MATHEMATICS

ON A CONJECTURE OF RYSER AND MINC

BY

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1. *Introduction*

Let A be an $n \times n$ matrix of zeros and ones, and suppose that A has r_i ones in its i^{th} row ($i=1, 2, \ldots, n$). It has been conjectured [2] that for the permanent of A we have the inequality

$$\text{Per } A \leqslant \prod_{i=1}^{n} (r_i!)^{1/r_i}. \tag{1}$$

The purpose of this paper is to prove

Theorem 1: *There is a universal constant* $\tau = .136708\ldots$ *such that*

$$\text{Per } A \leqslant \prod_{i=1}^{n} \{(r_i!)^{1/r_i} + \tau\}. \tag{2}$$

Previous work on inequalities of the form

$$\text{Per } A \leqslant \prod_{i=1}^{n} \varphi(r_i) \tag{3}$$

has been done by MINC [2] who showed that $\varphi(x) = \frac{1}{2}(x+1)$ is admissible in (3), MINC [3] who found $\varphi(x) = (1+\sqrt{2})^{-1}(x+\sqrt{2})$ for (3), JURKAT and RYSER [4], WILF [5], and others, but no previous $\varphi(x)$ has given even the correct first term of the asymptotic behaviour of

$$x!^{1/x} = \frac{x}{e} + \frac{\log x}{2e} + \frac{\log \sqrt{2\pi}}{e} + o(1) \qquad (x \to \infty) \tag{4}$$

whereas our theorem 1 above correctly gives the first two terms.

2. *The function* $\varphi(n)$

Let φ be a fixed function of the positive integers, and suppose (3) holds for all matrices A of zeros and ones of order $\leqslant n-1$. Now let A be an $n \times n$ matrix of this type, with row sums $r_1, \ldots, r_n$. If any $r_i = 1$, then (3) holds for A provided only that $\varphi(1) = 1$. Otherwise, suppose the rows and columns of A have been permuted, if necessary, so that the

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ones in the first column occur in the first c rows. Expanding by minors down the first column we find

$$\begin{aligned}\operatorname{Per} A &= \sum_{i=1}^{c} \operatorname{Per}(A^{i1}) \\ &\leqslant \sum_{i=1}^{c} \{ \prod_{\substack{k=1\\k\neq i}}^{c} \varphi(r_k-1) \prod_{k=c+1}^{n} \varphi(r_k)\} \\ &= \sum_{i=1}^{c} \frac{1}{\varphi(r_i-1)} \prod_{k=1}^{c} \varphi(r_k-1) \prod_{k=c+1}^{n} \varphi(r_k).\end{aligned}$$

Comparing this expression with the right side of (3) we see that in order to prove (3) for A it is enough to exhibit a function φ such that

$$\sum_{i=1}^{c} \frac{1}{\varphi(r_i-1)} \prod_{k=1}^{c} \frac{\varphi(r_k-1)}{\varphi(r_k)} \leqslant 1 \tag{5}$$

for all positive integers $c, r_1, \ldots, r_c$.

Consider the function φ which is recursively defined by

$$\begin{cases} \text{(a)} \quad \varphi(1)=1 \\ \text{(b)} \quad \varphi(n+1)=\varphi(n)e^{1/e\varphi(n)}. \end{cases} \tag{6}$$

For this φ, the left side of (5) is

$$\begin{aligned}&\sum_{i=1}^{c} \frac{1}{\varphi(r_i-1)} \prod_{k=1}^{c} e^{-1/e\varphi(r_k-1)} \\ &= \left\{\sum_{i=1}^{c} \frac{1}{\varphi(r_i-1)}\right\} \exp\left\{-\frac{1}{e} \sum_{k=1}^{c} \frac{1}{\varphi(r_k-1)}\right\} \\ &= xe^{-x/e} \\ &\leqslant \max_{x\geqslant 0} xe^{-x/e} \\ &= 1.\end{aligned}$$

It follows that the function φ of (6) is admissible for the inequality (3). The remainder of this paper is devoted to a close study of the recurrance (6) with a view to establishing the relations

$$\varphi(n) = \frac{n}{e} + \frac{\log n}{2e} + \frac{A}{e} + o(1) \qquad (n \to \infty) \tag{7}$$

and

$$\varphi(n) \leqslant n!^{1/n} + \frac{A - \log\sqrt{2\pi}}{e} \quad (\text{all } n \geqslant 1). \tag{8}$$

3. *Asymptotic behaviour of* φ.

We remark first that putting $b_n = (e\varphi(n))^{-1}$ in (6) (b) yields

$$b_{n+1} = b_n e^{-b_n} = F(b_n)$$

where $F(x)=xe^{-x}$. The asymptotic relation (7) then follows from well-known theorems about the successive iterates of functions F which have the form

$$F(x)=x-ax^2+\dots \qquad (a>0)$$

near the origin (see, *e.g.*, [6]). Nonetheless, we prove (7) independently of those results.

