THE INTERCHANGE GRAPH OF A FINITE GRAPH

By

A. C. M. van ROOIJ and H. S. WILF* (Philadelphia, USA)

(*Presented by* G. HAJÓS)

Let G be a finite graph. The *interchange* graph G' of G, has a vertex corresponding to each edge of G, two vertices of G' being connected if the corresponding edges of G have a common vertex in G. In reference [1], the questions are raised of when G' can be isomorphic to G, and how to describe the sequence of iterated interchange graphs of a given G. We suppose that G has no loops and no multiple edges.

The *local degree* of a vertex a of a graph G is $\varrho(a)$, the number of edges of G emanating from a. If $\nu, \nu', \varepsilon, \varepsilon'$ denote the number of vertices and edges of G and G' respectively, then clearly $\nu' = \varepsilon$ and

$$\varepsilon' = \sum_{a \in G} \frac{\varrho(a)(\varrho(a)-1)}{2} = \frac{1}{2} \sum_{a \in G} \varrho(a)^2 - \frac{1}{2} \sum_{a \in G} \varrho(a) = \frac{1}{2} \sum_{a \in G} \varrho(a)^2 - \varepsilon. \tag{1}$$

Now if G is isomorphic to G' then in particular $\nu' = \nu$, $\varepsilon' = \varepsilon$ and so

$$\nu = \nu' = \varepsilon = \varepsilon' = \frac{1}{2} \sum_{a \in G} \varrho(a)^2 - \varepsilon$$

whence

$$\left\{ \frac{1}{\nu} \sum_{a \in G} \varrho(a)^2 \right\}^{\frac{1}{2}} = \left\{ \frac{4\varepsilon}{\nu} \right\}^{\frac{1}{2}} = 2. \tag{2}$$

But also

$$\frac{1}{\nu} \sum_{a \in G} \varrho(a) = \frac{2\varepsilon}{\nu} = 2. \tag{3}$$

Thus the positive numbers $\varrho(a)$ $(a \in G)$ have the same arithmetic mean and root mean square, and so they are all equal, their common value being 2 and we have

THEOREM 1. *$G = G'$ if and only if G is regular of degree 2.*

To investigate the sequence of iterated interchange graphs $G, G', G'', \dots$ we first consider three mutually exclusive (but not exhaustive) families of graphs:

I. All $\varrho(a) = 2$ $(a \in G)$

II. All $\varrho(a) \leqq 2$ $(a \in G)$ and some $\varrho(a) < 2$.

III. All $\varrho(a) \geqq 2$ $(a \in G)$ and some $\varrho(a) > 2$.

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Now if a' is a vertex of the interchange graph G' of a graph G, suppose a' "comes from" an edge of G whose endpoints are a_1, a_2 in G. Then the local degree of a' in G' is

$$\varrho(a_1)+\varrho(a_2)-2.$$

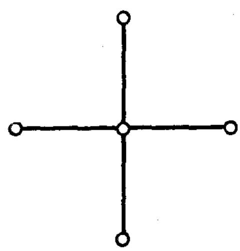
It follows that the graph mapping $G \to G'$ is an endomorphism of each of the classes I, II, III above. Suppose $G \in \text{II}$. Then

$$v' = \varepsilon = \frac{1}{2}\sum_{a \in G} \varrho(a) < \frac{1}{2}\sum_{a \in G} 2 = v.$$

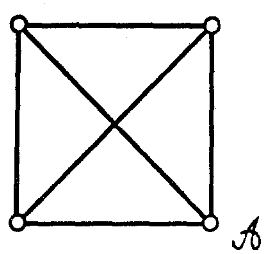
Hence for $G \in \text{II}$, the sequence of graphs $G, G', G'', G''', \ldots$ degenerates to the null graph after finitely many steps.

Similarly if $G \in \text{III}$ we find $v' > v$ and so the sequence $G, G', G'', \ldots$ increases without bound.

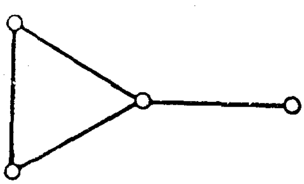
Now let G be arbitrary but connected. If any $\varrho(a) \geqq 4$ then G has the subgraph

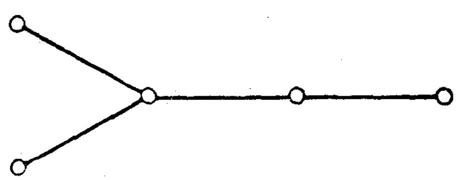


whose interchange

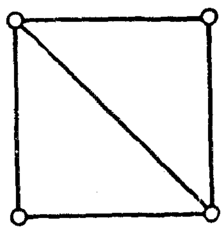


is of type III and so again $G, G', G'', \ldots$ increases without bound. Now suppose all $\varrho(a) \leqq 3$. If any point with $\varrho(a)=3$ is connected to a point with $\varrho(a) \geqq 2$ then G contains one of the subgraphs



