JOURNAL OF NUMBER THEORY 4, 98–106 (1972)

Representations of Integers by Linear Forms in Nonnegative Integers*

ALBERT NIJENHUIS AND HERBERT S. WILF

Department of Mathematics, University of Pennsylvania, Philadelphia, Pennsylvania 19104

Communicated by P. T. Bateman

Received June 22, 1970

Let Ω be the set of positive integers that are omitted values of the form $f = \sum_{i=1}^{n} a_i x_i$, where the a_i are fixed and relatively prime natural numbers and the x_i are variable nonnegative integers. Set $\omega = \#\Omega$ and $\kappa = \max \Omega + 1$ (the conductor). Properties of ω and κ are studied, such as an estimate for ω (similar to one found by Brauer) and the inequality $2\omega \geqslant \kappa$. The so-called Gorenstein condition is shown to be equivalent to $2\omega = \kappa$.

1. INTRODUCTION

Let $a_1, \ldots, a_n$ be positive integers, and let

$$\begin{aligned} d_i &= \text{g.c.d.}\,(a_1, \ldots, a_i) \qquad (i = 1, \ldots, n), \\ d_0 &= 0. \end{aligned} \tag{1}$$

As $x_1, \ldots, x_n$ run independently over the nonnegative integers, the values of the form

$$f = a_1 x_1 + \cdots + a_n x_n \tag{2}$$

run over a certain set of nonnegative integers. This set of assumed values is clearly a semigroup. If $d_n = 1$, it is well known that there is an m_0 such that all $m \geqslant m_0$ are assumed by f.

The purpose of this paper is to study the following two properties of the form f:

(a) $\kappa(f)$, the *conductor* of f, is the least positive m_0 for which f assumes all values $\geqslant m_0$.

(b) $\Omega = \Omega(f)$, the set of *omitted values* of f, and in particular, $\omega(f) = \#\Omega$.

* Research supported in part by the National Science Foundation.

For example, if

$$f = 5x_1 + 3x_2,$$

we have $\kappa(f) = 8$, $\Omega = \{1, 2, 4, 7\}$, $\omega = 4$.

A classical theorem of Sylvester [1] states that, if $n = 2$, then

$$\kappa(f) = (a_1 - 1)(a_2 - 1), \tag{3}$$

$$\omega(f) = \tfrac{1}{2}(a_1 - 1)(a_2 - 1), \tag{4}$$

and so, in particular,

$$\omega(f) = \tfrac{1}{2}\kappa(f). \tag{5}$$

We give another proof of (3)–(5) in Section 2 below, to introduce the methods which will be used here for $n > 2$.

In 1942, A. Brauer investigated this problem [2], and he showed that under the condition

(I) *For each $i = 2,..., n$, the number a_i/d_i is an assumed value of the form*

$$f_{i-1} = \frac{1}{d_{i-1}}(a_1x_1 + \cdots + a_{i-1}x_{i-1}), \tag{6}$$

it follows that

(II) $$\kappa(f) = \sum_{k=1}^{n}\left(\frac{d_{k-1}}{d_k} - 1\right) a_k + 1,$$

and he showed also that the right side of (II) is always an upper bound for $\kappa(f)$.

We will show below that under the same condition (I), the formula

(III) $$\omega(f) = \frac{1}{2}\left\{\sum_{k=2}^{n}\left(\frac{d_{k-1}}{d_k} - 1\right) a_k + 1\right\}$$

holds and that the right side of (III) is always an upper bound for $\omega(f)$. Our proof yields (II) also, and considerably more, namely, that the Conditions (I), (II), (III) above are actually *equivalent*.[1]

It will follow, then, that under any of these three conditions,

(IV) $$\omega(f) = \tfrac{1}{2}\kappa(f).$$

We further investigate the relationship of these conditions to another proposition which arises in the theory of Gorenstein rings [3]. Suppose S

[1] The equivalence of (I) and (II) was shown by Brauer and Seelbinder [4].

is the set of integers m which are assumed by f and in which we have $x_n = 0$ for every representation, i.e.,

$$S = \{m \in R \mid m - a_n \notin R\}, \tag{7}$$

where R is the set of assumed values of f.

Then define a set

$$T = \{m \in S \mid (\forall i = 1,..., n)\, m + a_i \notin S\}. \tag{8}$$

The Gorenstein condition is the property

(V) $$\#T = 1.$$

We will show that (IV) and (V) are equivalent.[1] The full collection of interrelationships among our conditions will then be

$$(\mathrm{I}) \Leftrightarrow (\mathrm{II}) \Leftrightarrow (\mathrm{III}) \Rightarrow (\mathrm{IV}) \Leftrightarrow (\mathrm{V}).$$

The example $(a_1, a_2, a_3) = (6, 7, 8)$ shows that the missing implication cannot be included in general.

