BUDAN'S THEOREM FOR A CLASS OF ENTIRE FUNCTIONS

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I. **Functions with negative real zeros.** A classical theorem of Budan asserts that the number of zeros of the polynomial

$$f(z) = a_0 + a_1 z + \cdots + a_n z^n$$

in the interval (a, b) is either equal to $V(a) - V(b)$, or less by an even number, where $V(x)$ is the number of variations of sign in the sequence

$$f(x), f'(x), \cdots, f^{(n)}(x).$$

It is easy to see that the "even number" qualification can be omitted if all the zeros of $f(z)$ are real, so that in this case we get exact information on every interval.

In this note we describe a class of entire functions for which the natural generalization of Budan's rule likewise gives exact information about the number of zeros in a real interval. This class K is the set of entire functions $f(z)$ which are real for real z, have negative real zeros and positive Taylor coefficients, and are of finite exponential type. Such functions can be written in any of the forms

$$\text{(I)} \qquad f(z) = \sum_{\nu=0}^{\infty} a_\nu z^\nu = \sum_{\nu=0}^{\infty} \frac{b_\nu}{\nu!} z^\nu \qquad (a_\nu, b_\nu > 0, \nu = 0, 1, \cdots),$$

$$\text{(II)}' \qquad f(z) = Ae^{\alpha z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{z_n}\right) e^{-z/z_n} \qquad (z_n > 0, n = 1, 2, \cdots),$$

$$\text{(II)}'' \qquad f(z) = A \prod_{n=1}^{\infty} \left(1 + \frac{z}{z_n}\right) \qquad (z_n > 0, n = 1, 2, \cdots).$$

LEMMA 1. *Let $f(z) \in K$. Then*

$$\text{(1)} \qquad z_1^2 \geq a_0^2 (a_1^2 - 2a_0 a_2)^{-1}.$$

PROOF. Suppose $f(z)$ has the form (II)$'$, i.e., is of genus one. Then

$$\text{(2)} \qquad \frac{f'(z)}{f(z)} = \alpha - \sum_{n=1}^{\infty} \frac{z}{z_n} \frac{1}{1 + z/z_n} = \alpha + \sum_{m=1}^{\infty} (-1)^m \sigma_{m+1} z^m$$

where

$$\sigma_m = \sum_{n=1}^{\infty} z_n^{-m} \qquad (m \geq 2)$$

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and the representation (2) converges for $|z| < z_1$. Matching coefficients in (2)

$$\sigma_2 = a_0^{-2}(a_1^2 - 2a_0a_2)$$

and since $\sigma_2 \geqq z_1^{-2}$, the result follows.[1] If $f(z)$ is of genus zero the proof is identical.

LEMMA 2. *Let $f(z) \in K$, and let $-z_{p1}$ denote the zero of $f^{(p)}(z)$ nearest the origin. Then, as $p \to \infty$, $z_{p1} \to \infty$.*

PROOF. Applying Lemma 1 to $f^{(p)}(z) \in K$, we find

$$(3) \qquad \begin{aligned} z_{p1}^2 &\geqq \frac{a_p^2}{p+1}\left\{(p+1)a_{p+1}^2 - (p+2)a_pa_{p+2}\right\}^{-1} \\ &= \frac{b_p}{b_{p+1}}\left\{\frac{b_{p+1}}{b_p} - \frac{b_{p+2}}{b_{p+1}}\right\}^{-1} \qquad (p = 0, 1, 2, \cdots). \end{aligned}$$

The quantity in braces is well known to be positive [2, p. 24] and therefore the bound is nontrivial. Further, since

$$\frac{b_{p+2}}{b_{p+1}} < \frac{b_{p+1}}{b_p}$$

the sequence $\{b_{p+1}/b_p\}_0^\infty$ is a decreasing sequence of positive numbers, and therefore approaches a limit, which by the ratio test is actually the type τ of $f(z)$. It is now evident that the right member of (3) tends to $+\infty$ as $p \to \infty$, proving the lemma.

Next, if $f(z) \in K$ it is well known that [2, p. 24]

$$f^{(p-1)}(x)f^{(p+1)}(x) < \{f^{(p)}(x)\}^2$$

and hence that at any zero of $f^{(p)}(x)$, $f^{(p-1)}(x)$ and $f^{(p+1)}(x)$ have opposite signs. Let (a, b) be a fixed, finite interval of the real axis. By Lemma 2, there is an integer p_0 such that for $p \geqq p_0$, the interval (a, b) is free of zeros of $f^{(p)}(z)$, i.e., $f^{(p)}(z)$ is of constant sign on (a, b). For $p \geqq p_0$ then, the sequence (e.g. [5]) $f(x), f'(x), \cdots, f^{(p)}(x)$ is a Sturm sequence for (a, b). Hence if $V(\xi_1, \xi_2, \cdots, \xi_n)$ denotes the number of changes of sign in the sequence $\xi_1, \xi_2, \cdots, \xi_n$, the number of zeros of $f(z)$ in (a, b) is

$$V(f(a), f'(a), \cdots, f^{(p)}(a)) - V(f(b), f'(b), \cdots, f^{(p)}(b)) \equiv V_p(a) - V_p(b)$$

provided $p \geqq p_0$. We have shown

THEOREM 1. *Let $f(z) \in K$. Then the number of zeros of $f(z)$ in (a, b) is precisely*