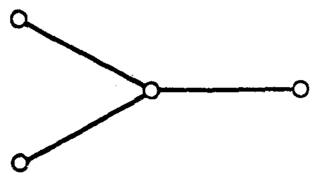


whose first or second interchange graph, respectively is $\mathscr{B}$,



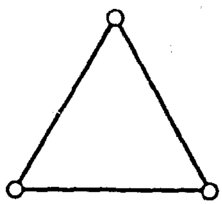
$\mathscr{B}$

of type III, and we have divergence once more. Finally, if all points with $\varrho(v)=3$ are connected only to points with $\varrho(v)=1$ then G is precisely the graph $\mathscr{C}$



$\mathscr{C}$

whose interchange



is of type I which is regular of degree 2. We summarize with

THEOREM 2. *The sequence of graphs $G, G', G'', \ldots$, where G is connected,*

(a) *has ultimately steadily increasing numbers of vertices.*
(b) *is of the form $G, G, G, G, \ldots$,*
(c) *is of the form $G, H, H, H, H, \ldots$,*
(d) *is of the form $G_1, G_2, \ldots, G_n, \Phi, \Phi, \Phi, \ldots$,*
depending on whether

(a)′ *G has at least one $\varrho(a) \geqq 3$ and is not the special graph $\mathscr{C}$,*
(b)′ *G is regular of degree 2,*
(c)′ *G is the special graph $\mathscr{C}$,*
(d)′ *G has all $\varrho(a) \leqq 2$ and some $\varrho(a) < 2$,*
respectively.

Naturally if G is not connected we may apply theorem 2 to each component.
Concerning the rate at which the number of edges grows under the iteration operation, let v_m, ε_m denote the number of vertices and edges in $G^{(m)}$. Then, from

(1), and the inequality

$$\sum_{a\in G}\varrho^2(a)\geqq\frac{1}{\nu}\left(\sum\varrho(a)\right)^2=\frac{4\varepsilon^2}{\nu}$$

we find that

$$\varepsilon_{m+1}\geqq 2\frac{\varepsilon_m^2}{\nu_m}-\varepsilon_m$$

and since $\nu_{m+1}=\varepsilon_m$, we have

$$\frac{\nu_{m+1}}{\nu_m}\geqq 2\frac{\nu_m}{\nu_{m-1}}-1.$$

By induction, then,

$$\frac{\nu_m}{\nu_{m-1}}\geqq 2^{m-1}\left(\frac{\varepsilon}{\nu}-1\right)+1 \qquad (m=1,2,3,\dots)$$

where $\varepsilon=\varepsilon_0$, $\nu=\nu_0$, and finally,

(4) $$\nu_m\geqq\nu\prod_{\varkappa=0}^{m-1}\{1+2^{\varkappa}\xi\} \qquad (m=1,2,3,\dots)$$

where

$$\xi=\frac{\varepsilon}{\nu}-1.$$

The sign of equality holds in (4) for some $m>0$ if and only if G is regular, and then it holds for all m. We may suppose $\xi>0$ without loss of generality. For $m\to\infty$ the right side of (4) is

$$\sim\nu f\left(\frac{1}{\xi}\right)\xi^m 2^{\frac{1}{2}m(m-1)}$$

where

$$f(x)=\prod_{\varkappa=0}^{\infty}\left(1+\frac{x}{2^{\varkappa}}\right).$$

For a regular graph of local degree ζ, $\xi=\frac{1}{2}\zeta-1$, so

$$\nu_m=\nu\prod_{\varkappa=0}^{m-1}[1+(\zeta-2)2^{\varkappa-1}]\leqq\nu 2^{\frac{1}{2}m(m-3)}(\zeta-2)^m f\left(\frac{2}{\zeta-2}\right).$$

Any graph G that contains more than three vertices can be extended to a complete, hence regular, graph of local degree $\nu-1>2$, so

(5) $$\log_2\nu_m\leqq\frac{1}{2}m^2+m\left[\log_2(-3)-\frac{3}{2}\right]+\log_2 f\left(\frac{2}{\nu-3}\right)+\log_2\nu.$$

