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Sieve Equivalence in Generalized Partition Theory

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1. INTRODUCTION

The sieve method (principle of inclusion–exclusion) is one of the most powerful tools of combinatorics. Given a set Ω of objects and a list $\mathscr{L}$ of properties of those objects, we determine the number $N(\mathscr{L}; \supseteq S)$ of objects in Ω whose set of properties contains (those in $\mathscr{L}$ that are indexed by) the set S. We are then able to find the number of objects that have no properties, exactly k properties, etc.

Now consider two lists $\mathscr{A}, \mathscr{B}$ of properties of objects. We will say that the lists are *sieve-equivalent*, written $\mathscr{A} \sim \mathscr{B}$, if for every set S we have

$$N(\mathscr{A}; \supseteq S) = N(\mathscr{B}; \supseteq S). \tag{1}$$

Clearly, if $\mathscr{A} \sim \mathscr{B}$, then all questions that the sieve method can answer will have the same answer relative to list $\mathscr{A}$ and to list $\mathscr{B}$.

A form of this idea appears first in [8]. Daniel Cohen [1, 2] has deduced many identities for the classical partition function in that way. The following argument is perhaps typical: If Ω is the set of partitions of n, let

$$\mathscr{A} = \{1+1, 2+2, 3+3, 4+4, \ldots\},$$
$$\mathscr{B} = \{\ \ 2\ \ ,\ \ 4\ \ ,\ \ 6\ \ ,\ \ 8\ \ , \ldots\}.$$

Since (1) obviously holds, we have proved

THEOREM 1 (Euler). *The number of partitions of n with distinct parts is equal to the number with no even part.*

The work of Cohen was done in the context of certain kinds of multisets. Remmel [3] extended these results by weakening the conditions of [1, 2],

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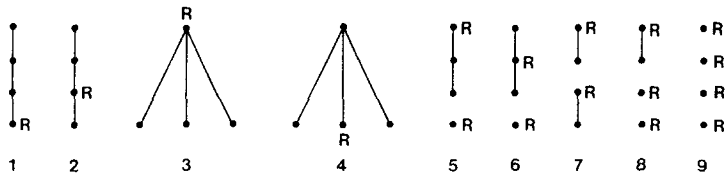


FIGURE 1

and by giving explicit bijections between the multisets with no properties, the latter by means of the *involution principle* of Garsia–Milne [4].

Our purposes in this paper are

(a) to identify the concept of sieve equivalence in the context of the sieve method itself, rather than being limited to multisets;

(b) to prove results (Theorems 2 and 3) that follow from sieve equivalence in the enlarged context referred to in (a);

(c) to discuss unique factorization of rooted trees and unlabeled graphs (Section 3) and thereby to give an explicit bijective proof of Theorem 2 in addition to the one implied by the general involution principle; and

(d) to find, in Section 5, necessary and sufficient conditions for sieve equivalence in *prefabs*.

The following result is an application of the generalized notion of sieve equivalence.

THEOREM 2. *The number of rooted forests of n vertices whose trees are all different* (*distinct parts*) *is equal to the number of such forests with no even tree* (*odd parts*).

A tree is *even* if it can be obtained by taking two copies of some rooted tree, joining the roots by a new edge, and rooting the resulting tree at one of the previous roots.

There are, for instance, nine rooted forests of four vertices (see Fig. 1).

Of these, forests 1, 2, 3, 4, 5, 6 have trees that are all distinct, and forests 1, 3, 4, 5, 6, 9 have no even tree.

2. BIJECTIONS WITH THE GENERAL SIEVE

We begin by recalling the involution principle, in the form given in [3]. Let A, B be two finite sets, each partitioned into subsets $A+$, $A-$, $B+$, $B-$ of *positive* and *negative* elements. Let $\alpha: A \leftrightarrow A$ be a sign-reversing bijection:

for each $a \in A$, either α fixes a, or $\alpha(a)$ and a are of opposite sign. Similarly, we have $\beta: B \leftrightarrow B$.

Further suppose that the fixed set F_α of α lies in $A+$, and $F_\beta \subseteq B+$. Finally, we let $f: A \leftrightarrow B$ be a sign-preserving bijection of A and B.

Then we can construct a bijection of F_α and F_β: for $\omega \in F_\alpha$ we carry out the maps

$$\omega \xrightarrow{f \circ \alpha} \omega' \xrightarrow{f^{-1} \circ \beta} \omega'' \xrightarrow{f \circ \alpha} \omega''' \xrightarrow{f^{-1} \circ \beta} \cdots \tag{2}$$

until for the first time we encounter an object $\varphi(\omega) \in F_\beta$. It is shown, e.g., in [3], that φ is well defined and bijective.