[1] The idea goes back to Euler; compare [1, p. 500].

$$\lim_{n\to\infty} \{V_n(a) - V_n(b)\}. \tag{4}$$

We note that for n large enough, $V_n(a)-V_n(b)$ is constant, and that, in particular cases, the size of n which is "large enough" can be estimated from (3).

Taking $b=0$, we find easily

THEOREM 2. *Let $f(z)\in K$ have the Taylor expansion*

$$f(z) = \sum_{\nu=0}^{\infty} c_\nu(z+a)^\nu$$

about $z=-a$. Then the number of zeros of $f(z)$ in $(-a, 0)$ is exactly the number of changes of sign in the coefficient sequence $\{c_\nu\}_0^\infty$.

II. **Generalizations.** The hypotheses of positive coefficients and negative real zeros were really necessary only to insure that $z_{1p}\to\infty$. Hence the conclusion of Theorem 1 remains true for functions of order $\rho<2$ with only real zeros (of arbitrary signs) and for which $|z_{1p}|\to\infty$. The behavior of $|z_{1p}|$ has been well studied (e.g. [3; 4]) with the conclusion that $|z_{1p}|$ does not $\to\infty$ if $\rho>1$, and need not if $\rho\leqq 1$, as the example $\sin \tau z$ shows.

Gontcharoff [4] has shown, however, that if $\rho<1$, or if $\rho=1$, $\tau=0$, then, for any fixed interval (a, b) there is a sequence $\{p_k\}$ tending to infinity such that $f^{(p_k)}(z)$ is not zero in (a, b). Hence the number of zeros of a function of zero exponential type (including $\rho<1$) with only real zeros is exactly

$$V_{p_k}(a) - V_{p_k}(b),$$

k being arbitrary. In the absence of any knowledge of the sequence $\{p_k\}$, all we can say is that among the numbers $\{V_n(a)-V_n(b)\}_0^\infty$, the number of zeros of $f(z)$ in (a, b) appears infinitely often, if $f(z)$ is of zero exponential type with only real zeros.

In conclusion we give another example of a situation where the zeros of $f^{(k)}(z)$ ultimately lie outside of any fixed compact set, namely the case where $f(z)$ is the product of $e^{-\alpha z}$ by a polynomial. The main interest of this result lies in the fact that the asymptotic distribution of *all* the zeros of the kth derivative is independent of the polynomial chosen.

THEOREM 3. *Let*

$$P(z) = \sum_{\nu=0}^{p} a_\nu z^\nu = \sum_{\nu=0}^{p} \frac{b_\nu}{\nu!} z^\nu$$

be a given polynomial, and let $P_n(z)$ be defined by

$$\left(\frac{d}{dz}\right)^n \{e^{-z}P(z)\} = (-1)^n e^{-z} P_n(z).$$

Then

$$\frac{1}{a_p} \lim_{n\to\infty} \{n^{-p}P_n(nz)\} = (z-1)^p.$$

PROOF. We have

$$\left(\frac{d}{dz}\right)^n \{e^{-z}P(z)\} = e^{-z} \sum_{k=0}^{n} C_{n,k}(-1)^{n-k}P^{(k)}(z)$$

and therefore for $n \geqq p$,

$$\begin{aligned} P_n(z) &= \sum_{k=0}^{p} C_{n,k}(-1)^k P^{(k)}(z) \\ &= \sum_{r=0}^{p} \frac{z^r}{r!} \sum_{k=0}^{p-r} (-1)^k C_{n,k} b_{r+k} \equiv \sum_{r=0}^{p} \frac{b_r^{(n)}}{r!} z^r. \end{aligned}$$

Hence for each fixed ν,

$$b_{p-\nu}^{(n)} \sim \frac{n^\nu}{\nu!}(-1)^\nu b_p \qquad (n \to \infty)$$

and the result follows by an obvious calculation.

In particular, if $z_{1k}, z_{2k}, \cdots, z_{pk}$ are the zeros of the kth derivative of $e^{-z}P(z)$, where $P(z)$ is any polynomial of degree p, then for each fixed ν, $z_{\nu k} \sim k$ $(k \to \infty)$. As an example, if $p=2$, a more exact calculation shows that the zeros of $P_n(z)$ are

$$n \pm n^{1/2} + O(1) \qquad (n \to \infty)$$

the dependence on $P(z)$ being $O(1)$.

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