Pattern avoidance in compositions and multiset permutations

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1 Introduction

One of the most arresting phenomena in the theory of pattern avoidance by permutations is the fact that the number of permutations of n letters that avoid a pattern π of 3 letters is independent of π. In this note we exhibit two generalizations of this fact, to ordered partitions, a.k.a. compositions, of an integer, and to permutations of multisets. It is remarkable that the conclusions are in those cases identical to those of the original case.

Further, the number of permutations of a multiset $S = 1^{a_1}2^{a_2}\ldots k^{a_k}$ that avoid a given pattern $\pi \in S_3$ is a symmetric function of the a_i's, and we will give here a bijective proof of this fact for $\pi = (123)$.

By a composition of an integer n into k parts we mean an integer representation

$$n = x_1 + x_2 + \ldots + x_k \qquad (\forall i : x_i \geq 0)$$

where two compositions are regarded as distinct even if they differ only in their order of the summands. If in fact we have $x_i \geq 1$ for all i then we speak of a composition into positive parts. A composition is said to contain the pattern, e.g., $\pi = (132)$ if $\exists\, i_1 < i_2 < i_3$ such that $x_{i_1} < x_{i_3} < x_{i_2}$, and similarly for other patterns π. Note the strict inequalities that we use, which, in view of the repeated part sizes that can occur, are quite material, though of course variants of these problems can also be considered in which some of the inequalities might not be strict.

Theorem 1 *Among the 2^{n-1} compositions of n into positive parts, the number that avoid a given pattern π of 3 letters is independent of π.*

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These numbers, which play a role analogous to that of the Catalan numbers in the case of permutations, are, for $n = 1, 2, \ldots$,

$$1, 2, 4, 8, 16, 31, 60, 114, 214, 398, 732, 1334, 2410, \ldots.$$

Below we will find the ordinary power series generating function of the above sequence, in the form

$$f(x) = \sum_{i \geq 1} \frac{1}{1-x^i} \prod_{j \neq i} \left\{ \frac{1-x^i}{(1-x^{j-i})(1-x^i-x^j)} \right\}. \tag{1}$$

A more refined version of Theorem 1 is also true.

Theorem 2 *Among the* $\binom{n-1}{k-1}$ *compositions of* n *into* k *positive parts, the number that avoid a given pattern* π *of 3 letters is independent of* π*. Likewise, among the* $\binom{n+k-1}{n}$ *compositions of* n *into* k *nonnegative parts the same conclusion holds.*

Finally, at the root of all of the above is the following finer gradation.

Theorem 3 *Fix a multiset* S*. The number of permutations of* S *that avoid* π *is independent of the choice of* $\pi \in S_3$*.*

The above results are all easy consequences of the enumeration of the permutations of a *multiset* that avoid a pattern. This enumeration was accomplished for the pattern (132) in [1, 2] by an elegant recursive construction of the sequence of generating functions involved. Those authors found that the number of permutations of the multiset $S = 1^{a_1}2^{a_2}\ldots k^{a_k}$ that avoid the pattern (132) is the coefficient of $x_1^{a_1}\ldots x_k^{a_k}$ in the generating function

$$g_k(\mathbf{x}) = \sum_{i=1}^{k} \frac{x_i^{k-1}(1-x_i)^{k-2}}{\prod_{1 \leq j \leq k;\, j \neq i} \{(x_i - x_j)(1-x_i-x_j)\}}. \tag{2}$$

We remark that, in g_k, for fixed $i < j$, the coefficient of $1/(x_i - x_j)$ is a skew-symmetric function of x_i, x_j, which therefore contains a factor of $x_i - x_j$ to cancel those factors that seem to appear in the denominators. Hence g_k is an analytic function of $\mathbf{x}$ in a neighborhood of the origin in R^k. For example,

$$g_2(\mathbf{x}) = \frac{1}{1-x_1-x_2},$$

and

$$g_3(\mathbf{x}) = \frac{1-x_1-x_2-x_3+x_1x_2+x_1x_3+x_2x_3}{(1-x_1-x_2)(1-x_1-x_3)(1-x_2-x_3)}.$$

2 Proofs

If we assume, for a moment, that Theorem 3 has been proved then Theorems 1 and 2 follow easily. Indeed, in each case define an equivalence relation on the set of compositions involved by declaring that two compositions are equivalent if they have the same multisets of parts. Then Theorem 3 implies that the desired pattern-independence conclusion holds separately within each equivalence class, so *a fortiori* it holds on the full set of compositions being considered.

