The joint distribution of descent and major index over restricted sets of permutations

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Abstract

We compute the joint distribution of descent and major index over permutations of $\{1,\ldots,n\}$ with no descents in positions $\{n-i, n-i+1,\ldots,n-1\}$ for fixed $i \geq 0$. This was motivated by the problem of enumerating symmetrically constrained compositions and generalizes Carlitz's q-Eulerian polynomial.

1 Introduction

In [**?**], S. Lee and the second author of this paper consider the problem of enumerating *symmetrically constrained compositions*. This study was motivated by problems in [**?**]. These symmetrically constrained compositions are integer sequences defined by linear constraints that are symmetric in the variables. For example, the integer sequences $(\lambda_1, \lambda_2, \lambda_3)$ satisfying

$$\lambda_{\pi(1)} + \lambda_{\pi(2)} \geq \lambda_{\pi(3)} \tag{1}$$

for every permutation π of $\{1,2,3\}$, are known as *integer-sided triangles* [**?**, **?**, **?**, **?**]. However, in contrast to other treatments, we are counting the number of *ordered* solutions, a harder problem. Generalizing to n dimensions, one could ask for the integer sequences $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ satisfying the constraint (1) for every permutation π of $[n] = \{1, 2, \ldots n\}$. Or, more generally, given positive integers k, ℓ, m with $k \geq \ell$, the integer sequences $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ satisfying

$$k\lambda_{\pi(1)} + \ell\lambda_{\pi(2)} \geq m\lambda_{\pi(3)} \tag{2}$$

for every permutation π of $[n]$.

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If the constraints are symmetric in the λ_i, then the generating function

$$G(x_1, x_2, \ldots, x_n) = \sum_{\lambda} x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_n^{\lambda_n}$$

will be a symmetric function of the x_i. The work in [**?**] is to show how to exploit the symmetry to compute the generating function $G_n(q) = G(q, q, \ldots, q)$.

For example, the generating function for (2) has the following explicit form when $m = k + \ell - 1$.

Proposition 1 [**?**] *If $m = k + \ell - 1$, then for $n \geq 3$, the generating function for the solutions to* (2) *is*

$$G_n(q) \quad = \quad \frac{1}{(1 - q^n)(1 - q^{n\ell-1}) \prod_{j=1}^{n-2}(1 - q^{j+nm})} \sum_{\pi \in S_n} \prod_{i \in D(\pi)} q^{i+nb_i}, \qquad (3)$$

where $b_i = m$, $1 \leq i \leq n-2$ and $b_{n-1} = m - k$ and $D(\pi)$ is the set of descents of π:

$$D(\pi) \quad = \quad \{i \mid 1 \leq i < n \text{ and } \pi_i > \pi_{i+1}\}.$$

In order to simplify this generating function, we consider a new twist on the problem of computing the distribution of permutation statistics. For a permutation π, $\mathrm{des}(\pi) = |D(\pi)|$ is the number of descents of π and the *major index* of π is the sum of the descent positions: $\mathrm{maj}(\pi) = \sum_{i \in D(\pi)} i$. The joint distribution of $\mathrm{des}(\pi)$ and $\mathrm{maj}(\pi)$ over the set S_n of all permutations of $[n]$ is given by Carlitz's q-Eulerian polynomial [**?**, **?**]:

$$C_n(x, q) = \sum_{\pi \in S_n} x^{\mathrm{des}(\pi)} q^{\mathrm{maj}(\pi)} \quad = \quad \prod_{i=0}^{n} (1 - xq^i) \sum_{j=1}^{\infty} ([\,j\,]_q)^n x^{j-1}, \qquad (4)$$

where $[\,j\,]_q = (1 - q^j)/(1 - q)$. (This distribution was also computed in [**?**], Vol. 2, Chapter 4.)

For $i \leq n - 1$, let $S_n^{(i)}$ be the set of permutations of $[n]$ that have no descent in positions $\{n - i, n - i + 1, ..., n - 1\}$. Let $C_n^{(i)}(x, q)$ be the joint distribution of *maj* and *des* over $S_n^{(i)}$:

$$C_n^{(i)}(x, q) \quad = \quad \sum_{\pi \in S_n^{(i)}} x^{\mathrm{des}(\pi)} q^{\mathrm{maj}(\pi)}. \qquad (5)$$

Then $C_n^{(0)}(x, q) = C_n(x, q)$. Now we can express $G_n(q)$ in (3) via the following.

