Extreme Palindromes

Kathy Q. Ji and Herbert S. Wilf

Abstract

A recursively palindromic (RP) word is one that is a palindrome and whose left half-word and right half-word are each RP. Thus ABACABA is, and MADAM is not, an RP word. We count RP words of given length over a finite alphabet and RP compositions of an integer. We use the same method to determine the parity of the Catalan numbers.

1 Introduction

Suppose we have a collection of tuples and we are interested in the parity of the size of the collection. One obvious method of reducing the problem would be to define an involution on the tuples, and then study the parity of the set of its fixed points. If for each tuple $(a_1, .., a_n)$ in our collection, the reversed tuple $(a_n, ..., a_1)$ is also in the collection, then one way to define such an involution would be to associate each tuple with its reverse, and discard the pair. That would leave us with only the palindromic tuples to study. But we can go further.

Suppose $(a_1, a_2, \ldots, a_{2k})$ is one of the remaining tuples of even length. Even though it is palindromic, it might be that its left half $(a_1, \ldots, a_k)$ is not, in which case its right half is also not palindromic. In that case we could pair this tuple with the one obtained by reversing its left half and reversing its right half, and similarly, recursively, at all levels. If $(a_1, a_2, \ldots, a_{2k+1})$ is one of the remaining tuples of odd length, then even though it is palindromic, it might be that its left half $(a_1, \ldots, a_k)$ is not, etc. as before.

The end result of this pairing and discarding process, i.e., the set of unpaired tuples, would be the collection of tuples that are *recursively palindromic* (RP), assuming that the collection of tuples is closed under the various reversal operations.

Thus instead of studying the parity of the number of all tuples in the given collection, it would suffice to study the parity of the collection of RP tuples.

Definition 1 *A tuple is recursively palindromic if it is empty, or it is a palindrome and its left half and its right half are recursively palindromic.*

The words WOWWOW or ABACABA are RP but the word MADAM is not.

2 The pairing

In general, we are given a collection $\mathcal{C}$ of words over some alphabet. We suppose that for every $w \in \mathcal{C}$, if $\sigma(w)$ is some rearrangement of the letters of w, then $\sigma(w) \in \mathcal{C}$ also. Then we associate with each word $w \in \mathcal{C}$ a binary tree $T(w)$, as follows.

By the *left half* of a word $w = (a_1 a_2 \ldots a_{2k})$ we mean the word $w_L = (a_1 a_2 \ldots a_k)$, while the left half of $w = (a_1 a_2 \ldots a_{2k+1})$ will also be $w_L = (a_1 a_2 \ldots a_k)$, and similarly for the right half w_R of a word. Then a labeled binary tree $T(w)$, associated with the word w, is constructed recurrently as follows: for the word $w = (a_1 a_2 \ldots a_{2k})$, label the root of T by $\emptyset$, while for the word $w = (a_1 a_2 \ldots a_{2k+1})$, label the root by a_{k+1}. Further, the left subtree at the root of $T(w)$ is $T(w_L)$ and the right subtree at the root of $T(w)$ is $T(w_R)$.

The binary tree T(MADAMIMADAM), for example, is shown in Fig. 1. If the tree $T(w)$ is given then we can recover the word w by concatenating the labels at the nodes of $T(w)$ when they are traversed in inorder (left-root-right).

Here we remark that the binary tree of an RP word is a binary tree such that all nodes on the same level have the same label.

We now use the tree $T(w)$ of a word to define an involution on the collection $\mathcal{C}$. Given a word w which is not RP, to find the word w' that is paired with w, proceed as follows. Begin with the root of $T(w)$, and go down the levels of T until for the first time reaching a level L such that the labels of the nodes on that level are not all the same. Then interchange the right and left subtrees at every node on level $L-1$, to obtain $T(w')$. Then we obtain w' by inorder traversal of $T(w')$. It's obvious that this map is an involution.

In the case of MADAMIMADAM, for example, we would exchange the left and right subtrees at each node on level 2 of the tree in Fig. 1 because the labels on level 3 are not all the same. By visiting the nodes of the new tree in *inorder*, we would find that MADAMIMADAM has been paired with AMDMAIAMDMA by this mapping.

3 RP words

Suppose we have an alphabet of K letters. Of the K^n possible words of length n, how many are RP words?

