The Patterns of Permutations

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To Dan Kleitman, on his birthday, with all good wishes.

Let n, k be positive integers, with $k \leq n$, and let τ be a fixed permutation of $\{1, \ldots, k\}$.[1] We will call τ the *pattern*.

We will look for the pattern τ in permutations σ of n letters. A pattern τ is said to occur in a permutation σ if there are integers $1 \leq i_1 < i_2 < \ldots < i_k \leq n$ such that for all $1 \leq r < s \leq k$ we have $\tau(r) < \tau(s)$ if and only if $\sigma(i_r) < \sigma(i_s)$.

Example: Suppose $\tau = (132)$. Then this pattern of $k = 3$ letters occurs several times in the following permutation σ, of $n = 14$ letters (one such occurrence is underlined):

$$\sigma = (5\ 2\ 9\ \underline{4}\ 14\ \underline{10}\ 1\ 3\ 6\ 15\ \underline{8}\ 11\ 7\ 13\ 12)$$

1 Some areas of research and recent results

Among the active areas of research are the following.

1. For a given pattern τ, let $f(n, \tau)$ be the number of τ-free permutations of n letters. Describe the equivalence classes of patterns that have the same f.

2. What can be said about the asymptotics of $f(n, \tau)$ for $n \to \infty$ and fixed τ?

3. For fixed τ what is the maximum number of occurrences of τ in a permutation of n letters? Call this $g(\tau, n)$. Which permutation has the maximum?

[1] We will often refer to $\{1, 2, \ldots, k\}$ as the *letters* on which the permutation acts, however their numerical sizes will be very relevant.

2 Packing density

First, as regards the question of stuffing in as many τ's as possible, Fred Galvin (unpublished) has shown the following.

Theorem 1 (Galvin) *For fixed* $\tau \in S_k$*,*

$$\left\{ \frac{g(\tau, n)}{\binom{n}{k}} \right\}_{n=k}^{\infty}$$

is decreasing, and thus

$$\lim_{n \to \infty} \frac{g(\tau, n)}{\binom{n}{k}}$$

exists.

Galvin's proof is reproduced here with his permission: Let τ be a fixed pattern of length k. If x is any sequence of distinct numbers, of length $\geq k$, let $g(x)$ be the number of τ-subsequences of x, and let $h(x) = g(x)/\binom{|x|}{k}$. For $n \geq k$, let

$$H(n) = \max\{h(x) : |x| = n\} = \frac{f(n, \tau)}{\binom{n}{k}}.$$

Suppose $k \leq m < n$, and let x be a permutation of length n with $h(x) = H(n)$. Note that $h(x)$ is equal to the average of $h(y)$ over all m-termed subsequences y of x, and therefore cannot exceed $H(m)$. $\square$

Definition: Given $\tau \in S_k$, let

$$\rho(\tau) = \lim_{n \to \infty} \frac{g(\tau, n)}{\binom{n}{k}}.$$

The limit $\rho(\tau)$ is called the *packing density* of the pattern τ.

Galvin's theorem asserts that every permutation (pattern) has a packing density. It is easy to see that it is always positive.

3 Questions and answers ...

- One open question concerns the rate of growth of $f(n, \tau)$, the number of permutations of n letters that avoid the pattern τ. In all known cases the growth is exponential. Stanley and Wilf have conjectured that the limit

$$\lim_{n \to \infty} f(n; \tau)^{1/n}$$

exists, is finite, and is nonzero. (In all known cases it is an integer). This conjecture had been considered a "sure thing," but the results of Alon and Friedgut (see below) seem to make it somewhat less certain because a similar bound, involving the Ackermann function, in the Davenport-Schinzel theory turns out to be best possible.

- Gessel, Weinstein and Wilf [12] relate permutations that avoid long increasing subsequences to lattice walks. Can this be extended to other patterns?