Proposition 2

$$\sum_{\pi \in S_n} \prod_{i \in D(\pi)} q^{i+nb_i} \quad = \quad C_n^{(1)}(q^{nm}, q) + q^{-nk}(C_n(q^{nm}, q) - C_n^{(1)}(q^{nm}, q)). \qquad (6)$$

Proof. If $\pi \in S_n$ does not have a descent in position $n - 1$, then $\pi \in S_n^{(1)}$ and

$$\begin{aligned} \sum_{\pi \in S_n^{(1)}} \prod_{i \in D(\pi)} q^{i+nb_i} \quad &= \quad \sum_{\pi \in S_n^{(1)}} \prod_{i \in D(\pi)} q^{i+nm} \\ &= \quad \sum_{\pi \in S_n^{(1)}} q^{\mathrm{maj}(\pi)} (q^{nm})^{\mathrm{des}(\pi)} \\ &= \quad C_n^{(1)}(q^{nm}, q). \end{aligned}$$

If $\pi \in S_n$ has a descent in position $n-1$, then $\pi \in S_n - S_n^{(1)}$ and

$$\begin{aligned}\sum_{\pi \in S_n - S_n^{(1)}} \prod_{i \in D(\pi)} q^{i+nb_i} &= \textstyle\sum_{\pi \in S_n - S_n^{(1)}} \quad q^{-nk} \displaystyle\prod_{i \in D(\pi)} q^{i+nm} \\ &= \quad q^{-nk} \left(\sum_{\pi \in S_n} \prod_{i \in D(\pi)} q^{i+nm} - \sum_{\pi \in S_n^{(1)}} \prod_{i \in D(\pi)} q^{i+nm} \right) \\ &= \quad q^{-nk} (C_n(q^{nm}, q) - C_n^{(1)}(q^{nm}, q)).\end{aligned}$$

Putting these two cases together gives the result. □

Further simplification of (6) requires computing $C_n^{(1)}(x,q)$. In this paper, we compute $C_n^{(i)}(x,q)$, for general i, in two ways. The first approach derives a recurrence for $C_n^{(i)}(x,q)$ and solves it in terms of Carlitz polynomials. The second, suggested by the referee, is a "P-partitions" approach.

In the last section, the results are applied to $G_n(q)$ of (3) to enumerate the sequences satisfying the symmetric constraints (2).

Throughout the paper, the following notation is used: $[n] = \{1, 2, \ldots, n\}$; $[\,n\,]_q = (1-q^n)/(1-q)$; $[\,n\,]_q! = \prod_{i=1}^{n} [\,i\,]_q$; and $(a;q)_n = \prod_{i=0}^{n-1} (1 - aq^i)$.

2 The joint distribution of maj and des over $S_n^{(i)}$

2.1 Recursive approach

In this section, we give a recursive approach to the problem of computing the joint distribution of *inv* and *maj* over the permutations of $[n]$ with no descent in positions $\{n-1, n-2, \ldots, n-i\}$.

A standard approach for counting the number of permutations with k descents is to derive a recurrence (see, for example, Bóna [**?**], Theorem 1.7). What we need is a q-analog of this count, refined to consider only permutations in $S_n^{(i)}$.

Proposition 3 *Define* $e_{n,k}^{(i)}(q)$ *by*

$$\sum_{k=0}^{n-1} e_{n,k}^{(i)}(q) x^k \quad = \quad \sum_{\pi \in S_n^{(i)}} x^{\mathrm{des}(\pi)} q^{\mathrm{maj}(\pi)} \quad = \quad C_n^{(i)}(x,q).$$

Then $e_{n,k}^{(i)}(q)$ *satisfies*

$$e_{n,k}^{(i)}(q) \quad = \quad q[\,k\,]_q e_{n-1,k}^{(i)}(q) + e_{n-1,k}^{(i-1)}(q) + ([\,n-i\,]_q - [\,k\,]_q) e_{n-1,k-1}^{(i)}(q),$$

with initial conditions $e_{n,k}^{(-1)} = e_{n,k}^{(0)}$; $e_{n,0}^{(i)} = 1$; *and* $e_{n,k}^{(i)} = 0$ *if* $k \geq n-i$.