First, from (6b),

$$\varphi(n+1)\geqslant\varphi(n)+\frac{1}{e}$$

and so

(a) $$\varphi(n)\geqslant\frac{n}{e}.$$

Next, from (6b),

$$\varphi(n+1)\leqslant\varphi(n)+\frac{1}{e}+\frac{1}{e^2\varphi(n)}$$

which, with (9), yields

(10) $$\varphi(n)\leqslant\frac{n}{e}+O(\log n) \qquad (n\to\infty).$$

Now write

(11) $$\begin{cases} H(x)=x\left\{e^{1/x}-1-\dfrac{1}{x}-\dfrac{1}{2x^2}\right\} \\ \qquad =O(x^{-2}) \qquad (x\to 0). \end{cases}$$

Then, with $y_n=e\varphi(n)$, we have

$$y_{n+1}=y_n+1+\frac{1}{2y_n}+H(y_n).$$

Summing,

(11a) $$\begin{cases} y_n=y_1+n-1+\frac{1}{2}\sum\limits_1^{n-1}\dfrac{1}{y_k}+\sum\limits_1^{n-1}H(y_k) \\ \quad =n+e-1+\frac{1}{2}\sum\limits_1^{n-1}\dfrac{1}{k}+\frac{1}{2}\sum\limits_1^{n-1}\left(\dfrac{k-y_k}{ky_k}\right)+\sum\limits_1^{n-1}H(y_k) \\ \quad =n+e-1+\frac{1}{2}\log n+\dfrac{\gamma}{2}+\frac{1}{2}\sum\limits_1^{\infty}\left(\dfrac{k-y_k}{ky_k}\right)+\sum\limits_1^{\infty}H(y_k)+o(1) \\ \quad =n+\frac{1}{2}\log n+A+o(1) \end{cases}$$

which proves (7), with

(11b) $$A=e-1+\frac{\gamma}{2}+\frac{1}{2}\sum_1^{\infty}\left(\frac{k-y_k}{ky_k}\right)+\sum_1^{\infty}H(y_k).$$

4. *Proof of* (8)

We claim that the differences

$$\{\varphi(n) - n!^{1/n}\}_1^\infty$$

increase monotonically. In view of (7) and (4), this will establish (8). We have first,

$$(12)\quad \begin{cases} (n+1)!^{1/n+1} - n!^{1/n} = \int_n^{n+1} \frac{d}{dx}\{\Gamma(x+1)\}^{1/x}\,dx \\ \qquad = \int_n^{n+1} \Gamma(x+1)^{1/x}\left\{\frac{1}{x}\,\frac{\Gamma'(x+1)}{\Gamma(x+1)} - \frac{1}{x^2}\log\Gamma(x+1)\right\}dx. \end{cases}$$

From [1], p. 18, eq. (27),

$$(13)\quad \begin{cases} \frac{\Gamma'(x+1)}{\Gamma(x+1)} \leqslant \log(x+1) - \frac{1}{2(x+1)} \\ \qquad \leqslant \log x + \frac{1}{2x} + \frac{1}{2x(x+1)}. \end{cases}$$

From [1], p. 22, eq. (9),

$$(14)\quad \begin{cases} \log\,\Gamma(x+1) \geqslant \left(x+\frac{1}{2}\right)\log(x+1) - x - 1 + C \\ \qquad\qquad (C = \log\sqrt{2\pi}) \\ \qquad \geqslant x\log x - x + \frac{1}{2}\log\,x + C - \frac{1}{4x^2}. \end{cases}$$

The quantity in braces in (12) is therefore

$$\{\quad\} \leqslant \frac{1}{x} - \frac{\log x}{2x^2} + \frac{\frac{1}{2} - C}{x^2} + \frac{1}{x^3} \qquad (x \geqslant 2).$$

Next, from [1], p. 21, eq. (8),

$$\log\Gamma(x+1) \leqslant \left(x+\frac{1}{2}\right)\log(x+1) - x - 1 + C + \frac{K}{x}$$

where

$$K = \max_{t>0} \frac{1}{t}\left|\frac{1}{e^t-1} - \frac{1}{t} + \frac{1}{2}\right|$$

and after some calculation, one finds that

$$(15)\quad \begin{cases} \Gamma(x+1)^{1/x} \leqslant \frac{x}{e} + \frac{\log x}{2e} + \frac{C}{e} + \frac{(\log x)^2}{8ex} + \frac{C\log x}{2ex} + \frac{A_2}{x} \\ \qquad\qquad (x > x_0) \end{cases}$$

where x_0, A_2 are any pair such that

$$A_2 \geqslant \max_{x \geqslant x_0} \left\{ \frac{A_1}{e} + \frac{C (\log x)^2}{8ex} + \frac{A_1 \log x}{2ex} + \frac{A_1 (\log x)^2}{4ex^2} + \frac{(\log x)^3}{8ex} \left[1 + \frac{C}{x} + \frac{A_1}{x^2} \right] \right\}$$

and $A_1 = 2K + 1$.