The proof of Theorem 3 is an easy extension of the results in [1] and [2]. Let $\mathbf{a} = (a_1, \ldots, a_k)$ be a given vector of k positive integers. The multiset $M(\mathbf{a})$ is the one that contains exactly a_i copies of the letter i, for each $i = 1, \ldots, k$. Define, for each pattern π, $f(\mathbf{a}, \pi)$ to be the number of permutations of the multiset $M(\mathbf{a})$ that avoid the pattern π. Then, in [1] and [2] it was shown that

1. For every $\mathbf{a}$ we have $f(\mathbf{a}, (123)) = f(\mathbf{a}, (132))$, and
2. $f(\mathbf{a}, (132))$ is a symmetric function of $(a_1, \ldots, a_k)$. This follows easily from (2) above. However it seems to be quite a remarkable fact, in that one might imagine that the number of permutations that avoid a pattern might change drastically if we swap the available supplies of large and small letters. We give a bijective proof of this symmetry for the pattern (123) in Section 4 below.

Now the reversal map shows that $f(\mathbf{a}, \pi)$ is constant on each of the pairs

$$((123), (321)), ((132), (231)), ((213), (312))$$

of patterns of length 3. Further, item 1 above shows that the first two of these three classes coincide, so it suffices to show that for all $\mathbf{a}$ we have $f(\mathbf{a}, (132)) = f(\mathbf{a}, (312))$.

The complement σ' of a permutation σ of n letters is obtained by replacing each letter i in the values of σ by $n + 1 - i$. Evidently we have $f(\mathbf{a}, \sigma) = f(\bar{\mathbf{a}}, \sigma')$, where $\bar{a}$ is the reversal of the vector $\mathbf{a}$. Thus

$$f(\mathbf{a}, (312)) = f(\bar{\mathbf{a}}, (132)).$$

Since all four patterns $(123), (321), (132), (231)$ have the same counting function, it follows from item 2 above that all four counting functions are symmetric functions of the a_i's. Thus $f(\bar{\mathbf{a}}, (132)) = f(\mathbf{a}, (132))$, completing the proof of Theorem 3.

A proof of Theorem 3 that is independent of the results of [1] and [2], relying instead on a generalization of the methods of Simion and Schmidt [6], has been given by Amy Myers [4].

3 Counting compositions that avoid a pattern

Let π be a fixed pattern of three letters. We will specialize the generating function (2) of the previous section to the enumeration of compositions of the integer n, into positive parts, that avoid the pattern π. By Theorem 1, of course, these results will be valid for all $\pi \in S_3$.

In the generating function (2), put $x_i = x^i$, for $i = 1, \ldots, k$, and consider the coefficient of x^n in $g_k(x, x^2, \ldots, x^k)$. This will be the sum of the numbers of multiset permutations that avoid π, summed over all multisets for which $a_1 + 2a_2 + \ldots + ka_k = n$, i.e., over all compositions of n into positive parts. This completes the proof of eq. (1).

Doron Zeilberger has remarked that since the generating function has infinitely many singularities, the sequence that it generates is not P-recursive. Hence pattern avoidance in compositions of an integer provides a simple and natural example of an avoidance problem whose solution sequence is not P-recursive.

If, in the generating function (1) we halt the outer sum at $i = k$ then we will be looking at the generating function for the π-avoiding compositions of n whose parts are all $\leq k$.

Now we can deal with the asymptotic growth rate of $c(n, k)$, the number of compositions of n, into positive parts $\leq k$, that avoid the pattern π. Since the generating function (1), with the upper limit of the outer i-sum replaced by k, is a rational function, what we need to do is to determine which one of the trinomial equations

$$x^i + x^j = 1 \qquad (1 \leq i < j \leq k)$$

has a root closest to the origin in the complex plane.

