THEOREM 4. *For most functions* $f \in \mathfrak{M}$, $f' = 0$ *a.e.*

Analogous phenomena happen for first-order Lipschitz maps. We consider those functions $f \in \mathcal{C}([0,1])$ for which

$$\alpha \leqslant \frac{f(y)-f(x)}{y-x} \leqslant \beta$$

for all pairs of distinct points x,y in $[0,1]$, α and β being fixed real numbers ($\alpha < \beta$). Let $\mathcal{V}_{\alpha,\beta}$ be the family of all such functions; obviously $\mathcal{V}_{\alpha,\beta} \subset \mathcal{V}(\max\{|\alpha|,|\beta|\})$. $\mathcal{V}_{\alpha,\beta}$ is of the second category. We get analogously:

THEOREM 5. *For most functions* $f \in \mathcal{V}_{\alpha,\beta}$, *we have, at each point* $x \in (0,1]$,

$$f_i^-(x) = \alpha \quad \text{or} \quad f_s^-(x) = \beta,$$

and, at each point $x \in [0,1)$,

$$f_i^+(x) = \alpha \quad \text{or} \quad f_s^+(x) = \beta.$$

THEOREM 6. *For most functions* $f \in \mathcal{V}_{\alpha,\beta}$, *the set*

$$f'^{-1}(\alpha) \cup f'^{-1}(\beta)$$

has measure 1.

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AN ALGORITHM-INSPIRED PROOF OF THE SPECTRAL THEOREM IN E^n

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THEOREM. *If A is a real symmetric matrix, there is a real orthogonal matrix Q such that* Q^TAQ *is diagonal.*

Of course, this is the spectral theorem. It implies that the eigenvalues are real, that there is a pairwise orthogonal complete set of eigenvectors—namely, the columns of Q—and that the dimension of an eigenspace is equal to the algebraic multiplicity of the eigenvalue.

Many proofs grapple with the question of finding enough independent eigenvectors for a multiple eigenvalue, usually one at a time. We shall find the whole matrix Q at once by using the main idea of Jacobi's numerical method for calculating the eigenvalues and vectors, together with a little compactness.

For an $n \times n$ real matrix A we shall use $\mathrm{Od}(A)$ for the sum of the squares of the off-diagonal elements of A, and $O(n)$ will denote the set (group) of $n \times n$ orthogonal matrices.

Suppose we can prove the following.

LEMMA. *If A is a nondiagonal real symmetric matrix, then there is a real orthogonal matrix J such that* $\mathrm{Od}(J^TAJ) < \mathrm{Od}(A)$.

Then the theorem would follow quickly, for let A be real and symmetric. Consider the mapping f that sends an orthogonal matrix P into $f(P) = P^TAP$. For fixed A this is a continuous mapping of $O(n)$, a compact set, and so $f(O(n))$ is compact. Let $f(Q) = D$ be a point at which the continuous function Od attains its minimum value on the image set of f. This value must be zero, else D could play the role of A in the above lemma, and so it would not minimize Od, concluding the proof.

It remains to prove the lemma. This was done by Jacobi, in his celebrated method of plane rotations for computing eigenvalues and eigenvectors, as follows: Suppose $a_{pq} \neq 0$, $p \neq q$. Then take for J the matrix that agrees with the $n \times n$ identity matrix except in the four positions $(p,p),(p,q);(q,p),(q,q)$, where the entries are $\cos\theta$, $\sin\theta$, $-\sin\theta, \cos\theta$, and the real angle θ is chosen to make $(J^TAJ)_{pq} = 0$. It is easy to check that $\mathrm{Od}(J^TAJ) = \mathrm{Od}(A) - 2a_{pq}^2$, and we are finished.

The proof readily generalizes to the complex Hermitian case.

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UNSOLVED PROBLEMS

EDITED BY RICHARD GUY

In this department the MONTHLY *presents easily stated unsolved problems dealing with notions ordinarily encountered in undergraduate mathematics. Each problem should be accompanied by relevant references (if any are known to the author) and by a brief description of known partial results. Manuscripts should be sent to Richard Guy, Department of Mathematics and Statistics, The University of Calgary, Calgary, Alberta, Canada, T2N 1N4.*

ARE π, e, AND $\sqrt{2}$ EQUALLY DIFFICULT TO COMPUTE?

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The general problem is to find efficient methods for calculating fundamental mathematical constants. The simplest of methods is chosen, namely, iterative sequences expressed using the operations of $+$, $-$, $\times$, $\div$, and $\sqrt{\ }$. Whereas iterative sequences exist which show that the time to compute n digits of $\sqrt{2}$ and π are essentially the same, $O(n \cdot \log n \cdot \log\log n)$, no equally efficient method of computing e is known.

The efficient computation of $\sqrt{2}$ and π compared to that of e depends on two factors: rate of convergence and the complexity of multiplication. The iterative sequences converge quadratically, i.e., the number of significant digits doubles with each iteration. Consequently the time to compute n digits of the constant is essentially the time to perform the last iteration. Since this involves multiplications, the Schönhage-Strassen algorithm is used which multiplies two n digit numbers in $O(n \cdot \log n \cdot \log\log n)$ time [**1**, Section 7.5]. (Note that division or extraction of square roots also has this complexity.) If either the convergence is linear or the classical $O(n^2)$ multiplication algorithm is used, then the time to compute n digits is at least $O(n^2)$, which theoretically is as bad as summing the appropriate series.

Specifically, Newton's sequence $x_i = \frac{1}{2}(x_{i-1} + 2/x_{i-1})$ or the more convenient $x_i = x_{i-1}(6 - x_{i-1}^2)/4$ converge quadratically to $\sqrt{2}$. These methods have essentially the same complexity as that based on continued fractions [**2**]. Vieta's formula for π,

$$\frac{2}{\pi} = \sqrt{\tfrac{1}{2}} \cdot \sqrt{\left(\tfrac{1}{2} + \tfrac{1}{2}\sqrt{\tfrac{1}{2}}\right)} \cdot \sqrt{\left(\tfrac{1}{2} + \tfrac{1}{2}\sqrt{\left(\tfrac{1}{2} + \tfrac{1}{2}\sqrt{\tfrac{1}{2}}\right)}\right)} \cdots,$$

can be defined by the two iterative sequences

$$x_i = \sqrt{\left(\tfrac{1}{2} + \tfrac{1}{2}x_{i-1}\right)}; \qquad y_i = y_{i-1}x_i, \qquad x_1 = 0, \quad y_1 = 1,$$