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THE DISK WITH THE COLLEGE EDUCATION

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The title is somewhat exaggerated, but the calculators-or-no-calculators dilemma that haunts the teaching of elementary school mathematics is heading in the direction of college mathematics, and this article is intended as a distant early-warning signal.

I have in my home a small personal computer. About 500,000 small personal computers have been sold in this country, of which a healthy fraction are owned by individuals. I use mine primarily for word processing (this article was written on it), for writing programs that do various mathematical jobs related to my teaching or to my research, for playing games, for keeping class rolls, etc.

A new program has recently been made available for my little computer, one whose talents seem worthy of comment here because it knows calculus; in fact, as you read these words, some of

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your students may be doing their homework with it.

The program is called muMATH; it was written by the Soft Warehouse, and is distributed in the United States by Microsoft Consumer Products of Bellevue, Washington. It costs about $75 and is supplied on a 5-inch floppy disk with an (inadequate) instruction manual.

The program on the disk does numerical calculation to high precision, or symbolic manipulation of expressions. The numerical calculation, which is less important as far as this article is concerned, is in rational arithmetic and is done with 611-digit accuracy. Thus, for example, when the program is loaded, the question

?30!;

yields the instant answer

@265252859812191058636308480000000

The question

?1 + 1/2 + 1/3 + 1/4 + 1/5 + 1/6 + 1/7;

elicits

@363/140

and so forth.

But these are fairly standard calculator-type questions. The first glimmer that a nontrivial intelligence lives on the disk comes with the request for $\sqrt{12}$,

?12 ↑ (1/2);

(the up arrow means "to the power"), whence the response

@2*3 ↑ (1/2)

At least the disk has been to junior high school. The next few samples show it in grades 9–12:

?(X + 2*Y) ↑ 3;

@12*X*Y ↑ 2 + 6*X ↑ 2*Y + X ↑ 3 + 8*Y ↑ 3

?COS(5*Y);

@ −20*COS(Y) ↑ 3 + 16*COS(Y) ↑ 5 + 5*COS(Y)

(Tschebycheff polynomials, anyone?).

The disk, however, has graduated from high school. Here it is in a freshman calculus course. To differentiate $x \sin x$ with respect to x just ask

?DIF(X*SIN(X),X);

to obtain

@X*COS(X) + SIN(X)

At the risk of some eyestrain, we might even ask it to

?DIF((X ↑ 3 + COS(X)) ↑ (1/2),X);

to which it replies

(3*X ↑ 2*(X ↑ 3 + COS(X)) ↑ (1/2)/2 − (X ↑ 3 + COS(X) ↑ (1/2)*SIN(X)/2)/ (X ↑ 3 + COS(X))

In fairness to it, I should say that there are some control variables that tell it how we would like our expressions to be simplified, and I didn't experiment to find the best settings of these variables; so there probably is a way to get the answer in more elegant form.

Since we can differentiate with respect to x, it seems that the thing should know how to differentiate partially; and so we ask

?DIF((1 + T*X ↑ 2) ↑ (1/2),T);

Our burgeoning inferiority complex is promptly reinforced with

@X ↑ 2/ (2*(1 + X ↑ 2*T) ↑ (1/2))

Next we come to integration, and the picture is a bit uneven. To integrate (antidifferentiate) $1/(x+7)$ with respect to x, enter

?INT(1/ (X + 7), X);

and of course,

@LN(7 + X)

Our self-esteem recovers somewhat, though, when we note that the simple question

?INT(X*SIN(X), X)

gets no helpful response (it can't do everything) but plunges again when the answer to

?INT((1/ (3 + 4*X + 5*X ↑ 2)), X):

appears instantly as

@ATAN(5*X/11 ↑ (1/2) + 2/11 ↑ (1/2))/11 ↑ (1/2)

which is its way of saying

$$\frac{1}{\sqrt{11}} \arctan\left(\frac{5x+2}{\sqrt{11}}\right).$$

There's yet another dimension to this disk. So far we've discussed its calculator mode of operation, in which we ask just a single question and it gives just a single response, however clever. But a whole computer language called muSIMP (similar to LISP) also lives on the disk, and we can write programs in muSIMP that use all of the capabilities above plus decision-making, looping, etc. The possibilities are numerous, and here are a few of them.