We claim that the "golden ratio" equation $x + x^2 = 1$ is the winner of this competition, with its root $x_0 = 2/(1+\sqrt{5})$ being the closest to the origin. Indeed, suppose x is a root of some $x^i + x^j = 1$, in which $1 \leq i < j \leq k$ and $j > 2$. If $|x| \geq 1$ then surely $|x| > x_0$ so there is nothing more to prove. If, on the other hand, $|x| < 1$ then we have

$$1 = |x^i + x^j| \leq |x|^i + |x|^j < |x| + |x|^2,$$

whence $|x| > 2/(1 + \sqrt{5})$, as claimed. It follows that if $c(n, k)$ is the number of π-avoiding compositions of n into positive parts $\leq k$, then

$$c(n, k) \sim K(k) \left(\frac{1 + \sqrt{5}}{2}\right)^n \qquad (n \to \infty). \tag{3}$$

where

$$K(k) = \frac{r}{(r-1)(r-s)} \left(r \prod_{j=3}^{k} \frac{1 - \frac{1}{r}}{(1 - r^{1-j}) \left(1 - \frac{1}{r} - r^{-j}\right)} - \prod_{j=3}^{k} \frac{1 - r^{-2}}{(1 - r^{2-j}) \,(1 - r^{-2} - r^{-j})} \right),$$

and $r = (1 + \sqrt{5})/2$, $s = (1 - \sqrt{5})/2$. Numerically, the values of $K(k)$ for $k = 5, 10, 20$, and ∞ are 9.95025, 17.9099, 18.9314, and 18.9399867..

4 A bijection to show symmetry

As in Section 2, for a given vector $\mathbf{a} = (a_1, \ldots, a_k)$ of k positive integers, let $M(\mathbf{a})$ be the multiset containing exactly a_i copies of the letter i, for each $i = 1, \ldots, k$, and let $f(\mathbf{a}, \pi)$ be the number of permutations of $M(\mathbf{a})$ that avoid the pattern π.

In this section we give an explicit bijection to show that $f(\mathbf{a}, (123))$ is a symmetric function of $(a_1, \ldots, a_k)$. For any permutation $\mathbf{b}$ of $\mathbf{a}$, we show there is a bijection

$$\Theta : S = M(\mathbf{a}) \quad \leftrightarrow T = M(\mathbf{b})$$

with the property that $x \in S$ avoids (123) if and only $\Theta(x) \in T$ avoids (123). Since any permutation of $\mathbf{a} = (a_1, a_2, \ldots, a_k)$ can be achieved by a sequence of transpositions of adjacent elements, it suffices to consider the case where $\mathbf{b}$ is obtained from $\mathbf{a}$ by exchanging a_i and a_{i+1}:

$$\mathbf{b} = (a_1, a_2, \ldots, a_{i-1}, a_{i+1}, a_i, a_{i+2}, \ldots, a_k).$$

We will do this by making use of a bijection between permutations of $i^{a_i}(i+1)^{a_{i+1}}$ and permutations of $i^{a_{i+1}}(i+1)^{a_i}$ which derives from the Greene-Kleitman symmetric chain decomposition in the Boolean lattice [3].

4.1 Description of Θ

If $a_i = a_{i+1}$, the mapping $\Theta : S \to T$ is the identity. Otherwise, if $a_i < a_{i+1}$, interchange S and T. If $a_i > a_{i+1}$, we proceed as follows, illustrating with the example:

$$S = (1^2)(2^1)(3^1)(4^5)(5^2)(6^7)(7^1) \to T = (1^2)(2^1)(3^1)(4^2)(5^5)(6^7)(7^1),$$

where T is obtained from S by interchanging $a_4 = 5$ and $a_5 = 2$.
Start with a string $x \in S$ (since $a_i > a_{i+1}$).

```
7 5 6 6 4 6 6 4 6 6 4 6 5 3 2 4 1 1 4
```

Replace i by '(' and $i + 1$ by ')'

```
7 ) 6 6 ( 6 6 ( 6 6 ( 6 ) 3 2 ( 1 1 (
```

Match parentheses in the usual way. Mark unmatched left parentheses as 'U'.

```
7 ) 6 6 U 6 6 U 6 6 ( 6 ) 3 2 U 1 1 U
```

Change the leftmost $a_i - a_{i+1}$ of the 'U's to ')'. Then change the remaining 'U's back to '('.

```
7 ) 6 6 ) 6 6 ) 6 6 ( 6 ) 3 2 ) 1 1 (
```

Change ')' back to $i + 1$ and change '(' back to i.

```
7 5 6 6 5 6 6 5 6 6 4 6 5 3 2 5 1 1 4
```

This is the string $\Theta(x) \in T$.