Proof. We get a permutation in $S_n^{(i)}$ with k descents by inserting n into

(i) a permutation in $S_{n-1}^{(i)}$ with k descents, immediately following a descent; or

(ii) a permutation in $S_{n-1}^{(i-1)}$ at the end of the permutation; or

(iii) a permutation in $S_{n-1}^{(i)}$ with $k-1$ descents, immediately following any of the positions $0, 1, \ldots, n-i-2$ that are not descents (leaving a total of $(n-i-1)-(k-1) = n-k-i$ positions for inserting n).

In (i) above, if n is inserted after the ith descent, maj increases by 1 for that descent and for every later descent, i.e., by $k+1-i$. This gives the first term in the recurrence, with its factor $q + q^2 + \cdots q^k$.

In (ii) above, if n is inserted at the end of the permutation, maj does not increase at all, giving the second term.

To see how (iii) gives rise to the third term in the recurrence, let $\pi = \pi(1)\pi(2)\ldots\pi(n-1)$ be a permutation in $S_{n-1}^{(i)}$ with $k-1$ descents. Let $0 = j_1 < j_2 < \cdots < j_{n-k-i} \leq n-i-2$ be the $n-k-i$ positions where n can be inserted into π to create a permutation in $S_n^{(i)}$ with k descents. We claim that inserting n immediately following any of $j_1, j_2, \ldots, j_{n-k-i}$ increases maj by $k, k+1, \ldots, n-i-1$, respectively. This will give the third term of the recurrence with its factor $q^k + q^{k+1} + \cdots + q^{n-i} = [\,n-i\,]_q - [\,k\,]_q$.

To prove this claim, let t_ℓ be the number of descents of π that are greater than j_ℓ. Inserting n just after π_{j_ℓ} creates a descent in position $j_\ell + 1$ and increases by 1 the position of each of the t_ℓ descents in π that are greater than j_ℓ. Thus maj increases by

$$m = j_\ell + 1 + t_\ell.$$

Let $d = j_{\ell+1} - j_\ell$. Then all of the positions $j_\ell+1, j_\ell+2, \ldots, j_\ell+d-1$ are descents. So $t_{\ell+1} = t_\ell - (d-1)$ and inserting n just after $\pi(j_{\ell+1})$ increases maj by

$$(j_{\ell+1} + 1) + t_{\ell+1} = (d + j_\ell + 1) + (t_\ell - (d-1)) = m + 1.$$

Since inserting n at the front of π would increase maj by $1 + (k-1) = k$) the claim is proved. □

From Proposition 3, we can derive a recurrence for $C_n^{(i)}(x,q)$. First, the $i = 0$ case gives the following recurrence for $C_n(x,q)$. It is straighforward to verify that (4) satisfies the recurrence, giving a simple proof of Carlitz's formula.

Proposition 4

$$C_n(x,q) = \frac{1-xq^n}{1-q} C_{n-1}(x,q) - \frac{q(1-x)}{1-q} C_{n-1}(qx,q), \tag{7}$$

with $C_0(x,q) = 1$.

Proof. Let $e_{n,k}(q) = e_{n,k}^{(0)}(q)$. Since $S_n^{(-1)} = S_n^{(0)}$, setting $i = 0$ in Proposition 3 gives

$$e_{n,k}(q) = [\,k+1\,]_q e_{n-1,k}(q) + ([\,n\,]_q - [\,k\,]_q) e_{n-1,k-1}(q) \tag{8}$$

(which also appears in [**?**]). Multiply (8) by x^k and sum over k. Substitute the definition of $[\,..]_q$ and (7) results immediately. □

Similarly, from Proposition 3, we get a recurrence for general i.

Proposition 5 *For $n \geq 0$ and $0 \leq i < n$,*

$$C_n^{(i)}(x,q) = \left(\frac{q - xq^{n-i}}{1-q}\right) C_{n-1}^{(i)}(x,q) - \left(\frac{q(1-x)}{1-q}\right) C_{n-1}^{(i)}(xq,q) + C_{n-1}^{(i-1)}(x,q),$$

with $C_n^{(-1)}(x,q) = C_n^{(0)}(x,q) = C_n(x,q)$, and $C_n^{(i)}(x,q) = 1$ if $i \geq n$.

To solve the recurrence of Proposition 5, first observe that it can be simplified as follows.