On the other hand, as we remarked before, if G is a graph with property (a)′, then either G' contains $\mathscr{A}$ or G'' contains $\mathscr{B}$. Using (4) for $\mathscr{A}$ and $\mathscr{B}$ we see that for G either

$$\nu_m\geqq 4\prod_{\mu=0}^{m-2}\left(1+2^{\mu}\cdot\frac{1}{2}\right)\geqq\prod_{\mu=0}^{m-2}2^{\mu-1}=2^{\frac{1}{2}(m-1)(m-5)}$$

or

$$v_m \geqq 4 \prod_{\mu=0}^{m-3} \left(1 + 2^{\mu} \cdot \frac{1}{4}\right) \geqq \prod_{\mu=0}^{m-3} 2^{\mu-2} = 2^{\frac{1}{2}(m-2)(m-7)}.$$

Combining this with (5) we find:

THEOREM 3. *For any graph G that has property* (a)′ *of theorem* 2,

$$\log_2 v_m = \tfrac{1}{2} m^2 + O(m) \qquad (m \to \infty).$$

Finally we give a necessary and sufficient condition for a graph H to be an interchange graph. The basic idea is the following. We can identify a vertex a of a graph G with the "pencil" of all edges containing a. Then we see that its image in G' is a complete subgraph, which is maximal if $\varrho(a) \geqq 3$. Conversely, every maximal complete subgraph in G' is the image of such a pencil, unless it is a triangle. In the latter case it is the image of either a pencil or a triangle in G. To distinguish these two cases we introduce the concepts of "even" and "odd" triangles.

A triangle D in a graph H is *even* if every vertex of H is joined with either 0 or 2 vertices of D; otherwise D is *odd.* Every triangle contained in a larger complete section of H is odd. The image in H' of any triangle of H is an even triangle in H'.

Before stating our theorem we define a notation: if a, b are vertices of a graph, we write $a \sim b$ for "a and b are joined", and $a \nsim b$ for "a and b are not joined".

The section of a graph G determined by the vertices $a_1, \ldots, a_n$ of G will be written as $\{a_1, \ldots, a_n\}$.

THEOREM 4. *A graph H is the interchange graph of some graph G if and only if*

(i) *If H contains a section graph isomorphic with* $\mathfrak{D}$, *then one of its two triangles is even. In other words: If* $\{a, b, c\}$ *and* $\{a, b, d\}$ *are odd triangles in H, then* $c = d$ *or* $c \sim d$.

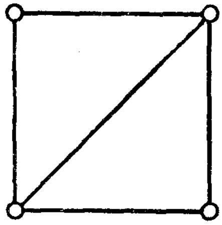
(ii) *H contains no section graph isomorphic with* $\mathfrak{E}$.

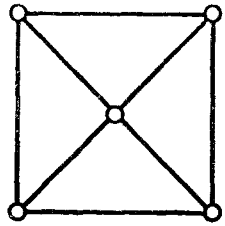
PROOF. The "only if" is easily verified. In proving the "if" clearly we may suppose H to be connected. One of the two following statements is true:

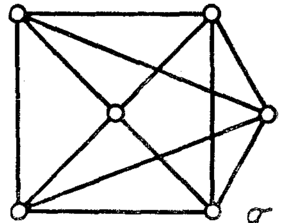
(I) H contains two even adjacent triangles.

(II) If two triangles in H have an edge in common, then one of them is odd.

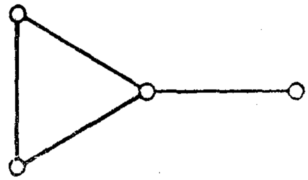
We will prove that only three graphs have properties (i), (ii), and (I), viz. $\mathfrak{D}$, $\mathscr{E}$, and $\mathscr{F}$.


$\mathfrak{D}$


$\mathfrak{E}$


$\mathfrak{F}$

These are the interchange graphs of $\mathcal{G}$, $\mathfrak{D}$, and $\mathscr{A}$, respectively.



$\mathcal{G}$

After that, for any H that has the properties (i), (ii), and (II), we construct a G such that $H=G'$.

First suppose that H contains two even adjacent triangles $\{a, b, c\}$ and $\{a, b, d\}$. Then $c \not\sim d$, so $\{a, b, c, d\} = \mathfrak{D}$. If H is not $\mathfrak{D}$, it must contain a fifth vertex, e, joined to either a, b, c or d. We may assume that either $e \sim a$ or $e \sim c$.

$e \not\sim c$ implies $e \sim a$, so $e \sim b$ because $\{a, b, c\}$ is even. But $\{a, b, d\}$ is even, so $e \sim a$ and $e \sim b$ together imply $e \not\sim d$. Then $\{a, c, d, e\}$ is isomorphic with $\mathfrak{C}$, contrary to our assumption (ii).

