Graphical Combinatorial Families and Unique Representations of Integers

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In 1978 I published ([1], [2]) two papers on a unified approach to several different algorithmic problems in combinatorics, which grew out of joint work [3] (particularly chapter 13) with A. Nijenhuis. Aside from these algorithmic uses, however, the method has purely mathematical consequences, and these have not been sufficiently brought out. So I would like to take this opportunity to review the unified approach here, this time discussing the graph theoretical side of the story, and its implications, more fully.

Let G be a strongly connected acyclic digraph. By a *terminal vertex* of G we mean a vertex of outdegree 0. We assume that G has exactly one terminal vertex, τ. We assume further that the edges of G are numbered, computer–science style. That is to say, if v is a fixed vertex, with ρ outgoing edges, then these edges are numbered 0, 1, ..., $\rho - 1$.

Such a digraph will be called a graphical combinatorial family.

Let $v \in V(G)$. By a *combinatorial object of order* v we mean a walk from v to τ. Associated with each combinatorial object of order v is its edge code word, which is the sequence of edge numbers that occur in the walk. Evidently the order v of the walk, and the code word determine the walk uniquely.

Many garden–variety combinatorial families are special cases of these structures. Take V(G) to the set of lattice points (n, k) $(n \geq k \geq 0)$ in the plane. If $v = (n, k)$ with $n > k \geq 1$ then out of v there go two edges, one, numbered 0, goes to $(n-1, k)$, and the other, numbered 1, goes to $(n-1, k-1)$. If $n = k \geq 1$ there is just one edge, numbered 0, and it goes to $(n-1, k-1)$. If $n > 0$ and $k = 0$ there is also just one edge, numbered 0, and it goes to $(n-1, k)$. The unique terminal vertex is $\tau = (0, 0)$.

Now, for a fixed $v = (n, k)$, there is a 1–1 correspondence between the combinatorial objects of order v and the subsets of k elements chosen from n: given a

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walk from v to (0, 0), insert into the set the x–coordinate of the initial vertex of every edge numbered 1 that occurs in the walk.

That was one example. There is an extensive list of combinatorial objects that can be brought into the fold, including set partitions, number partitions, permutations by cycles, permutations by runs, Young tableaux, etc. etc. Hence it is a worthwhile matter to study the general properties of these systems.

The original reason for studying them was to carry out four algorithms: listing, ranking, unranking, and random selection. It turns out that we can list the walks from v to τ in lex order in any such digraph. Hence we can list the objects in any such family. For example, we can make a list of all of the permutations of 19 letters that have 8 cycles, or of all of the partitions of [1..15] that have 8 classes, etc., by the same algorithm.

The ranking problem is this: given an object in a family of objects. Determine where it is on the list. More precisely, determine its position in the lex ordered list of all such objects. For instance, in the list of all 225 permutations of 6 letters that have 3 cycles, where is (14) (523) (6)?

Ranking problems can also be done by a general method, in the graph–theoretic setting of combinatorial families. The problem is the, given a walk from v to τ. Where is it in the lex ordered list of all walks from v to τ?

In order to present the new applications of the method, it will be helpful to review the ranking procedure. Suppose we are given a path from v to τ in a graphical combinatorial family, as shown in Fig. 1.

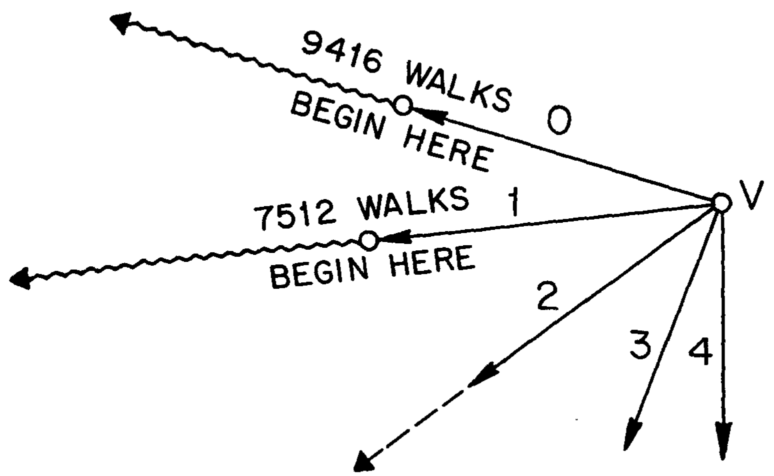

Fig. 1: This step costs $16,928.

Suppose the walk has arrived at v, and that it next uses edge number 2 outbound from v. It will then ignore all of the walks that would have used edges 0 or 1 next, instead. Furthermore, all of those walks *precede* the walk that uses edge 2 since the letters 0, 1 precede 2. Therefore, if a total of 9416 walks begin at the terminal vertex of edge 0, and if 7512 walks begin at the terminal vertex of edge 1, we will have jumped to a place in the alphabet that is below all 16, 928 of those walks. Hence, if edge 2 is in the walk that we are considering, its rank is augmented by 16, 928 on that single step.

Furthermore, the rank of a complete walk from v to τ is the sum of all of the numbers, like 16,928, that occur on every edge of the walk. We call 16,928 the *cost* of the edge. The costs of all the edges can be precomputed, without reference to some particular walk. The cost of an edge e, whose initial vertex is v and whose terminal vertex is w, is the sum of the numbers of walks from v' to τ, over all vertices v' such that $(v, v') \in E(G)$, and edge–number of edge (v, v') is < e.