Proposition 6 *For $i > 0$ and $n \geq i$*

$$C_n^{(i)}(x,q) = \frac{C_n^{(i-1)}(x,q) - \binom{n}{i} xq^{n-i} C_{n-i}(x,q)}{1 - xq^{n-i}}. \tag{9}$$

Proof. For $n = i$ the proposition is true, as both sides are equal to 1. For $n > i$, apply the recurrence of Proposition 5 and then induction:

$$\begin{aligned}
C_n^{(i)}(x,q) &= \frac{q}{1-q}\Big((1-xq^{n-i-1})C_{n-1}^{(i)}(x,q) - (1-x)C_{n-1}^{(i)}(xq,q)\Big) + C_{n-1}^{(i-1)}(x,q) \\
&= \frac{q(1-xq^{n-i-1})}{1-q} \frac{C_{n-1}^{(i-1)}(x,q) - \binom{n-1}{i} xq^{n-i-1} C_{n-i-1}(x,q)}{1-xq^{n-i-1}} \\
&\quad - \frac{q(1-x)}{1-q} \frac{C_{n-1}^{(i-1)}(xq,q) - \binom{n-1}{i} xq^{n-i} C_{n-i-1}(xq,q)}{1-xq^{n-i}} + C_{n-1}^{(i-1)}(x,q).
\end{aligned}$$

Rearranging terms,

$$\begin{aligned}
C_n^{(i)}(x,q) &= \frac{q}{1-q} \frac{(1-xq^{n-i})C_{n-1}^{(i-1)}(x,q) - (1-x)C_{n-1}^{(i-1)}(xq,q)}{1-xq^{n-i}} + \frac{C_{n-1}^{(i-2)}(x,q)}{1-xq^{n-i}} \\
&\quad + C_{n-1}^{(i-1)}(x,q) - \frac{C_{n-1}^{(i-2)}(x,q)}{1-xq^{n-i}} \\
&\quad - \frac{\binom{n-1}{i} xq^{n-i}}{1-q} \frac{(1-xq^{n-i})C_{n-i-1}(x,q) - q(1-x)C_{n-i-1}(xq,q)}{1-xq^{n-i}}.
\end{aligned}$$

Apply Proposition 5 to the first line, the induction hypothesis to the second line, and Proposition 7 to the last line to obtain

$$\begin{aligned}
C_n^{(i)}(x,q) &= \frac{C_n^{(i-1)}(x,q)}{1-xq^{n-i}} - \frac{\binom{n-1}{i-1} xq^{n-i} C_{n-i}(x,q)}{1-xq^{n-i}} - \frac{\binom{n-1}{i} xq^{n-i} C_{n-i}(x,q)}{1-xq^{n-i}}. \\
&= \frac{C_n^{(i-1)}(x,q) - \binom{n}{i} xq^{n-i} C_{n-i}(x,q)}{1-xq^{n-i}}.
\end{aligned}$$

□

Finally, iterating the recurrence (9), we can solve for $C_n^{(i)}(x,q)$ in terms of Carlitz functions.

Theorem 1

$$C_n^{(i)}(x,q) = \frac{C_n(x,q)}{(xq^{n-i};q)_i} - \sum_{k=1}^{i} \binom{n}{k} xq^{n-k} \frac{C_{n-k}(x,q)}{(xq^{n-i};q)_{i-k+1}}. \tag{10}$$

In particular, the motivating problem of computing $C_n^{(1)}(x,q)$ is solved:

Corollary 1

$$C_n^{(1)}(x,q) \quad = \quad \frac{C_n(x,q) - nxq^{n-1}C_{n-1}(x,q)}{1-xq^{n-1}}. \tag{11}$$

Substituting (4) into (10) and simplifying gives $C_n^{(i)}(x,q)$ explicitly:

Corollary 2

$$C_n^{(i)}(x,q) = (x;q)_{n-i} \sum_{j\geq 1} \left((1-xq^n)[\, j\,]_q^n x^{j-1} - x^j \sum_{\ell=n-i}^{n-1} \binom{n}{\ell} (q[\, j\,]_q)^\ell \right).$$

