ldots) \end{aligned}\right\} \tag{10}$$

Now the solution of (10) is

$$\log \gamma_k^{1/p_k} = \sum_{j=2}^{k} \left(\frac{1}{p_{j-1}} - \frac{1}{p_j}\right) \log (2p_j)$$

$$= \sum_{j=2}^{k} \left(\frac{1}{p_{j-1}} - \frac{1}{p_j}\right) \log p_j + 0(1) \qquad (k \to \infty). \tag{11}$$

Suppose

$$\sum_{j=2}^{\infty} \left(\frac{1}{p_{j-1}} - \frac{1}{p_j}\right) \log p_j = \beta < \infty. \tag{12}$$

Then, $\log \gamma_k^{1/p_k} \leqq \beta$,
and (9) becomes

$$\lambda_j \leqq (e^{\beta}\lambda_1^{1/p_1})^{p_j} \qquad (j = 2, 3, \ldots) \tag{13}$$

On the other hand,

$$\begin{aligned} \lambda_j = \mu_{2p_j} &= \int_{-\infty}^{\infty} x^{2p_j} d\psi(x) \\ &= \left\{\int_{-\infty}^{-X} + \int_{-X}^{X} + \int_{X}^{\infty}\right\} x^{2p_j} d\psi(x) \\ &\geqq \left\{\int_{-\infty}^{-X} + \int_{X}^{\infty}\right\} x^{2p_j} d\psi(x) \\ &\geqq X^{2p_j}\left\{\int_{-\infty}^{-X} + \int_{X}^{\infty}\right\} d\psi(x) \\ &= X^{2p_j} F(X). \end{aligned} \tag{14}$$

Thus,

$$F(X) \leqq \left(\frac{e^{\beta}\lambda_1^{1/p_1}}{X^2}\right)^{p_j} \qquad (j = 2, 3, \ldots)$$

and therefore $d\psi(x)$ has zero mass outside of the finite interval

$$|x| \leqq e^{\beta/2}\lambda_1^{1/2p_1} \tag{15}$$

Returning to the series (12), we have first

$$\left(\frac{1}{p_{j-1}} - \frac{1}{p_j}\right) \log p_{j-1} \leqq \int_{1/p_j}^{1/p_{j-1}} \log \frac{1}{x}\, dx$$

and therefore, for any sequence p_j we have

$$\sum_{j=2}^{x} \left(\frac{1}{p_{j-1}} - \frac{1}{p_j}\right) \log p_{j-1} \leqq \int_{0}^{1/p_1} \log \frac{1}{x}\, dx < \infty. \tag{16}$$

Suppose there is an integer m such that

$$p_j \leqq p_{j-1}^{m} \qquad (j = 2, 3, \ldots) \tag{17}$$

Then,

$$\sum_{j=2}^{k}\left(\frac{1}{p_{j-1}}-\frac{1}{p_j}\right)\log p_j = \sum_{j=2}^{k}\left(\frac{1}{p_{j-1}}-\frac{1}{p_j}\right)\log p_{j-1} + \sum_{j=2}^{k}\left(\frac{1}{p_{j-1}}-\frac{1}{p_j}\right)\log\frac{p_j}{p_{j-1}} \leqq 0(1) + (m-1)\sum_{j=2}^{k}\left(\frac{1}{p_{j-1}}-\frac{1}{p_j}\right)\log p_{j-1} = 0(1)\ (k\to\infty)$$

and the series converges. The example

$$p_j \geqq e^{p_{j-1}} \qquad (j = 2, 3, \ldots),$$

for which (12) is easily seen to diverge, shows that some restriction of the type (17) is essential. The same argument can be applied to the odd subsequence of $\{n_j\}$ with the identical result. We summarize with

THEOREM 2. *Let $\{2p_j\}_1^\infty$, $\{2q_j + 1\}_1^\infty$ be the even and odd subsequences of the T-sequence $\{n_j\}_1^\infty$ of a measure $d\psi(x)$ which has positive mass outside of every finite interval. Then the assertions*

$$p_j \leqq p_{j-1}^m \qquad (j = 2, 3, \ldots) \tag{17}$$

$$q_j \leqq q_{j-1}^m \qquad (j = 2, 3, \ldots) \tag{18}$$

are false for every fixed integer m. If $d\psi(x)$ has zero mass on the negative real axis then actually

$$n_j \leqq n_{j-1}^m \qquad (j = 2, 3, \ldots) \tag{19}$$

is false for every fixed integer m.