2. The Case $n = 2$

The following short proof of (3) and (4) is based on methods that will be used several times. Since g.c.d. $(a_1, a_2) = 1$, every integer m can be written as $m = xa_1 + ya_2$ in many ways if x and y are allowed to be negative; the representation becomes unique if we demand that $0 \leqslant x < a_2$. Then m is assumed by f if $y \geqslant 0$; m is omitted if $y < 0$. The largest omitted value is therefore obtained for $x = a_2 - 1$, $y = -1$, and $\kappa(f)$ is one unit bigger:

$$\kappa(f) = (a_2 - 1)a_1 - a_2 + 1 = (a_1 - 1)(a_2 - 1).$$

Now, let $0 \leqslant m < \kappa(f)$, and let m be represented with $0 \leqslant x < a_2$, then

$$m' = \kappa(f) - 1 - m = (a_2 - 1 - x)\, a_1 + (-1 - y)a_2\,.$$

Here $0 \leqslant a_2 - 1 - x < a_2$, so if $y \geqslant 0$ then m is representable and m' is omitted, while if $y < 0$ the roles are reversed. This shows that precisely half of the numbers $0,..., \kappa(f) - 1$ are omitted by f, so (5) holds.

[1] (*Added in proof*) This was proved independently by E. Kunz [5].

3. A Map and an Inequality

We now return to general values of n.

THEOREM 1. *Under the hypothesis* $d_n = 1$ we have

$$\omega(f) \geqslant \tfrac{1}{2}\kappa(f). \tag{9}$$

Proof. Define

$$\rho(x) = \kappa(f) - 1 - x \tag{10}$$

(this reversal map will be used several more times); so $x + \rho(x) = \kappa(f) - 1$. The right side is not assumed by f; by the definition of $\kappa(f)$, hence not both terms on the left can be assumed (semigroup property!). So, if x is represented, then $\rho(x)$ is not. The set of omitted values among $0,\ldots, \kappa(f) - 1$ contains therefore a subset of the cardinality of that of the assumed values, so at least half of the numbers $0,\ldots, \kappa(f) - 1$ are omitted by f; i.e., (9) holds.

The same argument shows: If m is an omitted value then at least half the numbers $0,\ldots, m$ are omitted.

4. The Gorenstein Condition

Let S and T be as in (7), (8).

LEMMA 1. *The set* $W = \{x \mid x - a_n \in T\}$ *is given by*

$$W = \{x \mid x \notin R, \forall_i x + a_i \in R\}; \tag{11}$$

$\kappa(f) - 1$ *belongs to* W.

Proof. A number m belongs to T if and only if it satisfies all conditions $(o) \cdots (n)$:

(o) $m \in R$ *and* $m - a_n \notin R$,

(i) $(i = 1,\ldots, n)$ $m + a_i \notin R$ *or* $m - a_n + a_i \in R$.

The first condition in (i) is never satisfied if the first in (o) is, so the former may be deleted. The first condition in (o) is the same as the second in (n), hence the former may be deleted. Hence,

$$T = \{m \mid m - a_n \notin R, \forall_i \quad m - a_n + a_i \in R\}.$$

Formula (11) is now obvious, and $\kappa(f) - 1 \in W$ as it is the largest omitted number.

THEOREM 2. *The Gorenstein condition* (V) (*equivalent to* $\#W = 1$) *is satisfied if and only if* (IV) *holds.*

Proof. Let $\#W = 1$. We show that $\rho(x)$ is an assumed value if x is omitted; so exactly half the numbers $0,..., \kappa(f) - 1$ are omitted. Let x be omitted, and let y be the largest assumed value for which $x + y$ is omitted. As $y + a_i$ is an assumed value which exceeds y, it follows that $x + y + a_i$ is assumed; so $x + y \in W$. That means $x + y = \kappa(f) - 1$, i.e., $y = \rho(x)$ is an assumed value.

Conversely, let (IV) hold, then $\rho(x)$ is assumed if and only if x is not. Let $w \in W$, then w is omitted, hence $\kappa(f) - 1 - w$ is assumed. Suppose $\kappa(f) - 1 - w > 0$, then it equals $\sum \xi_i a_i$ with at least one $\xi_i > 0$, hence there is i such that $w' = \kappa(f) - 1 - w - a_i$ is assumed. Then $\rho(w') = w + a_i$ is omitted, contrary to one of the properties of the elements w of W. Hence, $w = \kappa(f) - 1$, and that is the only element of W.