Theorem 1 *There are exactly $K^{\alpha(n)}$ RP words of length n over an alphabet of K letters, where $\alpha(n)$ denotes the sum of the binary digits of n.*

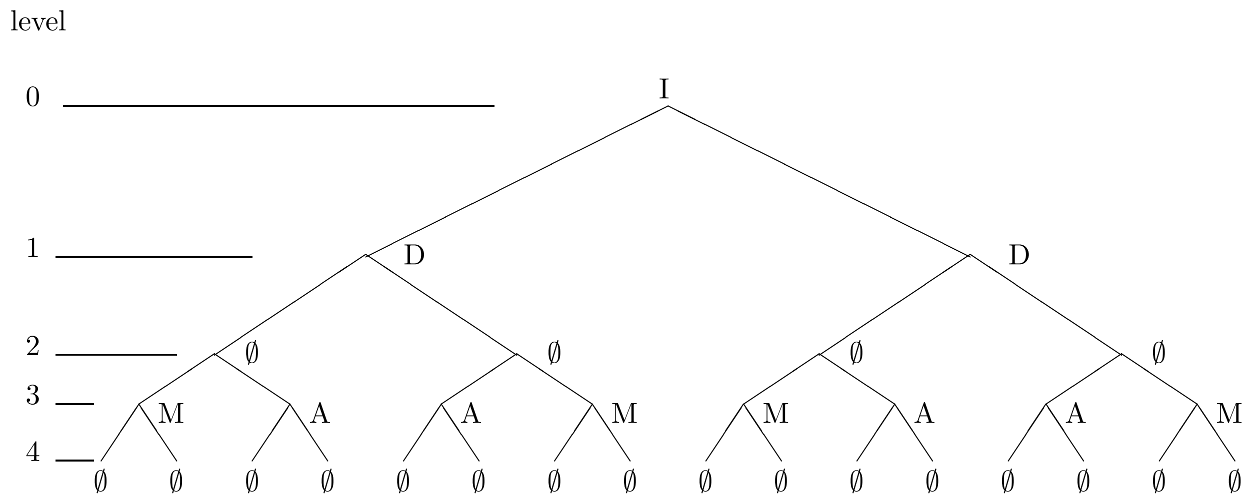

Figure 1: The binary tree of height 5 corresponding to "MADAMIMADAM"

The easy proof of this theorem is by recurrence. Let $f(n)$ be the required number of words. Then evidently $f(2n) = f(n)$, for $n \geq 1$, and $f(2n+1) = Kf(n)$, for $n \geq 0$, with $f(0) = 1$. The function $K^{\alpha(n)}$ satisfies the same recurrences with the same initial value. □

For example, among the 128 binary words of length 7, there are 8 RP words, viz.

0000000, 0001000, 0100010, 0101010, 1010101, 1011101, 1110111, 1111111.

3.1 Bijective proof

Given the binary representation of n, consider the collection of all sequences π that can be obtained from that binary representation by replacing each 1 by some letter in the given alphabet of K letters. Evidently $K^{\alpha(n)}$ is the number of such sequences. If π is such a sequence, let its length be $L = 1 + \lfloor \log_2 n \rfloor$. We will first build a bijection between these sequences π and complete binary trees $CT(\pi)$ of height L whose vertices are labeled with letters from our alphabet.

If $\pi = \emptyset$, then $CT(\pi) = \emptyset$. If $\pi \neq \emptyset$, then for each i, position i in the sequence π will correspond to level i of a complete binary tree. Furthermore if the letter in position i of the sequence is t , then all nodes on level i of that complete binary tree are labeled by t. Hence, for example, the sequence $\pi = (A0B0C)$ will correspond to the following complete binary tree of height $L = 5$:

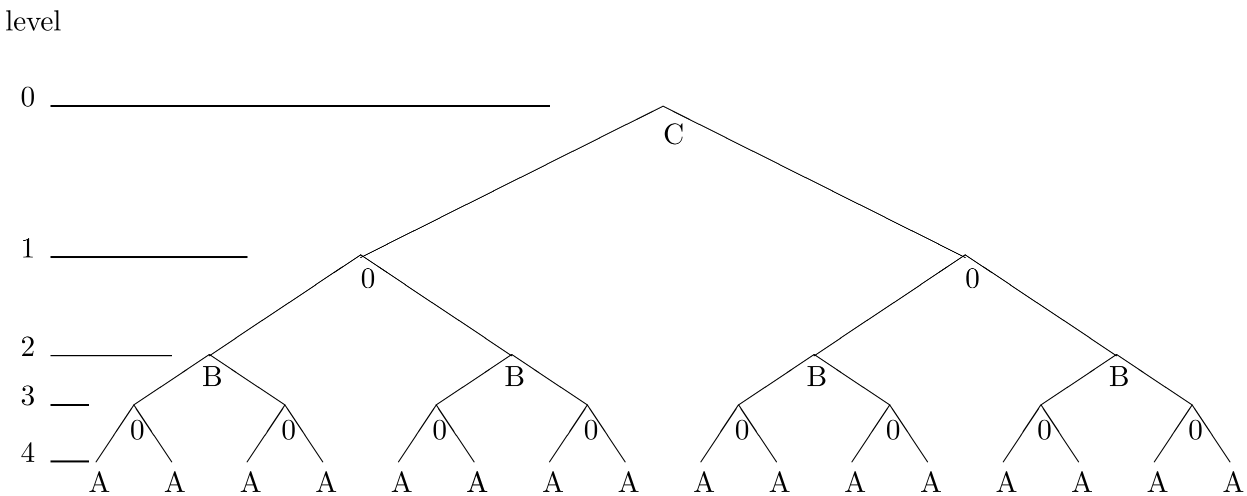

Figure 2: A complete binary tree of height 5 corresponding to an RP word.

Note that the number of *nodes labeled by a letter* in such a complete binary tree is equal to n ($= 21$ in the example), while the number of *levels labeled by a letter* is equal to the sum of the binary digits of n, namely $\alpha(n)$. Then, reading all letters on such a complete binary tree by inorder traversal will give an RP word of n letters over the alphabet $\{A, B, C, \ldots, K\}$. Thus, e.g., if $n = 21$, the sequence $(A0B0C)$ corresponds to the RP word $AABAAAABAACAABAAAABAA$ of 21 letters.

It's easy to see in general that the resulting $CT(\pi)$ is a complete binary tree, and furthermore all nodes on the same level have the same label. Hence it will correspond to one of the sequences that is counted by $K^{\alpha(n)}$. $\Box$

4 RP compositions

By a composition of n we will mean an ordered representation of n as a sum of positive integers, called the *parts* of the composition. For instance, $6 = 1+2+1+2$ is a composition of 6 into four parts. There are 2^{n-1} compositions of n. Notice that, for example, $2+1+2+6+2+1+2$ is an RP composition of $n = 16$. We ask for the number of compositions of n that are RP.

Let $f(n)$ be the required number. Now an RP composition of $2n$ is either of the form

1. ww, where w is an RP composition of n, or

2. $w(2j)w$, where $j > 0$ and w is an RP composition of $n - j$.

Thus $f(2n) = \sum_{j=0}^{n} f(n-j)$, and by subtraction, $f(2n) = f(2n-2) + f(n)$.

An RP composition of $2n+1$ is of the form $w(2j+1)w$, for some $j \geq 0$, where w is an RP composition of $n - j$. Thus $f(2n+1) = \sum_{j=0}^{n} f(n-j) = f(2n)$. Hence the counting function $f(n)$ satisfies

$$f(2n+1) = f(2n) \quad (n \geq 0); \qquad f(2n) = f(2n-2) + f(n) \quad (n \geq 1); \qquad f(0) = 1.$$

These recurrences and the initial value are identical with those satisfied by the sequence $b(n)$, the number of partitions of n into powers of 2. Indeed, a partition of $2n$ either uses at least one 2 or it does not. The latter case can be created by taking a partition of n into powers of 2, doubling each term, then replacing each of the resulting 2's with two 1's. Thus we have

Theorem 2 *The number of RP compositions of n is equal to the number of partitions of n into powers of 2 ("binary partitions").*

5 Bijective proof

For a partition λ of n into powers of 2, suppose that its largest part is 2^l. Similarly to the bijection on RP words, we will first build a bijection between the partitions λ of n into powers of 2 and complete binary trees $CT(\lambda)$ of height $l+1$ whose vertices are labeled with nonnegative integers.

If $\lambda = \emptyset$, then $CT(\lambda) = \emptyset$. If $\lambda \neq \emptyset$, then for each $i = 0, 1, \ldots, l$, the part 2^i of λ will correspond to the level i of the complete binary tree, which has 2^i nodes. Furthermore if the part 2^i of λ has multiplicity m_i, then all nodes on level i of the complete binary tree are labeled by m_i. Hence the partition $\lambda = (16, 4, 4, 4, 4, 1, 1, 1, 1, 1)$ of 37 will correspond to the complete binary tree of height 5 that is shown in Figure 2.

Note that the sum of the labels of the nodes in such a complete binary tree is equal to n. Then, reading all positive integers on such a complete binary tree by inorder traversal will give an RP composition of n. Thus, e.g., the partition $\lambda = (16, 4, 4, 4, 4, 1, 1, 1, 1, 1)$ corresponds to the RP composition of 37: $37 = 1+1+4+1+1+1+1+4+1+1+5+1+1+4+1+1+1+1+4+1+1$.