The Classical Measures.—With Theorem 1 we can now easily settle the classical cases of the Laguerre and Hermite measures. Indeed, in the Laguerre case,

$$\mu_r = [\Gamma(\alpha)]^{-1}\int_0^\infty x^{r+\alpha-1}e^{-x}dx = \frac{\Gamma(\alpha+r)}{\Gamma(\alpha)};$$

hence, by (3), the sequence

$$\sigma_r = \left[\frac{N\Gamma(\alpha+r)}{\Gamma(\alpha)}\right]^{1/r} \qquad (r = 1, 2, \ldots, N)$$

must decrease. But

$$\frac{\sigma_{N-1}}{\sigma_N} = \left\{\frac{N\Gamma(\alpha+N)}{\Gamma(\alpha)(\alpha+N-1)^N}\right\}^{1/N(N-1)}$$

which is less than unity as soon as

$$\frac{\Gamma(\alpha)}{N} > \frac{\Gamma(\alpha+N-1)}{(\alpha+N-1)^{N-1}} \sim AN^\alpha e^{-N} \qquad (N\to\infty). \tag{20}$$

If $\alpha = 1$, (20) holds as soon as $N \geqq 3$.

In the Hermite case, the analogue of (20) is

$$1.3.\ldots(2m-3) < \frac{(2m-1)^{m-1}}{2m}. \tag{21}$$

if $N = 2m$, which holds for $m \geqq 3$, or $N \geqq 6$. For N odd, we find $N \geqq 7$ by the same method, and using the result of reference 3, conclude that Tschebycheff quadrature is impossible for $N \geqq 4$.

THEOREM 3. *The measures of Laguerre and Hermite do not have property T. If* $\alpha = 1$ *in the former case, the T-sequence is* $\{1, 2\}$, *in the latter case, the T-sequence is* $\{1, 2, 3\}$.

Remarks on Bernstein's Method.—If $d\psi(x)$, (a, b) are given, let $\{\phi_n(x)\}_0^\infty$ be the sequence of orthogonal polynomials thereby generated. If $\xi_{1n} < \xi_{2n} < \ldots < \xi_{nn}$ are the zeros of $\phi_n(x)$ and $\lambda_{n1}, \lambda_{n2}, \ldots, \lambda_{nn}$, the associated Christoffel numbers for Gauss-Jacobi quadrature, Bernstein's criterion,[2] in the present terminology, is that

$$\lambda_{n1} = o(n^{-1}) \qquad (n \to \infty) \tag{22}$$

implies the absence of property T. We remark that Bernstein gives the proof for $d\psi(x) = dx$ on $(0, 1)$, but his argument is perfectly general. If this result is used in conjunction with the estimates of Winston[5] for λ_{n1}, one can show that Hermite's measure does not have property T, and that Laguerre's, for $\alpha > 1$ only, does not have property T. No information is obtained about the exact nature of the T-sequences because of the imprecision of known estimates for λ_{n1}. I would therefore regard the methods of the preceding sections and those of Bernstein as being complementary in that the usefulness of the former is restricted to infinite intervals and that of the latter to finite intervals, apparently.

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2 Bernstein, S., "Sur les formules de quadrature de Cotes et de Tschebycheff," Comptes Rendus de L'Académie des Sciences de L'URSS, **14,** 323–326 (1937).

3 Salzer, H., "Equally Weighted Quadrature Formulas over Semi-infinite and Infinite Intervals," *Jour. Math. and Phys.*, **34,** 54–63 (1955).

4 Hardy, Littlewood, and Pólya, *Inequalities* (Cambridge University Press, 2d ed., 1952).

5 Winston, C., "On Mechanical Quadratures Formulae Involving the Classical Orthogonal Polynomials," *Ann. Math.,* **35** (3), 658–677 (1934).