4.2 The Greene-Kleitman bijection

A consequence of the Greene-Kleitman symmetric chain decomposition of the Boolean lattice described in [3] is the following bijection between the t-subsets and the $(n-t)$-subsets of an n-element set.

If $t = n - t$ the bijection is the identity. If $t > n - t$, define the mapping from $(n - t)$-subsets to t-subsets, (i.e. from permutations of $0^t 1^{n-t}$ to permutations of $0^{n-t}1^t$). If $t < n - t$, define the map in the reverse direction. Assume that $t > n - t$.

For $x \in 0^t 1^{n-t}$. Regard '0' as '(' and '1' as ')' and match parentheses in the usual way. Define

$$\tau : 0^t 1^{n-t} \to 0^{t-1} 1^{n-t+1}$$

by: $\tau(x) = y$ where y is obtained from x by changing the leftmost unmatched '0' in x to '1'. So, τ can only be applied to a string with an unmatched '0'. If $t > n - t$, there are at least $t - (n - t) = 2t - n$ such.

> (***): As observed in [3], all unmatched '0's in a string are to the right of any unmatched '1's. Thus, the application of τ does not change the matching. That is, any '1' in x that was matched to a '0' is still matched in $y = \tau(x)$ to the same '0'. So, the unmatched '0's in y are the unmatched '0's of x with the leftmost removed. Also, changing the leftmost unmatched '0' to a '1' makes it the rightmost unmatched '1'.

Now if x has at least j unmatched '0's, we could apply τ at least j times. Let $\tau^j(x) = \tau(\tau(...(\tau(x))...))$ (j applications of τ.) It follows from (***) that $\tau^j(x) = y$ where y is obtained from x by changing the leftmost j unmatched '0's in x to '1's. The following is the desired bijection:

$$\tau^{2t-n} : 0^t 1^{n-t} \to 0^{n-t} 1^t.$$

To get the inverse mapping, start with a string $y \in 0^{n-t}1^t$. Then y has at least $t - (n - t) = 2t - n$ unmatched '1's. change the rightmost $2t - n$ unmatched '1's into '0's.

4.3 Proof of bijection Θ

We can rephrase the mapping $\Theta : S \to T$ described in Section 4.1 as follows:

$$\Theta = \tau^{a_i - a_{i+1}},$$

where $\tau(x) = y$ and y is obtained from x by changing the leftmost unmatched 'i' in x to '$i + 1$'. Then Θ is the Greene-Kleitman mapping of the preceding section, so Θ is a bijection. It remains to check whether Θ preserves "(123)-avoidance". We check this in steps:

$$S \to \tau(S) \to \tau(\tau(S)) \to \cdots \to \tau^{a_i - a_{i+1}}(S) = \Theta(S) = T$$

and show that in each step "(123)-avoidance" is preserved. We show that if x is a permutation with an unmatched 'i', then x avoids (123) iff $y = \tau(x)$ avoids (123).

Assume x avoids (123) and that $x_t = i$ is the leftmost unmatched i in x.

Note:

(a) If $j < t$ and $x_j = i$, then x_j must be matched in x to some $i + 1 = x_u$ with $j < u < t$.

(b) If $j > t$ and $x_j = i + 1$, then x_j must be matched in x to some $i = x_u$ with $t < u < j$.

Let $y = \tau(x)$. Then $y(t) = i + 1$ and $y(i) = x(i)$ if $i \neq t$. We show that y avoids (123). If y does contains a (123), it must involve y_t. So, there exist r, s such that one of the following holds:

Case (i): $r < s < t$ and $y_r < y_s < y_t = i+1$. We cannot have $y_s < i$, otherwise $x_r x_s x_t = y_r y_s i$ is a (123) in x. So, $y_s = i$. Apply (a) with $j = s$. Then $x_r x_s x_u$ is a (123) in x.