Remark. Let $C_n^{(i)}(q) = C_n^{(i)}(1,q)$. It is easy to see from Proposition 5 that $C_n^{(-1)}(q) = C_n^{(0)}(q)$ and that for $i \geq 0$, $C_n^{(i)}(q) = C_{n-1}^{(i-1)}(q) + q[\, n-i-1\,]_q C_{n-1}^{(i)}(q)$ if $n > i$ and $C_n^{(i)}(q) = 1$ otherwise. Therefore we have

Corollary 3

$$C_n^{(i)}(q) = \sum_{\pi \in S_n^{(i)}} q^{\mathrm{maj}(\pi)} = \prod_{j=1}^{n-i-1} [\, j\,]_q \sum_{k=1}^{n-i} \binom{k+i-1}{i} q^{k-1} = [\, n-i-1\,]_q! \sum_{k=1}^{n-i} \binom{k+i-1}{i} q^{k-1}.$$

Compare this with the distribution over all permutations: $C_n^{(0)}(q) = [\, n\,]_q!$.

2.2 Direct approach

Now we give a direct approach to the problem of computing the joint distribution of *inv* and *maj* over the permutations of $[n]$ with no descent in positions $\{n-1, n-2, \ldots, n-i\}$.

Theorem 2

$$C_n^{(i)}(x,q) = (x;q)_{n-i} \sum_{j=1}^{n-i} \binom{n-j}{i} q^{n-i-j} \sum_{k=0}^{\infty} [\, k+1\,]_q^{j-1} [\, k\,]_q^{n-i-j} x^k.$$

Proof. Let $S_n^{(i,j)}$ be the set of permutations π of $[n]$ with no descents in positions $\{n-i, n-i+1, \ldots, n-1\}$ with the additional property that $\pi_{n-i} = j$. To obtain a permutation in $S_n^{(i,j)}$ we first choose a subset $I \subseteq [n]$ of cardinality i to be the values of $\pi_{n-i+1}, \pi_{n-i+2}, \ldots, \pi_n$. Since $j = \pi_{n-i} < \pi_{n-i+1} < \ldots < \pi_n$ every element of I must be greater than j, so I must be a subset of $\{j+1, j+2, \ldots, n\}$, and $(\pi_{n-i+1}, \pi_{n-i+2}, \ldots, \pi_n)$ must consist of the elements of I arranged in increasing order. Then $(\pi_1, \pi_2, \ldots, \pi_{n-i-1})$ may be an arbitrary permutation of $[n] \setminus (I \cup \{j\})$.

Since π has no descents in positions $\{n-i, n-i+1, \ldots, n-1\}$, the descent number and major index of π are the same as for $(\pi_1, \pi_2, \ldots, \pi_{n-i})$, and these statistics are unchanged if we replace $(\pi_1, \pi_2, \ldots, \pi_{n-i})$ with the permutation of $[n-i]$ in which the entries have the same relative order. (i.e., we replace $(\pi_1, \pi_2, \ldots, \pi_{n-i})$ with its "pattern".) Note that this replacement does not change $\pi_{n-i} = j$, since $[j] \subseteq [n] \setminus I$ (so $1, 2, \ldots, j$ all occur in $(\pi_1, \pi_2, \ldots, \pi_{n-i})$.) Now let

$$A_m^{(j)}(x,q) = \sum_{\sigma} x^{\mathrm{des}(\sigma)} q^{\mathrm{maj}(\sigma)},$$

where the sum is over all permutations σ of $[m]$ with $\sigma_m = j$. Then the contribution to $C_n^{(i)}(x,q)$ from permutations π with $\pi_{n-i} = j$ corresponding to a given i-subset $I \subseteq \{j+1, j+2, \ldots, n\}$ is $A_{n-i}^{(j)}(x,q)$ independent of the choice of I. There are $\binom{n-j}{i}$ possible choices for such a subset I, so summing on j gives

$$C_n^{(i)}(x,q) = \sum_{j=1}^{n-i} \binom{n-j}{i} A_{n-i}^{(j)}(x,q).$$

We now derive a formula for $A_m^{(j)}(x,q)$ using the "P-partition" approach [**?**, **?**]. This relies on the observation that nonnegative integer sequences $(a_1, \ldots, a_n)$ in which $a_i \leq k$, $1 \leq i \leq n$ have generating function:

$$\sum_{\max(a) \leq k} q^{|a|} = [\,k+1\,]_q^n,$$

where $|a| = a_1 + \cdots + a_n$.