- There are 6 patterns τ when $k = 3$. The (Catalan) number $f(n; \tau)$ of τ-free permutations of n letters is independent of τ, when $k = 3$. A bijective proof of this has been given by Simion and Schmidt [21].

- We can define an equivalence relation among permutations (patterns) τ of length k: Say that $\tau \sim \tau'$ if $|f(n, \tau)| = |f(n, \tau')|$ for all $n = 0, 1, 2, 3, \ldots$.

  Thus, when $k = 3$ there is just one class.

  When $k = 4$, using obvious symmetry, there are at most 6 classes. This number was reduced by West and by Stankova. Finally Stankova proved that $(1234) \sim (4123)$, which shows that there are just 3 classes when $k = 4$. Very little is known about this number when $k \geq 5$.

4 Some new results

- Noga Alon and Ehud Friedgut [1] showed the following.

  Theorem 2 *For every pattern τ, $\exists C$ such that the number of n-permutations that avoid τ is at most $C^{n\alpha(n)}$, where $\alpha(n)$ is an inverse Ackermann function (grows extremely slowly).*

  The proof converts this problem to one about Davenport-Schinzel sequences [14], about *words* that avoid patterns of equalities.

  They also proved that the $O(C^n)$ conjecture holds for patterns that consist of an increasing sequence followed by a decreasing sequence, or vice-versa.

- Noonan and Zeilberger [16] found the exact number of permutations that have exactly 1 occurrence of (132). If $f_r(n)$ is the number of n-permutations that contain exactly r occurrences of a given pattern τ, they ask if f is P-recursive in n, for each fixed r and τ. Bóna [6] answered affirmatively for $\tau = (132)$ and all r.

  Richard Stanley is inclined to favor a negative answer to N-Z's question.

5 Results of Robertson, Wilf, and Zeilberger (1999)

They found the number of permutations of n letters that have exactly r (123)'s and s (132)'s, in the form of a Maple program that will output the required generating functions, in principle, on demand.

Let the weight of a permutation π of $\nu(\pi)$ letters be $z^{\nu(\pi)}q^{|123(\pi)|}t^{|12(\pi)|}$, in which $|123(\pi)|$ is the number of patterns (123) (ascending triples) in π, and $|12(\pi)|$ is the number of rising pairs in π. Let

$$P(q,z,t) = {\sum_\pi}' \mathrm{weight}(\pi),$$

where the sum extends over all (132)-avoiding permutations π. Then they showed that

$$P(q,z,t) = 1 + zP(q,zt,tq)P(q,z,t).$$

It follows from the above functional equation that

$$P(q,z,t) = \cfrac{1}{1-\cfrac{z}{1-\cfrac{zt}{1-\cfrac{zt^2q}{1-\cfrac{zt^3q^3}{1-\cfrac{zt^4q^6}{\cdots}}}}}}$$

in which the nth numerator is $zt^nq^{\binom{n}{2}}$.

If $f_r(n)$ denotes the number of permutations of n letters that contain no pattern (132) and have exactly r (123)'s, we write $\mathrm{AR}(r,z) := \sum_n f_r(n)z^n$. Then $\mathrm{AR}(r,z)$ is the coefficient of q^r in the series development of the continued fraction $P(q,z,1)$.

That is, we have

$$\cfrac{1}{1-\cfrac{z}{1-\cfrac{z}{1-\cfrac{zq}{1-\cfrac{zq^3}{1-\cfrac{zq^6}{\cdots}}}}}} = \sum_{r\geq 0} \mathrm{AR(r,z)}q^r$$

$$= \frac{1-z}{1-2z} + \frac{z^3q}{(1-2z)^2} + \frac{(1-z)z^4q^2}{(1-2z)^3} + \frac{(1-z)^2z^5q^3}{(1-2z)^4} + \frac{z^4\left(-1+6z-13z^2+11z^3-3z^4+z^5\right)q^4}{(-1+2z)^5} + \ldots$$