Case (ii): $r < t < s$ and $y_r < y_t < y_s$. Note then $y_s > i + 1$. It can't be that $y_r < i$, else $x_r x_t x_s = y_r i y_s$ is a (123) in x. So, $y_r = i$. But then $x_r = i$. Apply (a) with $j = r$. Then $x_r x_u x_s$ is a (123) in x.

Case (iii): $t < r < s$ and $i + 1 = y_t < y_r < y_s$. Then $x_t x_r x_s = i y_r y_s$ is a (123) in x.

Finally, we show the converse: if x *does* have a (123) then so does $\tau(x)$. Assume x has a (123) and that $x_t = i$ is the leftmost unmatched i in x. Again $y_t = i + 1$, $y_i = x_i$ if $i \neq t$. If the (123) pattern in x does not involve x_t, then $y = \tau(x)$ has the same (123) pattern. So suppose x has a (123) pattern which does involve x_t. Then there exist r, s such that one of the following holds:

Case (i): $r < s < t$ and $x_r < x_s < x_t = i$. Then $y_r y_s y_t = x_r x_s (i + 1)$ is a (123) pattern in y.

Case (ii): $r < t < s$ and $x_r < x_t < x_s$. Note then $x_r < i$. If $x_s > i+1$, then $y_r y_t y_s = x_r (i+1) x_s$ is a (123) in y. Otherwise, $x_s = i + 1$. Apply (b) with $j = s$. Then $y_r y_u y_s$ is a (123) in y.

Case (iii): $t < r < s$ and $i = x_t < x_r < x_s$. If $x_r > i+1$ then $y_t y_r y_s = (i+1) x_r x_s$ is a (123) in y. Otherwise, $x_r = i + 1$ and $x_s > i + 1$. Apply (b) with $j = r$. Then $y_u y_r y_s$ is a (123) in y. This completes the proof.

4.4 Other patterns

Although we can adapt Θ to show the symmetry of $f(\mathbf{a}, (321))$ in the variables $a_1, \ldots a_k$, we note that this approach does not work to show symmetry for $\pi = (132)$. For example, the permutations in $(1^2)(2^1)(3^1)$ containing (132) are:

$$\{1132, 1312, 1321\},$$

and the permutations in $(1^1)(2^2)(3^1)$ containing (132) are:

$$\{2132, 1232, 1322\}.$$

It is not possible to get from the strings in the first set to those in the second simply by changing a '1' to a '2'. The permutations (213), (231), and (312) have similar problems. However, the same mapping Θ provides a simple bijective proof that $f(\mathbf{a}, (12 \ldots r))$ is symmetric in the variables

$a_1, \ldots a_k$, a result that was shown in [1] using Schur functions and the Robinson-Schensted-Knuth correspondence.

However the map $\Theta = \Theta(\mathbf{a}, \mathbf{a}')$ can be composed with the bijections of Myers [4], which generalize earlier constructions of Simion and Schmidt [6] to give bijective proofs of symmetry for all six patterns of three letters. Indeed, $S(\mathbf{a}, \pi)$ be the set of all permutations of the multiset $1^{a_1}2^{a_2}\ldots$ that avoid the pattern $\pi \in S_3$, and let

$$SSM(\mathbf{a}, \pi, \pi') : S(\mathbf{a}, \pi) \to S(\mathbf{a}, \pi')$$

be the map of Simion-Schmidt-Myers. Then the map

$$SSM(\mathbf{a}', (123), \pi) \circ \Theta(\mathbf{a}, \mathbf{a}') \circ SSM(\mathbf{a}, \pi, (123))$$

is a bijection between permutations of a multiset $\mathbf{a}$ that avoid π to permutations of the multiset $\mathbf{a}'$ that avoid the same pattern π.

5 Some open questions

1. What is the asymptotic behavior of $c(n, n)$ (see (3) above)?
2. Investigate the equivalence classes of permutation patterns of length four under avoidance by permutations of multisets.
3. Investigate avoidance of patterns that are not themselves permutations.

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