Now we want to use this technique to obtain a bijective mapping between the sets of objects that have no properties on list $\mathscr{A}$ and those with no properties on list $\mathscr{B}$, where $\mathscr{B}$ and $\mathscr{A}$ are given sieve-equivalent lists. Let $H(S; \mathscr{A})$ (*resp.* $H(S; \mathscr{B})$) denote the set of objects whose properties on list $\mathscr{A}$ (*resp.* list $\mathscr{B}$) include the set (indexed by) S. Thus $|H(S; \mathscr{A})| = N(\mathscr{A}; \supseteq S)$, etc.

By sieve equivalence (1), we can assume that there is a family of bijections

$$f_S: H(S; \mathscr{A}) \leftrightarrow H(S; \mathscr{B}), \tag{3}$$

one for each subset S of positive integers.

Now we might apply the construction used by Remmel to this general situation. Let A denote the set of all ordered pairs (ω, S), where $\omega \in H(S; \mathscr{A})$, and similarly

$$B = \{(\omega, S) \mid \omega \in H(S; \mathscr{B})\}.$$

Next, the elements of $A+$ are those (ω, S) for which $|S|$ is even, and similarly for $B+$.

For $\omega \in \Omega$, let $S_{\mathscr{A}}(\omega)$, $S_{\mathscr{B}}(\omega)$ be the sets of properties of ω on lists $\mathscr{A}$, $\mathscr{B}$ respectively, let a_ω be the largest (index of a) property that ω has, on list $\mathscr{A}$ and similarly b_ω on list $\mathscr{B}$. Then the sign-reversing bijection of A is

$$\begin{aligned} \alpha(\omega, S) &= (\omega, S - \{a_\omega\}), && \text{if} \quad a_\omega \in S, \quad S_{\mathscr{A}}(\omega) \neq \varnothing, \\ &= (\omega, S \cup \{a_\omega\}), && \text{if} \quad a_\omega \notin S, \quad S_{\mathscr{A}}(\omega) \neq \varnothing, \\ &= (\omega, \varnothing), && \text{if} \quad S_{\mathscr{A}}(\omega) = \varnothing, \end{aligned} \tag{4}$$

and similarly β is defined for B.

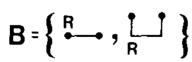
Finally, the sign-preserving bijection f of A and B is a mosaic of the sieve mappings (3), namely,

$$f(\omega, S) = (f_S(\omega), S) \tag{5}$$

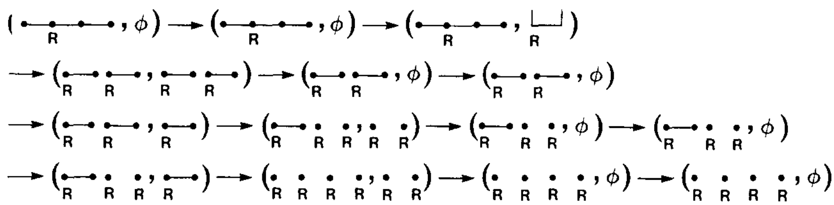
The machinery of the involution principle is now in place, and it follows that (2) yields a bijection of the sets $\{\omega \mid S_{\mathcal{A}}(\omega) = \varnothing\}$ and $\{\omega \mid S_{\mathcal{B}}(\omega) = \varnothing\}$. With small modifications, we would obtain bijections between the sets $\{\omega \mid S_{\mathcal{A}}(\omega) = T\}$ and $\{\omega \mid S_{\mathcal{B}}(\omega) = T\}$, for given T, or between $\{\omega \mid |S_{\mathcal{A}}(\omega)| = k\}$, $\{\omega \mid |S_{\mathcal{B}}(\omega)| = k\}$ for given $k \geqslant 0$, etc.

We illustrate the bijection in the case of Theorem 2 above, where we will obtain the image of forest number (2) of Fig. 1 under the mapping. Here, since $n = 4$, we can take the list $\mathcal{A}$ to be the two forests (*repeated parts*)

and the list $\mathcal{B}$ to be the two trees (*even parts*)



We begin with the ordered pair consisting of forest number (2) and the empty set of properties from list $\mathcal{A}$. Then the complete chain of mappings that leads to the ordered pair consisting of the image of forest (2) and the empty set of properties from list $\mathcal{B}$ is as follows:



We see, therefore, that the image of forest (2) is forest (9) of Fig. 1. A much quicker proof of this fact is given below.