The approach is to count nonnegative integer sequences $(a_1, \ldots, a_n)$ such that $a_j = 0$ and $a_i > 0$ for $i > j$ in two different ways.

First way: First count those in which $a_i \leq k$, $1 \leq i \leq n$:

$$\sum_{\max(a) \leq k} q^{|a|} = [\,k+1\,]_q^{j-1} (q[\,k\,]_q)^{n-j},$$

where $|a| = a_1 + \cdots + a_n$. Then multiply by x^k and sum over k:

$$\sum_{k=0}^{\infty} \sum_{\max(a) \leq k} q^{|a|} x^k = q^{n-j} \sum_{k=0}^{\infty} [\,k+1\,]_q^{j-1} [\,k\,]_q^{n-j} x^k.$$

If we only want those a such than $\max(a) = k$:

$$\sum_{a} q^{|a|} x^{\max(a)} = \sum_{k=0}^{\infty} \sum_{\max(a)=k} q^{|a|} x^k = (1-x) q^{n-j} \sum_{k=0}^{\infty} [\,k+1\,]_q^{j-1} [\,k\,]_q^{n-j} x^k. \tag{12}$$

Second way: For a permutation π, let $D(\pi)$ be the descent set of π. Now, let π be the unique permutation of $[n]$ satisfying: (i) $a_{\pi(1)} \geq a_{\pi(2)} \geq \ldots \geq a_{\pi(n)}$ and (ii) $a_{\pi(i)} > a_{\pi(i+1)}$ when $i \in D(\pi)$. As $a_j = 0$ and $a_i > 0$ for $i > j$. This implies that $\pi(n) = j$.

Let $\lambda_i = a_{\pi(i)} - a_{\pi(i+1)}$ for $1 \leq i < n$.

Then $(a_1, \ldots a_n) \leftrightarrow (\pi, \lambda)$ is a bijection between nonnegative integer sequences of length n such that $a_j = 0$ and $a_i > 0$ for $i > j$ and pairs (π, λ) where π is a permutation of n with $\pi(n) = j$ and λ is a nonnegative integer sequence of length $n-1$ satisfying $\lambda_i > \lambda_{i+1}$ when $i \in D(\pi)$. Then $\sum_{i=1}^n \lambda_i = \max(a)$ and $|a| = \sum_{i=1}^n i\lambda_i$. So,

$$\begin{aligned}
\sum_a q^{|a|} x^{\max(a)} &= \sum_{\{\pi \in S_n \,:\, \pi(n)=j\}} \sum_{\{(\lambda_1,\ldots,\lambda_{n-1}) \,:\, \mathrm{des}(\lambda)=\mathrm{des}(\pi)\}} q^{\sum_{i=1}^n i\lambda_i} x^{\sum_{i=1}^n \lambda_i} \\
&= \sum_{\{\pi \in S_n : \pi(n)=j\}} q^{\mathrm{maj}(\pi)} x^{\mathrm{des}(\pi)} \sum_{(\lambda'_1,\ldots,\lambda'_{n-1})} \prod_{i=1}^n (xq^i)^{\lambda'_i} \\
&= \frac{\sum_{\{\pi \in S_n : \pi(n)=j\}} q^{\mathrm{maj}(\pi)} x^{\mathrm{des}(\pi)}}{(1-xq)(1-xq^2)\cdots(1-xq^{n-1})}.
\end{aligned}$$

Equating this with (12) gives the desired formula for $A_m^{(j)}(x,q)$, namely,

$$A_m^{(j)}(x,q) = \sum_{\{\pi \in S_n : \pi(n)=j\}} q^{\mathrm{maj}(\pi)} x^{\mathrm{des}(\pi)} = (x;q)_n q^{n-j} \sum_{k=0}^{\infty} [\,k+1\,]_q^{j-1} [\,k\,]_q^{n-j} x^k.$$

□

This also establishes the identity obtained by equating expressions for $C_n^{(i)}(x,q)$ in Corollary 2 and Theorem 2.

3 Application

We apply the results of Section 2.1 to enumerate the sequences satisfying the constraints (2).