Suppose we want to develop a function, given as an expression, in a Taylor series about 0. We could evaluate the expression at 0, then replace the expression by its derivative (!!) and loop back. A complete subroutine to print out the successive derivatives of a given function (EXPR) evaluated at 0 might look like this:

```
?FUNCTION SERIES (EXPR),
LOOP
L:EVSUB(EXPR,X,0),
PRINT(L),
SPACES(2),
X:'X,
EXPR:DIF(EXPR,X),
ENDLOOP,
ENDFUN$
```

Inside the LOOP, we first evaluate EXPR by replacing each X by 0 and call the result L. Then we PRINT L and two spaces. The next instruction is a technicality that we will not discuss here, and finally we replace EXPR by its derivative.

To use this, we call for

?SERIES(SIN(X)):

and get

@0 1 0 −1 0 1 0 −1

etc. More interesting is to see the sequence of Bell numbers $b(n)$ $(n = 0, 1, 2, \dots)$, where $b(n)$ is the number of partitions of a set of n elements. The exponential generating function is well known to be $\exp(\exp(x) - 1)$, and so we ask

?SERIES(#E ↑ (#E ↑ X − 1));

and, as we watch, the screen gradually fills up with

@1 1 2 5 15 52 203 877 4140 21147

(after 4 minutes). By stopping the program here, we can peek at the symbolic ninth derivative of $\exp(\exp(x) - 1)$, and so, fearing the worst, we type

? EXPR ;

Instead of reproducing the output exactly as it appeared, I'll render it into standard notation as

$$\exp(e^x - 1 + x) + 255\exp(e^x - 1 + 2x) + 3025\exp(e^x - 1 + 3x)$$
$$+7770\exp(e^x - 1 + 4x) + \cdots + 36\exp(e^x - 1 + 8x) + \exp(e^x - 1 + 9x).$$

The coefficients seem to be Stirling numbers $S(9, k)$ of the second kind, and so we are led to conjecture that the nth derivative of $\exp(\exp(x) - 1)$ is

$$\exp(\exp(x) - 1)\sum_{k=1}^{n} S(n, k)e^{kx}$$

and a proof follows quickly by induction.

For a parting volley, note that we can ask for the Taylor series in powers of x of a function that depends on x and t, say, and the output coefficients will then be functions of t, printed symbolically. Thus, to see the Legendre polynomials (times $n!$), we inquire

?SERIES((1 − 2*X*T + X ↑ 2) ↑ (−1/2));

and across the screen there go marching

@1 T −1 + 3*T ↑ 2 −9*T + 15*T ↑ 3 9 − 90*T ↑ 2 + 105*T ↑ 4

Programs that do symbolic manipulation of mathematical expressions are not new. The MACSYMA program of MIT has been doing it for years. What is new is the sudden mass availability of a program with these capabilities, and the promise that more of the same is in prospect.

This year for the first time there are widely available pocket computers: objects about the size of calculators that have thousands of bytes of memory and speak BASIC. These pocket computers are not yet quite powerful enough to handle a sophisticated program like muMATH. In a few years, though, they probably will be.

As teachers of mathematics, our responses might range all the way from a declaration that "no computers are allowed in exams or to help with homework" to the if-you-can't-lick-'em-then-join-'em approach (teach the students how to use their clever little computers).

Will we allow students to bring them into exams? Use them to do homework? How will the content of calculus courses be affected? Will we take the advice that we have been dispensing to teachers in the primary grades: that they should teach more of concepts and less of mechanics? What happens when $29.95 pocket computers can do all of the above and solve standard forms of differential equations, do multiple integrals, vector analysis, and what-have-you?

Excuse me if I don't have answers. I wanted only to raise the questions and beat a hasty retreat.