We can write

$$P(q,z,t) = \frac{A(q,z,t)}{B(q,z,t)}$$

where $A(q,z,t) = B(q,zt,tq)$, and B satisfies the functional equation

$$B(q,z,t) = B(q,zt,tq) - zB(q,t^2z,t^2q)$$

$$\begin{aligned}
&= 1 + \left(-1 - t - q\,t^2 - q^3\,t^3 - q^6\,t^4 - q^{10}\,t^5\right) z \\
&\quad + \left(t^2 + q\,t^4 + q^2\,t^4 + q^3\,t^6 + q^5\,t^6 + q^6\,t^6 + q^6\,t^8 + q^9\,t^8 + q^{11}\,t^8 + \ldots\right) z^2 \\
&\quad + \left(-q^2\,t^8 - q^{10}\,t^{11} - q^5\,t^{12} - q^6\,t^{12} - q^{21}\,t^{14} - q^{17}\,t^{15} - q^{18}\,t^{15} + \ldots\right) z^3 \\
&\quad + \left(q^{10}\,t^{22} + q^{32}\,t^{26} + q^{21}\,t^{28} + q^{17}\,t^{30} + q^{18}\,t^{30} + q^{58}\,t^{30} + \ldots\right) z^4 \\
&\quad + \left(-q^{32}\,t^{52} - q^{84}\,t^{57} - q^{58}\,t^{60} - q^{49}\,t^{64} - q^{47}\,t^{68} - q^{48}\,t^{68} + \ldots\right) z^5 \\
&\quad + (q^{84}\,t^{114} + \ldots) z^6 + \ldots
\end{aligned}$$

- Bóna [5] evaluated exactly $f(n, 1342)$, and showed that $\lim_{n\to\infty} f(n, 1342)^{1/n} = 8$. In fact he found the generating function,

$$\frac{32x}{1 + 12x - 8x^2 - (1-8x)^{3/2}}.$$

  Since the same limit for (1234) is 9, the limit depends on more than the length of the pattern.

- A pattern is called *layered* if it is a disjoint union of substrings, each substring being decreasing, with the substrings (layers) increasing from one to the next. For instance (32148765) is layered.

  Theorem 3 *(Stromquist [24]) Among all patterns of length k, the maximum possible packing density is achieved on a layered pattern.*

- Kleitman, Galvin, and Stromquist, independently, found the packing density of (132), namely $2\sqrt{3}-3$. Price [17] found the packing density of (2143) to be 3/8, as well as that of two-layered patterns.

- Alex Burstein [9] generalized these questions to *words* (1998). Given a pattern τ, how many words of n letters, over an alphabet of k letters, avoid τ? He found the answer for all patterns of ≤ 4 letters and for some longer ones.
  Note: "pattern occurs in word" here has a different meaning from the Davenport-Schinzel theory. Here the pattern is a pattern of sizes of letters, and in the latter it is a pattern of multiplicities.

- Bóna [4] showed that if τ is layered then $f(n,\tau) = O(C^n)$ for some C.

6 The identity pattern $\iota_k = (1, 2, 3, \ldots, k)$

For fixed k, the identity pattern need not be extremal. For instance, 15793 permutations of 8 letters avoid the pattern (1324), 15767 avoid the identity (1234), and 15485 avoid (1342), so the identity is neither maximal nor minimal in that respect.

Gessel [11] found the generating function $U_{k+1}(x)$ for $f(n; \iota_{k+1})$, in the form

$$\sum_{n \geq 0} f(n; \iota_{k+1}) \frac{x^{2n}}{n!^2} = \det \left(I_{|i-j|}(2x) \right)_{i,j=1}^{k},$$

where I_ν is the Bessel function.

Regev [18] found the asymptotics of this f, and in particular that

$$f(n; \iota_{k+1})^{1/n} \to k^2 \qquad (n \to \infty).$$

Permutations that avoid ι_{k+1} are closely related to Young tableaux. By Schensted's algorithm we can associate with any such permutation a pair of tableaux of the same shape, whose first row has length $\leq k$, and conversely.