We have just seen that, if $w \in W$, $w < \kappa(f) - 1$, then $\rho(w)$ is omitted. Therefore we have

THEOREM 3. *The following inequality holds*

$$2\omega(f) - \kappa(f) \geqslant \#W - 1. \tag{12}$$

5. A COUNT OF OMITTED VALUES

To determine the set Ω of omitted values of $(a_1 ,..., a_n)$ with $d_n = 1$, it suffices to determine first the set D of omitted values of

$$(a_1/d_{n-1} ,..., a_{n-1}/d_{n-1})$$

and then study the values that are taken by the form $xd_{n-1} + ya_n$ $(x, y \geqslant 0)$. This idea is motivation for the following lemma:

LEMMA 2. *Suppose a and b are positive integers,* g.c.d. $(a, d) = 1$, *and D is a finite set of positive integers. Let D' be the set of positive integers z not of the form*

$$z = md + xa \qquad (m \geqslant 0, m \notin D, x \geqslant 0). \tag{13}$$

Furthermore, let D_a be the set

$$D_a = \left\{m \in D \mid m - ka \in D \quad \textit{for all} \quad k = 0,..., \left[\frac{m}{a}\right]\right\}.$$

Then

$$\#D' = \frac{(a-1)(d-1)}{2} + d \cdot \#D_a \leqslant \frac{(a-1)(d-1)}{2} + d \cdot \#D \quad (14)$$

and

$$\max D' = d \cdot \max D_a + (d-1)a \leqslant d \cdot \max D + (d-1)a. \quad (15)$$

In particular,

$$2\#D' - (\max D' + 1) = d(2\#D_a - (\max D_a + 1)). \quad (16)$$

If $\#D_a$ *or* $\#D = 0$ *replace* $\max D_a$ *or* $\max D$ *in* (15) *and* (16) *by* -1.

Proof. The numbers of D' are of two types:

I. Numbers z which have no representation of the form

$$z = md + xa \quad (m, x \geqslant 0); \quad (17)$$

their number is given by (4) with $a_1 = d$, $a_2 = a$.

II. Numbers z which have representations of the form (17) but for *every* such representation we have $m \in D$.

One of the representations of z of the form (17) has $x < d$; the others are then of the form

$$z = (m - ka)\, d + (x + kd)\, a \qquad k = 1,\dots, \left[\frac{m}{a}\right].$$

If z is of Type II, then $m - ka \in D$ for all $k = 0,\dots, [m/a]$; hence m must belong to D_a. For each such m there are precisely d numbers z not of the type (13), namely, for $x = 0,\dots, d-1$. Hence there are $d \cdot \#D_a$ many numbers z of Type II. This proves (14).

For (15) consider the largest element of Type I; it is $(d-1)(a-1) - 1$ by (3). The largest element of Type II is obtained from (17) by maximizing m in D_a and $x < d$; this gives $\max D_a \cdot d + (d-1)a$. The maximal element of Type II obviously exceeds that of Type I provided D_a is non-empty. Otherwise, setting $\max D_a = -1$ gives exactly the maximal element of Type I.

LEMMA 3. *If in Lemma* 2 *D is the set of omitted values of*

$$(a_1/d_{n-1}, \dots, a_{n-1}/d_{n-1}), \qquad a = a_n, \qquad d = d_{n-1},$$

and $d_n = 1$, *then* $\Omega = D'$ *is exactly the set of omitted values of* $(a_1, \dots, a_n)$. *If* a_n *is an assumed value of* $(a_1/d_{n-1}, \dots, a_{n-1}/d_{n-1})$, *then* $D_a = D$.

Proof. In the light of Lemma 2 the first statement is a precise formulation of the introductory remark of this section. The second part uses the fact that the difference $x - y$ between an omitted value $x \in D$ and an assumed value $y = a_n$ is itself always an omitted value, hence belongs to D if it is not negative.

Remark. Under the hypotheses of Lemma 3 (except that a_n need not be an assumed value) it is easy to see that the set D_a in Lemma 2 is exactly the set of omitted values of $(a_1/d_{n-1}, ..., a_{n-1}/d_{n-1}, a_n)$. By (16), $\{a_1, ..., a_n\}$ will satisfy $2\omega = \kappa$ if and only if $\{a_1/d_{n-1}, ..., a_{n-1}/d_{n-1}, a_n\}$ does. The ordering of the a's is actually irrelevant; this proves:

THEOREM 4. *Let d be the* g.c.d. *of the numbers $a_1, ..., a_n$ with the exception of a_i. Then $\{a_1, ..., a_n\}$ satisfies the Gorenstein condition if and only if $\{a_1/d, ..., a_{i-1}/d, a_i, a_{i+1}/d, ..., a_n/d\}$ satisfies the Gorenstein condition.*

For an application, consider $\{12, 13, 14\}$; we use self-explanatory notation.

$$\mathrm{Gor}(12, 13, 14) \Leftrightarrow \mathrm{Gor}(6, 13, 7) \Leftrightarrow \mathrm{Gor}(6, 7)\text{: satisfied.}$$