Substituting (15), (14) in (12),

$$(16) \quad \begin{cases} (n+1)!^{1/n+1} - n!^{1/n} \leqslant \int_n^{n+1} \left\{ \frac{1}{e} + \frac{1}{2ex} - \frac{(\log x)^2}{8ex^2} - \frac{(C - \frac{1}{2}) \log x}{2ex^2} + \frac{A_3}{x^2} \right\} dx \\ \qquad (n \geqslant x_0, x_1) \end{cases}$$

where x_1, A_3 are any pair such that

$$A_3 \geqslant \max_{x \geqslant x_1} \left\{ \left(A_2 + \frac{1}{e} \right) + \frac{C(\frac{5}{2} - C)}{2ex} + \frac{(\log x)^2}{8ex^2} + \frac{A_2}{x^2} + \frac{C \log x}{2ex^2} \right\}$$

Then (16) gives

$$(17) \quad \begin{cases} (n+1)!^{1/n+1} - n!^{1/n} \leqslant \frac{1}{e} + \frac{1}{2en} - \frac{(\log n)^2}{8en^2} - \left(\frac{C - \frac{1}{2}}{2e} \right) \frac{\log n}{n^2} + \frac{A_4}{n^2} \\ \qquad (n \geqslant n_2) \end{cases}$$

where n_2, A_4 are any pair such that $n \geqslant n_2$ implies

$$- \frac{[\log (n+1)]^2}{8e(n+1)^2} - \frac{C - \frac{1}{2}}{2e} \frac{\log (n+1)}{(n+1)^2} + \frac{A_3}{n^2} \leqslant - \frac{(C - \frac{1}{2})}{2e} \frac{\log n}{n^2} + \frac{A_4}{n^2} - \frac{(\log n)^2}{8en^2}.$$

Now from (6b),

$$\varphi(n+1) \geqslant \varphi(n) + \frac{1}{e} + \frac{1}{2e^2 \varphi(n)}$$

and since

$$\varphi(n) \leqslant \frac{n}{e} + \frac{\log n}{2e} + A_5$$

we find that

$$(18) \qquad \varphi(n+1) - \varphi(n) \geqslant \frac{1}{e} + \frac{1}{2ne} + \frac{\log n}{4en^2} - \frac{A_5}{2n^2}$$

Subtracting (17) from (18),

$$[\varphi(n+1)-(n+1)!^{1/n+1}]-[\varphi(n)-n!^{1/n}]$$

$$\geqslant \frac{(\log n)^2}{8en^2}-\frac{2C-1}{4e}\,\frac{\log\ n}{n^2}-\frac{A_5+2A_4}{n^2}$$

$$\geqslant 0$$

for $n \geqslant n^*$. By simple estimations, one can take, successively, $K=1/12$, $(x_0, A_2)=(1, 1.5)$, $(x_1, A_3)=(1, 1.9)$, $(n_2, A_4)=(1, 2)$, $A_5=1$, and finally $n^*=e^{12}<5000$, which proves the monotonicity of the sequence for $n \geqslant 5000$. By actual calculation [1] one observes the monotonicity for $n \leqslant 5000$ also, completing the proof.

5. *Conclusion*

Since $\{\varphi(n)-n!^{1/n}\}\uparrow$ we have from (7) and (4) the desired inequality (8). By computation we found, for example

$$\varphi(99{,}000)=36{,}422.65517926\ldots.$$

$$(99{,}000)!^{1/99{,}000}=36{,}422.5186529\ldots.$$

We remark that the constant A can be exhibited as the solution of a certain functional equation, as is characteristic of problems of functional iteration. Suppose we let $\varphi(n, \xi)$ denote the solution of the recurrence (6) (b) with the starting value $\varphi(1, \xi)=\xi$, in place of (6) (a). One has again the formula (7),

$$\varphi(n, \xi)=\frac{n}{e}+\frac{\log\ n}{2e}+\frac{A(\xi)}{e}+o(1) \qquad (n\to\infty).$$

Now, since

$$\varphi(n+1, \xi)=\varphi(n, \xi e^{1/e\xi})$$

there follows

$$A(\xi e^{1/e\xi})=1+A(\xi). \tag{19}$$

Thus, our constant A is $A(1)$, where $A(\xi)$ is a solution of (19). We remark finally that the method will not prove the conjecture in the sense that the function $\varphi(x)=x!^{1/x}$ does not satisfy the relation (5), for example when $r_1=\ldots=r_m=2m$; $r_{m+1}=\ldots=r_{m+n}=2n$.

The constant $A=1.2905502\ldots.$ was computed from (11b) in which the series shown are rapidly convergent.

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REFERENCES

1. Bateman, H. *et al*, Higher Transcendental Functions, vol. 1, McGraw Hill, 1953.
2. Minc, H., Upper Bounds for Permanents of (0,1)-matrices, Bull. Amer. Math. Soc. 69, 789–791 (1963).
3. ———, An inequality for permanents of (0,1)-matrices, J. Combinatorial Theory, 2, 312–6 (1967).
4. Jurkat, W. B. and H. J. Ryser, Matrix Factorizations of determinants and permanents, J. Algebra 3, 1–27 (1966).
5. Wilf, H. S., A mechanical counting method and combinatorial applications, J. Combinatorial Theory 4, 246–258 (1968).
6. Bruijn, N. G. de, Asymptotic methods in analysis, North-Holland Interscience, 1958.