We note that the mapping that we have described above, may not, in general, be canonical, i.e., the result may depend on the order in which the properties are listed. In the cases of partitions and of prefabs, this does not happen, and the image is independent of order. For a different way of producing the final mapping that is always canonical, see the paper of Basil Gordon, "Sieve-Equivalence and Explicit Bijections," later in this issue.

3. How to Double a Tree or a Graph

The bijection that we gave in the previous section was a mapping between forests of distinct trees and forests of odd trees. Now we will describe the mapping in simpler language that directly generalizes the classical *binary expansion* proof of Theorem 1.

What we need is to generalize the assertion that every *integer* is uniquely a power of two times an odd number. We now give such a generalization, to *rooted graphs*.

If G is any connected rooted graph, then by $2 * G$ we mean: make two copies of G, join them at their roots by a new edge, and root the result at one of the former roots. A rooted graph is *odd* if it is not of the form $2 * G$.

LEMMA 1. *Every connected rooted graph H is uniquely of the form $2^k * H'$, where H' is an odd rooted graph.*

Proof. By induction on the number of vertices, it will be enough to show that if $H = 2 * H'$, then the representation is unique. Indeed, suppose that removal of an edge (R, a) leaves two isomorphic copies of a rooted graph H'. If we had removed a different edge (R, b), then we would have found one whole copy of H' and the edges (R, b), (R, a) in the same connected component, hence the components could not have been of equal sizes. ∎

Direct Proof of Theorem 2

We give an explicit mapping between rooted forests of odd trees and rooted forests of distinct trees. Let F be a forest of odd trees. Then F is m_1 copies of some odd tree T_1 and m_2 copies of some odd tree T_2, etc. Let

$$m_1 = 2^{a_1} + 2^{a_2} + \cdots; \qquad m_2 = 2^{b_1} + 2^{b_2} + \cdots; \cdots;$$

be the binary expansions of the m_i. Then map F into one copy of the tree $2^{a_1} * T_1$, one copy of $2^{a_2} * T_1$,..., one copy of $2^{b_1} * T_2$,... . These trees are all distinct, by Lemma 1. We leave the inversion of the mapping as an exercise for the interested reader.

It is also easy to show, along the lines of [3], that the explicit bijection given above is always identical to that given by the involution principle, in the case of Theorem 2.

For another application of the doubling method, consider an unlabelled (and unrooted) connected graph G. By $2 * G$ we mean: make two copies of G and join every vertex in one copy with every vertex in the other copy of G. An unlabelled graph is *odd* if it is not of the form $2 * G$.

The analogue of Lemma 1 is true, namely,

LEMMA 2. *Every connected unlabelled graph H is uniquely of the form $2^k * H'$, where H' is odd.*

Proof. Again it suffices to show that if $G = 2 * H$, then it is uniquely so. If false, then the complement of G can be expressed in two different ways as the union of two identical lists of connected components, which is impossible. ∎

Now that the uniqueness of the "odd part" of a graph has been established, by the same binary expansion argument that was used in the direct proof of Theorem 2, we find a bijective proof of

THEOREM 3. *The number of connected graphs of n vertices whose connected components are all different is equal to the number whose components are all odd.*

My thanks go to Dr. Nijenhuis for several discussions that were quite helpful to me.

4. SIEVE EQUIVALENCE IN PREFABS

Now we want to specialize the notion of sieve-equivalence to the setting of *prefabs* (see Bender and Goldman [5] and Foata and Schützenberger [7], with further theoretical and algorithmic developments described in [6]). It will be seen that this context is still general enough to include all of the results of partition theory referred to above, and more (such as Theorem 2 above).

A prefab is a set X of objects together with a distinguished subset P of *prime* objects. Atached to each element $x \in X$ is its order $\Omega(x)$, a positive integer. Two objects x', x'' can be *synthesized* (stitched together) into an object $x'x''$, associatively and commutatively. Further, for all x', x'' we have $\Omega(x', x'') = \Omega(x') + \Omega(x'')$. Finally, every object $x \in X$ is uniquely a synthesis of primes,

$$x = p_1^{a_1} p_2^{a_2} \cdots . \tag{6}$$

As one example, take $X =$ all partitions of integers, $P =$ the partitions $n = n$ $(n = 1, 2, \ldots)$. If $x \in X$ is a partition of n, then $\Omega(x) = n$, and to synthesize two partitions is to construct the partition of the union of their parts, e.g.,

$$(3+3+1)(2+1+1) = 3+3+2+1+1+1.$$

As another example, if $X =$ all rooted forests, $P =$ all rooted trees, define $\Omega(x) =$ number of vertices of x. To synthesize two forests is to construct the union of their sets of trees.

