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Note

A Method and Two Algorithms on the Theory of Partitions

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In this note we describe a general principle for selecting at random from a collection of combinatorial objects, where "at random" means in such a way that each of the objects has equal probability, *a priori*, of being selected. We apply this principle by displaying two algorithms, the first of which will select a random partition of an integer n, and the second will choose a random partition of a set S of n elements.

We begin by stating the general principle, albeit rather vaguely. Let a_n be the number of combinatorial objects of order n. Suppose the numbers a_n satisfy a recurrence relation of the form

$$a_n = \sum_{m<n} \alpha_{m,n} a_m , \tag{1}$$

in which $\alpha_{m,n} \geqslant 0$. Suppose further that the recurrence (1) can be given a combinatorial interpretation, by which we mean that there is a proof of (1) in which, by an explicit construction, the a_m objects of order m are extended to $\alpha_{m,n} a_m$ objects of order n. Then we have an algorithm for selecting an object of order n at random: first choose a value of m, $0 \leqslant m \leqslant n - 1$ according to the probabilities

$$\text{Prob}(m) = \frac{\alpha_{m,n} a_m}{a_n} \qquad (m = 0, 1, \ldots, n-1). \tag{2}$$

Then, inductively, select a random object of order m, and extend it as described in the proof.

We make the above statement precise in two cases. First, if $p(n)$ is the

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number of partitions of the integer n, then an identity of Euler asserts that

$$np(n) = \sum_{m<n} \sigma(n-m)\,p(m) \qquad (p(0)=1), \tag{3}$$

where $\sigma(m)$ is the sum of the divisors of the integer m, while if a_n is the number of partitions of a *set* of n elements, then the a_n satisfy

$$a_n = \sum_{m=0}^{n-1} \binom{n-1}{m} a_m \qquad (a_0 = 1). \tag{4}$$

Next we must ask for the combinatorial interpretations of (3), (4). For (3), let π denote a fixed partition of an integer $m < n$. Let d be a divisor of $n-m$. With the pair (π, d) we associate exactly d copies of a single partition π', of n. Here π' is obtained by adjoining to the partition π, of m, exactly $(n-m)/d$ copies of d.

We claim that as d varies over all divisors of $n-m$ and π varies over all partitions of m ($m = 0, 1,..., n-1$), each partition of n is constructed exactly n times by this process, which will establish (3). Indeed, let

$$\pi' : n = \mu_1 r_1 + \mu_2 r_2 + \cdots + \mu_k r_k \tag{5}$$

be a fixed partition of n, where the r_i are the *distinct* parts of π' and the μ_i are their multiplicities. Then π' is constructed by adjoining t copies of r_i to a partition of $n - tr_i$ for each $1 \leqslant t \leqslant \mu_i$, $1 \leqslant i \leqslant k$, and by replicating the resulting partition of n r_i times, which gives a total of

$$\sum_{i=1}^{k} r_i \sum_{t=1}^{\mu_i} 1 = \mu_1 r_1 + \cdots + \mu_k r_k = n,$$

times altogether, as claimed.

As for partitions of sets, the interpretation of (4) is well known, but we include it for completeness: If π is a partition of $\{1, 2,..., m\}$, let S be a subset $\{s_1,..., s_m\}$ of m elements chosen from $\{1, 2,..., n-1\}$. With (π, S) we associate the partition of $\{1, 2,..., n\}$ in which elements s_i and s_j from S are in the same class iff their subscripts i and j are in the same class of π, while all elements not in S are in a single class. It is easy to check that each partition of $\{1, 2,..., n\}$ occurs exactly once. (We have assumed $s_1 < s_2 < \cdots < s_m$.)

We can now describe the algorithms.

ALGORITHM I. *Given n, select a random partition of n.*

(A) Set $n' \leftarrow n$, $P \leftarrow$ empty partition.

(B) Choose an integer $m < n'$ according to the probabilities

$$\text{Prob}(m) = \frac{\sigma(n'-m)\,p(m)}{n'p(n')} \qquad (m = 0, 1,..., n'-1).$$

(C) Choose a divisor d of $n' - m$ according to the probabilities

$$\text{Prob}\,(d) = d/\sigma(n'-m) \qquad (d \mid (n'-m)).$$

(D) Adjoin to the partition P $(n'-m)/d$ copies of d.

(E) Replace n' by m.

(F) If $n' = 0$, stop. Otherwise return to step (B).

ALGORITHM S. *Given n, select a random partition of $U = \{1, 2,..., n\}$.*

(A) Set $U' \leftarrow U$, $m \leftarrow n$, $P \leftarrow$ unique partition of empty set.

(B) Choose an integer $k < m$ according to the probabilities

$$\text{Prob}(k) = \binom{m-1}{k}\frac{a_k}{a_n} \qquad (0 \leqslant k \leqslant m-1).$$

(C) Let l be the largest element of U'. Choose a random k-subset S of $U' - \{l\}$.

(D) Adjoin $U' - S$ as a single class to P. Stop if $k = 0$. Otherwise set $m \leftarrow k$, $U' \leftarrow S$ and return to step (B).

It remains to show that all partitions have equal probability ($= p(n)^{-1}$ or a_n^{-1}, respectively). We show this in the case of partitions of an integer, the other case being similar.

Let

$$\pi' : n = \mu_1 r_1 + \cdots + \mu_k r_k \tag{6}$$

be a fixed partition of n. Inductively, suppose that for all $n' < n$ it has been shown that our algorithm produces partitions of n' with all equal probabilities. Then Prob(π') is a sum over all partitions π'' of integers $m < n$ of Prob(π'') multiplied by the probability that on the next step π'' is extended to π':

$$\begin{aligned}\text{Prob}(\pi') &= \textstyle\sum \text{Prob}(\pi'')\,\text{Prob}(\pi''_m \to \pi') \\ &= \textstyle\sum 1/p(m)\,\text{Prob}(\pi''_m \to \pi').\end{aligned} \tag{7}$$

The last factor vanishes unless π' can result from π''_m by adjunction of exactly t copies of one part r_i. In the latter case, if π' is the partition (6) then

$$\begin{aligned}\operatorname{Prob}(\pi''_m \to \pi') &= \operatorname{Prob}(m = n - tr_i)\operatorname{Prob}(d = r_i)\\ &= \left\{\frac{\sigma(tr_i)\,p(m)}{np(n)}\right\}\left\{\frac{r_i}{\sigma(tr_i)}\right\}\\ &= \frac{p(m)}{np(n)}\,r_i .\end{aligned}$$

The sum (7) over all candidates π''_m for extension to π' in a single step becomes just

$$\begin{aligned}\operatorname{Prob}(\pi') &= \sum_{i=1}^{k}\sum_{t=1}^{\mu_i}\frac{1}{p(m)}\frac{p(m)}{n\,p(n)}\,r_i\\ &= \frac{1}{n\,p(n)}\sum_{i=1}^{k}\mu_i r_i\\ &= \frac{1}{p(n)},\end{aligned}$$

as required.

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Note added in proof. A generalization of these ideas, along with other applications, appears in [1].

Reference

1. A. Nijenhuis and H. S. Wilf, "Combinatorial Algorithms," Academic Press, New York, 1975.