Another:

$$\mathrm{Gor}(6, 10, 15) \Leftrightarrow \mathrm{Gor}(2, 10, 3) \Leftrightarrow \mathrm{Gor}(1, 5, 3) \Leftrightarrow \mathrm{Gor}(1)\text{: satisfied.}$$

6. Upper Estimates for $\omega(f)$ and $\kappa(f)$.

THEOREM 5. *Let $a_1, ..., a_n$ be positive integers, $d_n = 1$, and let f be given by* (2). *Let $\omega(f)$ be the number of positive values omitted by f, and $\kappa(f)$ the conductor of f. Then*

$$\kappa(f) \leqslant 2\omega(f) \leqslant 1 + \sum_{k=1}^{n} a_k \left(\frac{d_{k-1}}{d_k} - 1\right) = \sum_{k=2}^{n} \frac{d_{k-1}}{d_k} a_k - \sum_{k=1}^{n} a_k + 1. \tag{18}$$

Equality between the second and third members implies equality throughout; this occurs if and only if condition (I) *holds.*

Proof. Let κ_k be short for $\kappa(a_1/d_k, ..., a_k/d_k)$, and similarly for ω_k. Then the inequalities (14) and (15) can be expressed as

$$2\omega_k \leqslant 2\frac{d_{k-1}}{d_k}\omega_{k-1} + \left(\frac{a_k}{d_k} - 1\right)\left(\frac{d_{k-1}}{d_k} - 1\right) \tag{19}$$

$$\kappa_k \leqslant \frac{d_{k-1}}{d_k}\kappa_{k-1} + \left(\frac{a_k}{d_k} - 1\right)\left(\frac{d_{k-1}}{d_k} - 1\right), \qquad (\kappa_1 = 0). \tag{20}$$

Equality in (19) occurs if $\omega_k^* = \omega_k$ [see (14), the starred quantity refers to $(a_1/d_{k-1}, ..., a_{k-1}/d_{k-1}, a_k/d_k)$]; then there is also equality in (20); all this happens if and only if a_k/d_k is assumed by $(a_1/d_{k-1}, ..., a_{k-1}/d_{k-1})$. Rewrite (19) as

$$2\omega_k d_k = 2\omega_{k-1}d_{k-1} + \left(\frac{a_k}{d_k} - 1\right)(d_{k-1} - d_k)$$

and sum on k; in view of $d_n = 1$ this gives

$$2\omega_n = \sum_{k=2}^{n} \left(\frac{a_k}{d_k} - 1\right)(d_{k-1} - d_k) = \sum_{k=2}^{n} \frac{a_k}{d_k}(d_{k-1} - d_k) - (d_1 - d_n)$$

$$= \sum_{k=1}^{n} a_k \left(\frac{d_{k-1}}{d_k} - 1\right) + 1$$

as required.

7. Consecutive Integers

An interesting special case is that in which

$$(a_1, ..., a_n) = (m, m+1, m+2, ..., m+n-1) \tag{21}$$

Brauer has given the formula for the conductor in this case. Indeed, the set of assumed values is clearly

$$\bigcup_{j=1}^{\infty} [jm, j(m+n-1)].$$

This yields at once

$$\kappa(f) = mJ, \tag{22}$$

$$\omega(f) = \frac{mJ}{2}\left\{\frac{(m-1)+\theta(k-1)}{m}\right\}, \tag{23}$$

$$\frac{\omega(f)}{\kappa(f)} = \frac{1}{2}\left\{\frac{(m-1)+\theta(k-1)}{m}\right\}, \tag{24}$$

where J is the least integer $\geqslant (m-1)/(k-1)$, and

$$J - 1 = \frac{m-1}{k-1} - \theta \qquad (0 < \theta \leqslant 1)$$

defines θ.

One sees in particular that $\omega/\kappa \geqslant \frac{1}{2}$ always, as required by Theorem 1, and that $\omega/\kappa = \frac{1}{2}$ if and only if $k-1$ divides $m-2$.

ACKNOWLEDGMENT

We acknowledge our gratitude to D. S. Rim for several stimulating discussions and suggested questions.

REFERENCES

1. J. SYLVESTER, Mathematical Questions, with their solutions, from the *Educational Times*, Vol. **41**, p. 21, 1884.
2. A. BRAUER, On a problem of partitions, *Amer. J. Math.* **64** (1942), 299–312.
3. D. GORENSTEIN, An arithmetic theory of adjoint plane curves, *Trans. Amer. Math. Soc.* **72** (1952), 414–436.
4. A. BRAUER AND B. M. SEELBINDER, On a problem of partitions II, *Amer. J. Math.* **76** (1954), 343–346.
5. A. KUNZ, The value-semigroup of a one-dimensional Gorenstein ring, *Proc. Amer. Math. Soc.* **25** (1970), 748–751.

Printed in Belgium