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On Dirichlet Series and Toeplitz Forms*

HERBERT S. WILF

University of Pennsylvania, Philadelphia, Pennsylvania

Submitted by Samuel Karlin

I. INTRODUCTION

Let $K(x, y)$ be nonnegative for nonnegative x and y. Then $K(x, y)$ is homogeneous of degree -1 if for every $\alpha > 0$ we have

$$K(\alpha x, \alpha y) = \alpha^{-1}K(x, y) \qquad (x, y > 0) \tag{1}$$

If $K(x, y)$ is also symmetric and decreasing we say that $K(x, y) \in \mathscr{H}$. Such a function defines an integral operator on $(1, n)$ by

$$Kf(x) = \int_1^n K(x, y) f(y)\, dy \tag{2}$$

It was shown implicitly in [1] and explicitly in [2] that the spectral theory of the operators (1), (2), and of Toeplitz integral operators

$$Gf = \int_{-A}^{A} G(x - y) f(y)\, dy \qquad (G(u) = G(-u)) \tag{3}$$

are two sides of the same coin in the sense that the kernel $K(x, y)$ of $\mathscr{H}$ on the interval $(1, n)$ has precisely the same eigenvalues as the Toeplitz kernel $K(e^{(x-y)/2}, e^{(y-x)/2})$ on the interval $(-\frac{1}{2}\log n, \frac{1}{2}\log n)$. Conversely, the Toeplitz kernel $G(x - y)$ on $(-A, A)$ has the same eigenvalues as

$$K(u, v) = \frac{1}{\sqrt{uv}}\, G\left(\log \frac{u}{v}\right) \tag{4}$$

which is evidently homogeneous of degree -1, on $(1, e^{2A})$. The symmetry of $K(x, y)$ reflects the evenness of $G(u)$. This identification permits the translation of spectral information about either class to corresponding information about the other.

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II. Spectral Density

First, it is well known that the $\mathscr{L}_2$ spectral theory of Toeplitz kernels depends on the behavior of the Fourier transform

$$F(\xi) = \int_{-\infty}^{\infty} e^{ix\xi}\, G(u)\, du \tag{5}$$

of the kernel on the real axis. Putting $G(u) = K(e^{u/2}, e^{-u/2})$ we find that

$$\begin{aligned} F(\xi) &= \int_{-\infty}^{\infty} e^{iu\xi}\, K(e^{u/2}, e^{-u/2})\, du \\ &= \int_0^{\infty} t^{-1+i\xi}\, K(t^{1/2}, t^{-1/2})\, dt \\ &= \int_0^{\infty} t^{-1/2+i\xi}\, K(t, 1)\, dt \\ &= \int_0^{\infty} t^{-s}\, K(t, 1)\, dt \qquad (s = \tfrac{1}{2} + i\xi) \end{aligned}$$

where the homogeneity of $K(x, y)$ was used. It follows that the spectral theory of $K(x, y)$ depends on the behavior of the Mellin transform of $K(t, 1)$ on the critical line, in the $\mathscr{L}_2$ case.

We mention a few applications of this idea. Let the Toeplitz kernel $G(x - y)$ have $\mathscr{L}_2$ bound M and let $0 < a < b \leqslant M$. Let $N_A(a, b)$ denote the number of eigenvalues of the operator (3) which lie in (a, b). Then Kac, Murdock, and Szegö [3] have shown that

$$\lim_{A\to\infty} \frac{N_A(a, b)}{2A} = \frac{1}{\pi} \mid E(\xi \mid a < F(\xi) < b) \mid \tag{6}$$

where $\mid E \mid$ is the measure of E. As an immediate corollary we have

Theorem 1. *Let $\mathscr{F}(s)$ denote the Mellin transform of $K(1, t)$, where $K(x, y) \in \mathscr{H}$. For $0 < \theta < 1$ let $f_n(\theta)$ denote the number of eigenvalues of the operator* (2) *which lie in the interval $(\theta M, M)$, where $M = \mathscr{F}(\frac{1}{2})$ is the bound of K. Then for fixed θ,*

$$f_n(\theta) \sim H(\theta) \log n \qquad (n \to \infty) \tag{7}$$

where

$$H(\theta) = \frac{1}{\pi} \mid E\{\xi \mid \theta\mathscr{F}(\tfrac{1}{2}) < \mathscr{F}(\tfrac{1}{2} + i\xi) < \mathscr{F}(\tfrac{1}{2})\} \mid$$

We remark that the functional equation $\mathscr{F}(s) = \mathscr{F}(1-s)$ is easily seen to hold for $\mathscr{F}(s)$ if $M < \infty$. Indeed,

$$\mathscr{F}(s) = \int_0^1 K(t, 1)\, t^{-s}\, dt + \int_0^1 K(t, 1)\, t^{s-1}\, dt$$

As an application of Theorem 1 we consider the Hilbert kernel $K(x, y) = (x+y)^{-1}$. We find

$$\mathscr{F}(s) = \pi \csc \pi s \qquad (0 < \operatorname{Re} s < 1) \tag{8}$$

and deduce

COROLLARY 1. *Let $f_n(\theta)$ be the number of eigenvalues of the equation*

$$\lambda\varphi(x) = \int_1^n \frac{\varphi(y)}{x+y}\, dy \tag{9}$$

which lie in the interval $(\theta\pi, \pi)$ $(0 < \theta < 1)$. Then

$$f_n(\theta) \sim \left(\frac{2}{\pi} \cosh^{-1} \frac{1}{\theta}\right) \log n \qquad (n \to \infty) \tag{10}$$

The Hilbert kernel happens also to be a Hankel kernel. The general theory of Hankel kernels (see [4, p. 89]) gives only that $f_n(\theta)$ is unbounded, in this case.