We can do arithmetic in a prefab. Not only can we multiply $x'x''$ but, for instance, $x' \mid x''$ if $a_i \leqslant b_i \ \forall i$, and

$$\mathrm{lcm}(x', x'') = \prod p_i^{\max(a_i, b_i)},$$

$$\gcd(x', x'') = \prod p_i^{\min(a_i, b_i)},$$

etc., if $x' = p_1^{a_1} p_2^{a_2} \cdots$ and $x'' = p_1^{b_1} p_2^{b_2} \cdots$.

Suppose now that we are given an infinite sequence of objects

$$\mathcal{A} = \{x_1', x_2', x_3', \ldots\} \tag{7}$$

from a prefab $\tilde{\mathscr{P}}$ and a nonnegative integer k. Let $G_n(\mathcal{A}; k)$ denote the set of objects in $\tilde{\mathscr{P}}$ of order n that are divisible by exactly k of the objects in the list $\mathcal{A}$.

To compute $|G_n(\mathcal{A}; k)|$ we might, of course, use the sieve method, and to begin with, we would compute for each subset S of positive integers, the number $H_n(S; \mathcal{A})$ of objects in $\tilde{\mathscr{P}}$ of order n that are divisible at least by all of the $\{x_i'\}_{i \in S}$.

We see that the two sequences $\mathcal{A}, \mathscr{B}$ of objects in $\tilde{\mathscr{P}}$ are sieve-equivalent if, for every subset S, we have $H_n(S; \mathcal{A}) = H_n(S; \mathscr{B})$ for all $n = 1, 2, 3, \ldots$.

Of course we will have $|G_n(\mathcal{A}; k)| = |G_n(\mathscr{B}; k)|$ for all $n \geqslant 1$, $k \geqslant 0$, in this case.

Now, for the sequences $\mathcal{A}$ of (7) and

$$\mathscr{B} = \{x_1'', x_2'', x_3'', \ldots\}, \tag{8}$$

under what conditions can we assert that $\mathcal{A} \sim \mathscr{B}$?

Fix a subset S. An object of order n that is divisible by all $\{x_i'\}_{i \in S}$ must be of the form

$$x = x * \mathrm{lcm}\{x_i'\}_{i \in S}. \tag{9}$$

By additivity of the order function, we have

$$n = \Omega(x) = \Omega(x^*) + \Omega(\mathrm{lcm}\{x_i'\}_{i \in S}),$$

and so x^* must be an object of order $n - \Omega(\mathrm{lcm}\{x_i'\}_{i \in S})$; indeed, any object x^* of that order will do.

It is now clear that we have

PROPOSITION 1. *If for every set S of positive integers the two elements*

$$\mathrm{lcm}\{x_i'\}_{i \in S} \quad \text{and} \quad \mathrm{lcm}\{x_i''\}_{i \in S} \tag{10}$$

are of the same order, then the sequences $\mathcal{A}$ and $\mathscr{B}$ are sieve-equivalent.

This proposition is just a lifting to prefabs of a proposition about partitions that was evolved by the authors cited above. It gives sufficient conditions for sieve-equivalence, as do the stronger hypotheses of

PROPOSITION 2. *Let $\mathscr{A}, \mathscr{B}$ as in (7), (8) be two sequences, each consisting of pairwise relatively prime objects. Then for all $i \geqslant 1$, the objects x_i' and x_i'' are of the same order iff $\mathscr{A}$ and $\mathscr{B}$ are sieve-equivalent.*

5. Conditions for Equivalence in Prefabs

Now we consider necessary and sufficient conditions for sieve equivalence of two sequences of prefab elements. Let $c(n)$ be the number of objects in $\tilde{\mathscr{P}}$ of order n, $n \geqslant 1$, with $c(0) = 1$.