If these costs have been precomputed, the *the rank of a walk is the sum of the costs of the edges in the walk.*

Corollary *In a graphical combinatorial family G, let v be a vertex, and suppose that there are exactly b(v) walks from v to τ. Then every integer r, $0 \le r \le b(v) - 1$ is uniquely representable as the sum of the costs on the edges of one of these walks.*

Proof Every such integer is the rank of exactly one such walk. ❑

This observation contains as special cases a number of well known, and some new, unique representation theorems for integers.

Example 1. Fibonacci Numbers

Consider the graph G whose vertices are the nonnegative integers. There is an edge from m to m – 1, numbered 0, and from m to m – 2, numbered 1, for each $m \ge 2$, and edge (1, 0), numbered 0. The number of walks from m to 0 is F_m, and consequently *every integer r, $0 \le r \le F_m - 1$ is uniquely representable as* $F_{m_1} + F_{m_2} + \ldots$, *where all $m_{i+1} \le m_i - 2$.* ❑

Example 2. Partitions of Integers

Define the graph G as follows. Its vertices are the plane lattice points (n, k) with $n \ge k \ge 1$ and the terminal vertex $\tau = (0, 0)$. There is an edge from (n, k) to (n - k, k), for $n \ge k \ge 1$, numbered 0, and from (n, k) to (n – 1, k – 1), numbered 1. The number of walks from (n, k) to (0, 0) is then p(n, k), the number of partitions of n into parts $\le$ k, because both p(n, k) and the number of walks satisfy the same recurrence relation with the same initial conditions, viz.

$$p(n, k) = p(n-1, k-1) + p(n-k, k).$$

It follows that *every integer* r *such that* $0 \le r \le p(n, k) - 1$ *is uniquely expressible as*

$$r = p(n_0, k) + p(n_1, k-1) + \ldots$$

where, for all $j \ge 0$, $n_{j+1} = n_j - t_j(k-j) - 1$, $n_0 = n - (t+1)k$, *and* $t, t_0, t_1, \ldots$ *are nonnegative integers.* ❑

Example 3. Binomial Coefficients

Here the vertices of G are the lattice points (n, k) with $n \ge k \ge 0$. There is an edge from each (n, k), $n > k \ge 1$, to $(n-1, k)$, numbered 0, and an edge from (n, k) to $(n-1, k-1)$, numbered 1. If $n > 0$ and $k = 0$ there is one edge, numbered 0, from (n, k) to $(n-1, k)$. The terminal vertex is (0, 0). The number of walks from (n, k) to τ is $\binom{n}{k}$.

We obtain at once the fact that *every integer* r, $0 \le r \le \binom{n}{k} - 1$, *is uniquely of the form*

$$r = \binom{n_1}{k} + \binom{n_2}{k-1} + \ldots + \binom{n_k}{1}$$

where $n > n_1 > \ldots \ge 0$. ❑

Example 4. Stirling Numbers

Let S(n, k) be the number of partitions of [n] into k classes (the Stirling numbers of the second kind.) The same method yields the result that *every integer* r, $0 \le r \le S(n, k) - 1$ *is uniquely expressible as*

$$r = j_1 S(n-1, k_1) + j_2 S(n-2, k_2) + \ldots$$

where for all i, $0 \le j_i \le k_i$, $2 \le k_i \le n - i$, *and*

$$k_{i+1} = \begin{cases} k_i & \text{if } j_i < k_i - 1 \\ k_i & \text{if } j_i = k_i, \end{cases}$$

and $k_1 = k$. ❑

Similar resluts hold for Stirling numbers of the first kind, q–binomial coefficients, hook–formula numbers, etc. All of these formulas simply express the fact that in a suitable lex ordered list, every element has a unique rank.

One other area of application of the method of combinatorial families deserves mention, because of interesting results that have been obtained by Bruce Sagan [4].

Let a sequence $\{c_j\}$ count certain sets $S_1, S_2, \ldots$ Suppose we want to prove that the sequence is log concave, i.e., that $c_{j+1}c_{j-1} < c_j^2$, for all j. One way to do that would be to exhibit an injection $S_{j+1} \times S_{j-1} \to S_j \times S_j$, for all j.

If the sets in question are representable as walks in a graphical combinatorial family, then we want an injection from a pair of walks with nearby initial vertices to a pair of walks with the same initial vertex. The geometric picture of walks on the plane lattice, which occurs in several important examples, opens the possibility of cutting and pasting pieces of the given pair of walks to obtain the final pair of walks, and yields injections that are natural to discuss in that context. Refer to [4] for the details, but here is one of the results that he obtained. It relates the concavity of the solution of a recurrence to the concavity of the coefficients of the recurrence.

Theorem *(Sagan [4]) Let* $\{t_{n,k}\}$ *satisfy a recurrence of the form*

$$t_{n,k} = c_{n,k}t_{n-1,k-1} + d_{n,k}t_{n-1,k} \quad (t, c, d, \geq 0)$$

where the $c_{n,k}$ *and the* $d_{n,k}$ *are themselves log concave in* k*, and further satisfy*

$$c_{n,k-1}d_{n,k+1} + c_{n,k+1}d_{n,k-1} \leq 2c_{n,k}d_{n,k}.$$

Then $\{t_{n,k}\}$ *is log concave in* k.

As a problem for future research, it would seem that the method has the capability of producing unimodality results, where log–concavity doesn't exist. This would require injections from some S_j to an S_{j+1}, that might easily be facilitated by cutting and pasting operations suggested by the lattice pictures.

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