III. APPROACH OF EIGENVALUES TO THE UNIFORM BOUND

As a second application of the duality between kernels of $\mathscr{H}$ and Toeplitz forms we mention the rate of approach of the νth eigenvalue of (2) to M, for fixed ν, as $n \to \infty$. The case $\nu = 1$ was treated in [1] and [2] and actually converted into a theorem about matrices. By virtue of the complete identity of the spectra, however, the result, at least for integral operators, persists for all ν, and we get the following translation of a theorem of H. Widom [5] into our present language:

THEOREM 2. *Let $K(x, y) \in \mathscr{H}$, let $\mathscr{F}(s)$ be the Mellin transform of $K(t, 1)$, and let $\lambda_\nu^{(n)}$ denote the νth eigenvalue, arranged in decreasing order of size, of the problem*

$$\lambda\varphi(x) = \int_1^n K(x, y)\, \varphi(y)\, dy \tag{11}$$

Then for fixed ν and $n \to \infty$ we have

$$\lambda_\nu^{(n)} = \mathscr{F}(\tfrac{1}{2}) - \frac{\nu^2\pi^2\gamma}{(\log n)^2} + O((\log n)^{-3})$$

where (12)

$$\gamma = \int_1^\infty (\log t)^2 K(1, t)\, t^{-1/2}\, dt.$$

IV. Dirichlet Series

Here we wish to observe that the spectral theory of matrices $K(\mu, \nu)]_{\mu,\nu=1}^n$ $(K(x, y) \in \mathscr{H})$ bears the same relation to Dirichlet series as the Toeplitz sections $G(\mu - \nu)]_{\mu,\nu=0}^n$ bear to trigonometric polynomials. Indeed suppose the relation

$$\mathscr{F}(s) = \int_0^\infty t^{-s} K(t, 1)\, dt \qquad 0 < \operatorname{Re} s < 1$$

is invertible, to give

$$K(t, 1) = \frac{1}{2\pi} \int_{-\infty}^\infty \mathscr{F}(\tfrac{1}{2} + i\xi)\, t^{-1/2-i\xi}\, d\xi$$

Then we have

$$\begin{aligned} K(u, v) &= v^{-1} K\left(\frac{u}{v}, 1\right) \\ &= \frac{1}{2\pi} \int_{-\infty}^\infty \mathscr{F}(\tfrac{1}{2} + i\xi)\, u^{-1/2-i\xi} v^{-1/2+i\xi}\, d\xi \end{aligned}$$

Hence if $\{x_\nu\}_1^\infty$ is any sequence of complex numbers it follows that

$$\sum_{1 \leqslant \mu,\nu \leqslant n} \bar{x}_\mu K(\mu, \nu)\, x_\nu = \frac{1}{2\pi} \int_{-\infty}^\infty \mathscr{F}(\tfrac{1}{2} + i\xi) \left| \sum_{\nu=1}^n \frac{x_\nu}{\nu^{1/2+i\xi}} \right|^2 d\xi \tag{13}$$

an interesting identity for Dirichlet series which is the analogue of the familiar relation

$$\sum_{\mu,\nu \leqslant n} \bar{x}_\mu G(\mu - \nu)\, x_\nu = \frac{1}{2\pi} \int_{-\pi}^\pi F(\theta) \left| \sum_{\nu \leqslant n} x_\nu e^{i\nu\theta} \right|^2 d\theta \tag{14}$$

for Toeplitz forms and trigonometric polynomials. Evidently the positive definiteness of $K(x, y)$ is bound up with the positivity of $\mathscr{F}(\tfrac{1}{2} + i\xi)$, a kind of "anti Riemann-hypothesis."

In particular, from (13) and Theorem 2 we have the following inequality for Dirichlet series:

THEOREM 3. *Let $\mathscr{F}(s)$ be the Mellin transform of $K(t, 1)$ for some $K(x, y)$ of $\mathscr{H}$ and suppose $\mathscr{F}(s)$ is invertible on the critical line. Then for arbitrary complex numbers $\{x_\nu\}_1^\infty$ we have*

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \mathscr{F}(\tfrac{1}{2} + i\xi) \left| \sum_{\nu=1}^{n} \frac{x_\nu}{\nu^{1/2+i\xi}} \right|^2 d\xi \leqslant \mathscr{F}(\tfrac{1}{2}) \sum_{\nu=1}^{n} | x_\nu |^2 \tag{15}$$