Hence, $e \sim c$. Again using the fact that $\{a, b, c\}$ and $\{a, b, d\}$ are even we see that either $e \sim a$ or $e \sim b$, but not both, and $e \sim d$. We may assume $e \sim a, e \not\sim b$. Then $\{a, b, c, d, e\} = \mathscr{E}$ and either $\{a, d, e\}$ or $\{a, c, e\}$ is even because of (i).

If H is not $\mathscr{E}$ there must exist a sixth vertex f of H, joined to one of the first five. It is easy to show now that H is necessarily $\mathscr{F}$.

Let G_1 be the class of all maximal complete subgraphs of H that are not even triangles.

Let G_2 be the class of those vertices $x \in H$ that are contained in some $A \in G_1$ and are not joined with any vertex outside A.

Let G_3 be the class of all edges of H that are contained in a unique *and* even triangle.

We define a graph G whose interchange graph is H as follows: The vertices of G are the elements of $G_1 \cup G_2 \cup G_3$; two of these vertices are joined if their intersection is non-void.

We construct a map $f\colon G' \to H$ as follows: Every $x \in G'$ corresponds to a pair (A, B) of vertices of G, and $A \cap B \neq \emptyset$. We will prove that $A \cap B$ contains only one point, and this point we call $f(x)$. This f turns out to be an isomorphism onto H. To show this we have to prove the following statements:

(α) For all $A, B \in G$ with $A \neq B$, $A \cap B$ contains at most one point. (Definition of f.)

(β) Every $a \in H$ is the intersection of certain $A, B \in G$ with $A \neq B$. (f is onto.)

(γ) These A and B are, except for their order, uniquely determined. (f is one-to-one.)

(δ) If $A_1, A_2, A_3 \in G$ are different, and $a \in A_1 \cap A_2, b \in A_2 \cap A_3$, then $a \sim b$. (If $x, y \in G'$ and $x \sim y$, then $f(x) \sim f(y)$.)

(ε) If $A_1, A_2, B_1, B_2 \in G$, $A_1 \neq A_2$, $B_1 \neq B_2$, $\{a\} = A_1 \cap A_2$, $\{b\} = B_1 \cap B_2$ and $a \sim b$, then one of the A_i's is one of the B_j's. (If $x, y \in G'$ and $f(x) \sim f(y)$, then $x \sim y$.)

(α) Suppose $A, B \in G, A \neq B, \{a, b\} \subset A \cap B, a \neq b$. Elements of G_3 are contained in no odd triangles, so $\{a, b\} \notin G_3$, and hence $A, B \in G_1$. Because A is maximal complete it must contain a c that is not in B. Then B must contain a d which is not joined to c. But then $\{a, b, c, d\}$ is a section, isomorphic with $\mathfrak{D}$.

(β) Take $a \in H$. First assume that a is in some even triangle $\{a, b, c\}$. Either $\{a, b\} \in G_3$ or $\{a, b\}$ is part of a triangle $\{a, b, d\}$ with $d \neq c$. Then by II, $\{a, b, d\}$ is odd, so it can be extended to a $C \in G_1$. Because $\{a, b, c\}$ is even, $d \not\sim c$, so $c \notin C$.

Hence, there exists an $A \in G$ (which is either $\{a, b\}$ or C) for which $a \in A$, $b \in A$, $c \notin A$. For the same reason there exists a $B \in G$ with $a \in B$, $b \notin B$, $c \in B$. By (α), these A and B cannot have a point $e \neq a$ in common.

Now assume that a is not contained in any even triangle. Let A be a maximal complete section containing a. If a is not joined to any vertex outside A, take $B = \{a\}$. If $a \sim b$ for some $b \notin A$, let B be any maximal complete section containing a and b.

(γ) The proof of this statement is simple but too tedious to give it here. The arguments involved are of the same kind as those used in proving (β).

(δ) This is trivial because every $A \in G$ is complete.

(ε) If $a \sim b$, then by (γ) and by the construction given in (β), $b \in A_1 \cup A_2$, say $b \in A_1$. But for the same reasons $a \in B_1 \cup B_2$, say $a \in B_1$. Then $A_1 = B_1$ because of (α).

Trivially any automorphism of a graph G induces an automorphism of G'. Conversely, every automorphism of G' is induced by an automorphism of G if and only if G' has property II. (This follows easily from the construction given above and from the fact that G is determined by G' unless G' is a triangle.) In this connection the graphs $\mathfrak{G}$, $\mathfrak{D}$, $\mathscr{A}$, whose interchange graphs do not have property II, have been mentioned by O. ORE [1] (Fig. 15.4.2).

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Bibliography

[1] O. ORE, *Theory of Graphs*, Amer. Math. Co. Coll. Publ. Vol. **38**, 1962.