Theorem 3

$$G_n(q) = \frac{C_n(q^{nm},q)(1-q^{n\ell-1}) - C_{n-1}(q^{nm},q)nq^{nm+n-1}(1-q^{-nk})}{(1-q^n)(1-q^{n\ell-1})(q^{nm+1};q)_{n-1}}. \qquad (13)$$

Proof. From Proposition 1,

$$G_n(q) = \frac{1}{(1-q^n)(1-q^{n\ell-1})(q^{nm+1};q)_{n-2}} \sum_{\pi \in S_n} \prod_{i \in D(\pi)} q^{i+nb_i}, \qquad (14)$$

where $b_i = m$, $1 \le i \le n-2$ and $b_{n-1} = m-k$, and $m = k+\ell-1$.

Start with (6), apply (11) and rearrange terms to get:

$$\begin{aligned}
\sum_{\pi\in S_n}\prod_{i\in D(\pi)} q^{i+nb_i} &= C_n^{(1)}(q^{nm},q)+q^{-nk}(C_n(q^{nm},q)-C_n^{(1)}(q^{nm},q))\\
&= (1-q^{-nk})C_n^{(1)}(q^{nm},q)+q^{-nk}C_n(q^{nm},q)\\
&= \frac{(1-q^{-nk})(C_n(q^{nm},q)-nq^{nm+n-1}C_{n-1}(q^{nm},q))}{(1-q^{nm+n-1})}+q^{-nk}C_n(q^{nm},q)\\
&= C_n(q^{nm},q)\frac{(1-q^{-nk})+q^{-nk}(1-q^{nm+n-1})}{(1-q^{nm+n-1})}-C_{n-1}(q^{nm},q)\frac{nq^{nm+n-1}(1-q^{-nk})}{(1-q^{nm+n-1})}\\
&= C_n(q^{nm},q)\frac{(1-q^{-nk+nm+n-1})}{(1-q^{nm+n-1})}-C_{n-1}(q^{nm},q)\frac{nq^{nm+n-1}(1-q^{-nk})}{(1-q^{nm+n-1})}\\
&= C_n(q^{nm},q)\frac{(1-q^{n\ell-1})}{(1-q^{nm+n-1})}-C_{n-1}(q^{nm},q)\frac{nq^{nm+n-1}(1-q^{-nk})}{(1-q^{nm+n-1})},
\end{aligned}$$

the last since $m=k+\ell-1$. Substitution into (14) gives the result. □

For example, since $C_2(x,q)=1+xq$ and $C_3(x,q)=1+2xq+2xq^2+x^2q^3$, solutions to

$$\{k\lambda_{\pi(1)}+\ell\lambda_{\pi(2)}\geq(k+\ell-1)\lambda_{\pi(3)},\ \mid \pi\in S_3\}$$

are given by:

$$G_3(q)=\frac{1+2q^{3\ell-1}+2q^{3(k+\ell)-2}+q^{6\ell+3(k-1)}}{(1-q^3)(1-q^{3\ell-1})(1-q^{3k+3\ell-2})}.$$

When $k=\ell$, this becomes:

$$G_3(q)=\frac{1+2q^{3k-1}+2q^{6k-2}+q^{9k-3}}{(1-q^3)(1-q^{3k-1})(1-q^{6k-2})}$$

and when $k=1$, this gives the generating function for (ordered) integer-sided triangles:

$$G_3(q)=\frac{1+2q^2+2q^4+q^6}{(1-q^3)(1-q^2)(1-q^4)}=\frac{1-q+q^2}{(1-q^2)^2(1-q)}.$$

When $n=4$, solutions to

$$\{k\lambda_{\pi(1)}+\ell\lambda_{\pi(2)}\geq(k+\ell-1)\lambda_{\pi(3)},\ \mid \pi\in S_4\}$$

are given by:

$$G_4(q)=\frac{1+3q^{4\ell-1}+3q^{4k+4\ell-3}+5q^{4k+4\ell-2}+5q^{8\ell+4k-4}+3q^{8\ell+4k-3}+3q^{8k+8\ell-5}+q^{12\ell+8k-6}}{(1-q^4)(1-q^{4\ell-1})(1-q^{4k+4\ell-3})(1-q^{4k+4\ell-2})}.$$

When $k=\ell$, this becomes:

$$G_4(q)=\frac{1+3q^{4k-1}+3q^{8k-3}+5q^{8k-2}+5q^{12k-4}+3q^{12k-3}+3q^{16k-5}+q^{20k-6}}{(1-q^4)(1-q^{4k-1})(1-q^{8k-3})(1-q^{8k-2})}.$$

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