As an illustration, take $x_\nu = \nu^{-1/2}$ ($\nu = 1, 2, \cdots$) and

$$K(x, y) = \{\max (x, y)\}^{-1} \tag{16}$$

Then

$$\mathscr{F}(s) = \frac{1}{s} + \frac{1}{1-s}$$

and (15) reads

$$\int_{-\infty}^{\infty} \left| \sum_{\nu=1}^{n} \frac{1}{\nu^{1+i\xi}} \right|^2 \frac{d\xi}{\xi^2 + \frac{1}{4}} \leqslant 8\pi \sum_{\nu=1}^{n} \frac{1}{\nu} = O(\log n) \quad (n \to \infty) \tag{17}$$

As an application of (13) take

$$x_\nu = \lambda(\nu)\, \nu^{-s} \qquad (\nu = 1, 2, \cdots) \tag{18}$$

where $\lambda(\nu)$ is Liouville's function

$$\lambda(p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_m^{\alpha_m}) = (-1)^{\alpha_1 + \cdots + \alpha_m}$$

Then the left side of (13) is

$$\begin{aligned}\sum_{\mu,\nu=1}^{n} \frac{\lambda(\mu)\,\lambda(\nu)}{\mu^{\bar{s}}\nu^{s}} K(\mu, \nu) &= \sum_{\mu,\nu=1}^{n} \frac{K(\mu, \nu)}{\mu^{\bar{s}}\nu^{s}} \lambda(\mu\nu) \\ &= \sum_{m=1}^{n^2} \lambda(m) \sum_{\substack{\mu \mid m \\ m/n \leqslant \mu \leqslant n}} K(\mu, m/\mu)\, \mu^{-\bar{s}} (m/\mu)^{-s} \\ &= \sum_{m=1}^{n^2} \frac{\lambda(m)}{m^s} \sum_{\substack{\mu \mid m \\ m/n \leqslant \mu \leqslant n}} K(\mu, m/\mu)\, \mu^{2it}\end{aligned}$$

Making $n \to \infty$ in (13) we find for $\operatorname{Re} s > \frac{1}{2}$,

$$\lim_{n\to\infty} \sum_{m=1}^{n^2} \frac{\lambda(m)}{m^s} \sum_{\substack{\mu|m \\ m/n \leqslant \mu \leqslant n}} K(\mu, m/\mu)\, \mu^{2it}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathscr{F}(\tfrac{1}{2} + i\xi) \left| \frac{\zeta(2s + 2s')}{\zeta(s + s')} \right|^2 d\xi \quad (s = \sigma + it,\ s' = \tfrac{1}{2} + i\xi) \tag{19}$$

If we take the Hilbert kernel, then the estimate

$$\left| \sum_{\substack{\mu|m \\ m/n \leqslant \mu \leqslant n}} \frac{\mu^{2it}}{\mu + (m/\mu)} \right| \leqslant \frac{1}{\sqrt{m}} \sum_{d|m} \frac{1}{(\mu/\sqrt{m}) + (\sqrt{m}/\mu)} \leqslant \frac{d(m)}{2\sqrt{m}}$$

justifies the interchange of limiting processes for $\sigma > \frac{1}{2}$, giving

$$\sum_{m=1}^{\infty} \frac{\lambda(m)}{m^s} \sum_{d|m} \frac{d^{2it}}{d + d'} = \int_0^{\infty} \frac{d\xi}{\cosh \pi\xi} \left| \frac{\zeta(2s + 2s')}{\zeta(s + s')} \right|^2 d\xi \tag{20}$$

where $d' = m/d$. A weaker, but still nontrivial statement is

$$0 < \sum_{m=1}^{\infty} \frac{\lambda(m)}{m^\sigma} \sum_{d|m} \frac{1}{d + d'} < \pi\zeta(2\sigma) \qquad (\sigma > \tfrac{1}{2}) \tag{21}$$

V. Open Questions

Among the many unsolved problems in this area we mention the following, which are probably arranged in increasing order of difficulty:

(a) Does Eq. (10) hold also for the Hilbert *matrix* $1/(\mu + \nu)]_{\mu,\nu=1}^n$?

(b) More generally does Theorem 1 hold for *matrices* $K(\mu, \nu)]_{\mu,\nu=1}^n$ where $K(x, y) \in \mathscr{H}$?

(c) Same as (b), for Theorem 2.

(d) What are the l_p, $\mathscr{L}_p$ generalizations of Theorem 2? Precisely, if $M_n^{(p)}$ is the $\mathscr{L}_p$ bound of $K(\mu, \nu)]_1^n$ then we know [6] that

$$M_n^{(p)} \to \int_0^{\infty} t^{-1/p} K(t, 1)\, dt = M^{(p)} \tag{22}$$

What can be said about the difference $M^{(p)} - M_n^{(p)}$? For $p = 2$ the answer turned on the behavior of the Mellin transform on the line $\sigma = \frac{1}{2}$. Equation (22) strongly suggests that in general the rate of approach will depend on the behavior of the Mellin transform on the lines $\sigma = 1/p$, $1/p'$ near the real axis.

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