If for a certain subset S we have $H_n(S; \mathscr{A}) = H_n(S; \mathscr{B})$ for all $n \geqslant 1$, then the number of objects of order n of the form

$$x^* \operatorname{lcm}\{x_i''\}_{i \in S}$$

is the same as the number of the form

$$x^{**} \operatorname{lcm}\{x_i'\}_{i \in S}.$$

Write $\sigma = \sigma(S) = \Omega(\operatorname{lcm}\{x_i'\}_{i \in S})$, and $\tau = \tau(S) = \Omega(\operatorname{lcm}\{x_i''\}_{i \in S})$, and suppose $\tau \leqslant \sigma$. Then we must have

$$c(n - \sigma) = c(n - \tau) \qquad (\text{all } n) \tag{11}$$

or

$$c(n + (\sigma - \tau)) = c(n) \qquad (\text{all } n). \tag{12}$$

Suppose $\sigma > \tau$. Then according to (12), the counting function of the prefab is periodic of period $\sigma - \tau$. On the other hand, in a prefab we always have (see, e.g., [6, p. 80])

$$\mathscr{C}(x) \stackrel{\text{def}}{=} \sum_n c(n) x^n = \prod_{\nu \geqslant 1} (1 - x^\nu)^{-d_\nu}, \tag{13}$$

where d_ν is the number of prime objects of order ν ($\nu \geqslant 1$). If $\sigma > \tau$, then

$$\mathscr{C}(x) = P(x)/(1 - x^{\sigma - \tau}), \tag{14}$$

where $P(x)$ is a polynomial of degree $< \sigma - \tau$, by periodicity of the $c(n)$'s. Then by (13) we have

$$P(x) = (1 - x^{\sigma - \tau}) \prod_{\nu \geqslant 1} (1 - x^\nu)^{-d_\nu}. \tag{15}$$

Since $P(x)$ is a polynomial, $P(x)/(1-x)$ has a pole of order 1 at $x=1$, hence the same is true of the product, which therefore consists of exactly one factor, and so must be of the form $(1-x^d)^{-1}$ for some $d \mid (\sigma-\tau)$.

Thus, if for a single subset S it is true that $H_n(S;\mathscr{A})=H_n(S;\mathscr{B})$ for all $n \geqslant 1$, and that $\sigma > \tau$, then the prefab $\tilde{\mathscr{P}}$ consists entirely of a single prime object p of order d dividing $\sigma-\tau$, and all of its powers.

Conversely, if $\tilde{\mathscr{P}}=\{p, p^2, p^3, \ldots\}$, then

$$H_n(S;\mathscr{A}) = 1, \qquad \text{if} \quad n \geqslant \max_{i \in S} i,$$
$$= 0, \qquad \text{else},$$

and $H_n(S;\mathscr{B})$ is the same.

If the H_n's agree for all n on several sets S, then this conclusion holds for each such S, and so the order of the single prime object p must divide the gcd of all of the $|\sigma-\tau|$. Let us introduce the notation

$$\sigma_S(\mathscr{A}) = \Omega(\operatorname{lcm}\{x_i'\}_{i \in S})$$

and similarly for $\mathscr{B}$. Then we have proved

Proposition 3. *For $\mathscr{A}, \mathscr{B}$ to be sieve-equivalent sequences over a prefab $\tilde{\mathscr{P}}$, it is necessary and sufficient that either*

(i) *for every subset S we have $\sigma_S(\mathscr{A})=\sigma_S(\mathscr{B})$, or else,*

(ii) *$\tilde{\mathscr{P}}$ contains just one prime object p, and its order divides* $\gcd(|\sigma_S(\mathscr{A})-\sigma_S(\mathscr{B})|)$, *where the* gcd *extends over all sets S such that $\sigma_S(\mathscr{A}) \neq \sigma_S(\mathscr{B})$.*

It is now easy to prove Theorem 2. In the prefab $\tilde{\mathscr{P}}$ of rooted forests, let the sequence $\mathscr{A}$ consist of all forests x of the form $x=p^2$, where p is a (rooted) tree, arranged in nondecreasing order of their numbers of vertices, but otherwise arbitrarily.

Let the list $\mathscr{B}$ consist of all trees T that can be obtained by taking two copies of a tree p, joining them at their roots by an edge, and rooting the result at one of the previous roots. List $\mathscr{B}$ is arranged so that T_{2p} is in the same position in $\mathscr{B}$ as p^2 is in $\mathscr{A}$. Now Proposition 2 applies, and Theorem 2 follows.

To complete the picture we need only to construct bijections in the case of prefabs. The general sieve construction discussed in Section 2 applies here, and we need only state what the family f_S of mappings in (3) is. In a prefab, let $x = p_1^{a_1} p_2^{a_2} \cdots$ be an object that has at least the set S of properties in list $\mathcal{A}$. With x we associate the object

$$f_S(x) = (x/\mathrm{lcm}\{x_i'\}_{i \in S}) \cdot \mathrm{lcm}\{x_i''\}_{i \in S}.$$

We see that $f_S(x)$ has at least the properties S in list $\mathcal{B}$, and we are finished.

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