If $y_k(n)$ is the number of such tableaux on n cells, and if $Y_k(x)$ is the exponential gf of $\{y_k(n)\}_{n \geq 0}$, then Wilf [27] showed that

$$U_k(x) = Y_k(x) Y_k(-x) \qquad (k = 2, 4, 6, \ldots),$$

where $U_k(x) = \sum_{n \geq 0} u_k(n) x^{2n}/n!^2$, and $u_k(n)$ is the number of permutations of n letters that have no ascending subsequence of length $> k$.

7 A listing algorithm

Schensted's correspondence yields the following algorithm for making a list of all permutations of n letters that avoid ι_{k+1}:

For each $i = 1$ to k
  For each partition π of n whose largest part is i
    For each pair $(T1, T2)$ of Young tableaux of shape π
      Output the permutation that corresponds to $(T1, T2)$

The partitions of n with largest part i can be efficiently listed, i.e., the full list can be made in time that is proportional to its length. Likewise the pairs of Young tableaux of given shape can be efficiently listed. So this algorithm will run in time that is proportional to (except for a polynomial factor) the number of permutations that are being listed, about k^{2n} of them, which is considerably faster than testing all $k!$ permutations. But –
How can we efficiently list the permutations that avoid other patterns?

8 A lower bound

Given a pattern τ, of k letters. Again, let $f(n;\tau)$ be the number of n-permutations that avoid τ. We want a lower estimate for f. The following is due to Marko Petkovšek (unpublished).

Theorem 4 *We have* $\liminf_{n\to\infty} f(n;\tau)^{1/n} \geq k-1$.

Proof. Suppose that k appears in the m-th position in the pattern τ. If π is any permutation of length $n-1$ which avoids τ, inserting n as the first, second, ..., $(m-1)$st element will not create an occurrence of τ because n could only correspond to k, but then it would need at least $m-1$ elements to its left. Likewise, it is safe to insert n as the nth, $(n-1)$st, ..., $(n-k+m+1)$st element of the new permutation of length n. So, by inserting n wherever possible in π, we will obtain from π at least $(m-1)+n-(n-k+m) = k-1$ permutations of length n which avoid τ. It follows that $f(n,\tau) \geq (k-1)f(n-1,\tau)$, whence the result.

9 Results of Richard Arratia

Richard Arratia [2] has shown that

$$\lim_{n\to\infty} f(n,\tau)^{1/n} = \sup_n f(n,\tau)^{1/n}.$$

First, wlog suppose that k precedes 1 in τ. Now superadditivity holds:

$$f(m+n,\tau) \geq f(m,\tau)f(n,\tau),$$

for if $\sigma' \in S_m$, $\sigma'' \in S_n$, are τ-free, then construct $\sigma = \sigma' \otimes \sigma''$: add m to the values of σ'' and glue them to the right of those of σ'.

Now σ is τ-free also. Indeed, τ can't occur in the σ' part or the σ'' part of σ, and if it straddles the boundary, then the smallest value of τ must occur in the σ' part and the largest value in the σ'' part, which contradicts the assumption that the biggest value in τ precedes its smallest value. $\square$

10 The poset

We can partially order the set of all permutations of all numbers of letters by declaring that $\sigma \leq \tau$ iff σ is contained as a pattern in τ. It would be interesting to study this as a poset. For example, what can be said about its Möbius function?

This suggests also the following question. Say that a collection $\mathcal{C}$ of permutations of k letters is unavoidable if every permutation of more than k letters contains at least one member of $\mathcal{C}$. Let $f(k)$ denote the minimum number of permutations in an unavoidable set of permutations of k letters. What can be said about $f(k)$? For instance, is $f(k) = o(k!)$? Michael Atkinson (unpublished) found an unavoidable set of 14 permutations of 4 letters, so $f(4) \leq 14$.

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