In the same way, the bijection is easily described in general. For an RP composition α of n, we construct a complete binary tree $CT(\alpha)$ labeled by nonnegative integers. Denote the number of parts of α by p. If $\alpha = \emptyset$, then $CT(\alpha) = \emptyset$. If $\alpha \neq \emptyset$ and p is odd, let the part in position $(p-1)/2$ be t. Thus $\alpha = \alpha' t \alpha'$ where α' is an RP composition of $\frac{n-t}{2}$. Now let t be

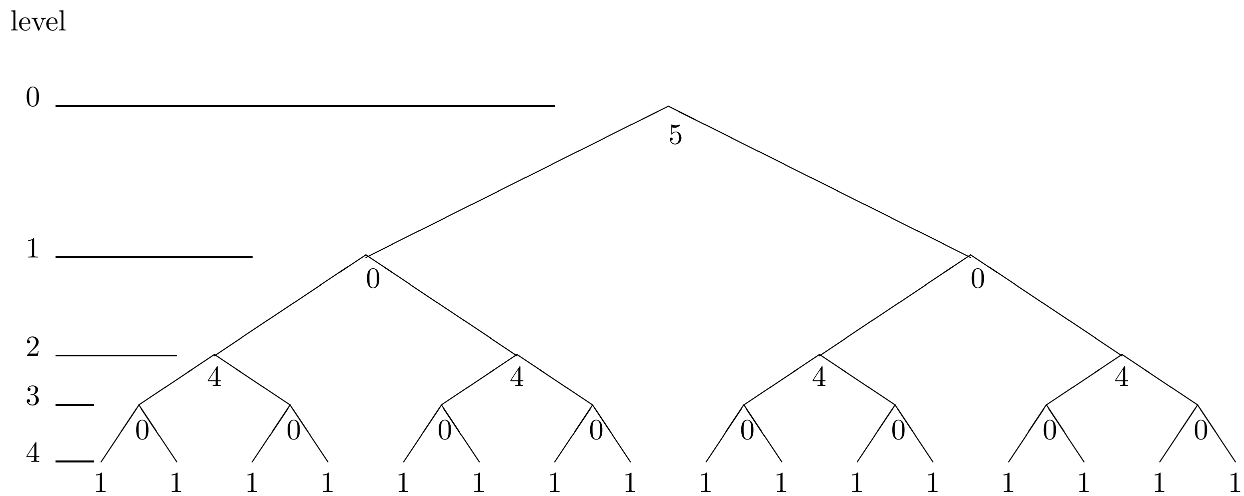

Figure 3: A complete binary tree of height 5 corresponding to an RP composition.

the root of $CT(\alpha)$ and let $CT(\alpha')$ be the left subtree at the root. If p is even, then $\alpha = \alpha'\alpha'$ where α' is an RP composition of $\frac{n}{2}$. Now let 0 be the root of $CT(\alpha)$, and let $CT(\alpha')$ be the left subtree as well as the right subtree. This yields an inductive definition of $CT(\alpha)$. It's easy to see that $CT(\alpha)$ obtained in this way is a complete binary tree, and furthermore all nodes on the same level have the same label. Hence it will correspond to a partition of n into powers of 2, namely the label on level i will correspond to the multiplicity of part 2^i. $\square$

6 The parity of the Catalan numbers

It is well known that the Catalan number C_n is odd iff $n = 2^k - 1$ for some k. This follows from arithmetic results of [1], and from bijective proofs of [2, 3, 4].

Our method of recursive palindromes provides a different, and nice bijective proof of this fact.

On the set of binary trees of n (internal) nodes we define an involution $\mathcal{I}$ as follows. If T is such a binary tree, we begin at the root of T, and we go down the levels of T until for the first time we reach a level L with the following property: there is a node v on level L whose right and left subtrees are not the same. We then interchange the right and left subtrees at *every node* on level L, to obtain the tree $\mathcal{I}(T)$.

This is evidently an involution. Its only fixed point is a complete binary tree, and these exist only if $n = 2^k - 1$ for some k. $\square$

We thank Professor Curtis Greene for calling this problem to our attention.

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Center for Combinatorics, LPMC, Nankai University, Tianjin 300071, P. R. China
`<ji@nankai.edu.cn>`

Department of Mathematics, University of Pennsylvania, Philadelphia, PA 19104-6395
`<wilf@math.upenn.edu>`