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Another Probabilistic Method in the Theory of Young Tableaux*

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1. INTRODUCTION

In a recent paper [1] we gave a probabilistic proof of the well-known "hook formula"

$$f(\lambda) = \frac{n!}{\pi h_{ab}} \tag{1}$$

for the number $f(\lambda)$ of Young tableaux of shape λ and size n. The proof emerged from a simple random walk, that we will now call a *hook walk*, on a board of shape λ.

In this paper we will show that a slight change in the hook walk leads to a proof of the Young–Frobenius formula

$$\sum_{|\lambda|=n} f(n)^2 = n! \tag{2}$$

The resulting procedure gives a supply of Young tableaux of size n such that each tableau of shape λ is produced with probability $f(\lambda)/n!$. Only n is given, however, in contrast to [1] where n and λ are prescribed. Thus each shape λ will occur with probability $f(\lambda)^2/n!$, proving (2). It is striking that the same

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hook-walk occurs both in the equidistributed case of [1] and in the present situation.

In Section 2 of this paper we give, for motivational purposes, a description of the line of reasoning that leads naturally (well, almost naturally) to the random walk proof technique.

In Section 3 we give the formal machinery that will be used to prove the theorem, and in Section 4 we illustrate that machinery by giving a new proof of the hook formula (1).

Finally, in Section 5 we completely describe the game that proves (2), and prove that it does so.

2. MOTIVATION

If we rewrite (2) as

$$\sum_{|\lambda|=n} \frac{f(\lambda)^2}{n!} = 1 \tag{3}$$

then the quantity $f(\lambda)^2/n!$ is seen to be the probability of occurrence of a shape λ. Hence what we need is a stochastic procedure that produces shapes λ (i.e., partitions of n) with the above frequencies.

Now we adopt an inductive approach to the question, and suppose that we have found a method that produces such shapes of $n-1$ cells. To get a shape of n cells, then, we will choose one of $n-1$ cells, and glue on to it a new corner cell κ with the right probability.

Let $\mathrm{Prob}(\lambda' \to \lambda)$ denote the probability that a shape λ' of $n-1$ cells is transformed to a shape λ of n cells. Of course $\mathrm{Prob}(\lambda' \to \lambda) = 0$ unless λ is obtained from λ' by adjunction of a single corner cell ("$\lambda' \subset \lambda$").

Now consider the probability $P(\lambda)$ with which a given shape λ is produced. It clearly satisfies

$$P(\lambda) = \sum_{\lambda' \subset \lambda} \mathrm{Prob}(\lambda' \to \lambda)\, P(\lambda')$$

and since we want $P(\lambda) = f(\lambda)^2/n!$, we must have

$$\frac{f(\lambda)^2}{n!} = \sum_{\lambda' \subset \lambda} \mathrm{Prob}(\lambda' \to \lambda) \frac{f(\lambda')^2}{(n-1)!}.$$

We can rewrite this as

$$f(\lambda) = \sum_{\lambda' \subset \lambda} \left\{ \frac{nf(\lambda')}{f(\lambda)} \mathrm{Prob}(\lambda' \to \lambda) \right\} f(\lambda'). \tag{4}$$

If we compare (4) with the well-known recurrence formula

$$f(\lambda) = \sum_{\lambda' \subset \lambda} f(\lambda') \tag{5}$$

we see that if

$$\text{Prob}(\lambda' \to \lambda) = \frac{f(\lambda)}{nf(\lambda')} \qquad (n = |\lambda| = 1 + |\lambda'|) \tag{6}$$

then (4) indeed holds.

But is (6) in fact a probability? That is, is it true that

$$\sum_{\lambda \supset \lambda'} f(\lambda) = nf(\lambda') \qquad (n = 1 + |\lambda'|). \tag{7}$$

In fact, the identity (7), a kind of dual to (5), is true, and it was known to Rutherford [2], who also gave a purely combinatorial proof of it.

However, instead of citing Rutherford's result, we want to give a random walk algorithm that will indeed attach a new corner of a given shape λ' with probability (6). Our proof of the validity of that algorithm will, of course, prove (7) at the same time.

3. The Random Walk Machinery

The basic random walk, the "hook walk," which underlies both the proof in [1] and the present one is as follows. Let λ be a shape of size n, i.e., a partition $(\lambda_1 \geqslant \lambda_2 \geqslant \cdots)$ of n. A random walk starts at cell (a, b) on the board and proceeds until a corner is reached. Generically, if a cell (u, v) is reached which is not a corner, the next cell in the walk is chosen uniformly at random from the other cells in the hook of (u, v).

The game of "Random Rook" is played on a shape λ by starting a hook walk at a cell chosen uniformly at random, and inserting "n" (where $n = |\lambda|$) into the terminal cell of the walk. This cell is then removed from the board, and the hook walk is performed repeatedly. As the shapes shrink, a Young tableau is constructed. This construction produces all Young tableaux of shape λ with equal probability, as shown in [1].

The modified game that is played to prove (2) is described in Section 5.

A board of shape $\lambda = (\lambda_1 \geqslant \lambda_2 \geqslant \cdots)$ consists of all cells [with row and column numbers] (a, b) such that $0 < b \leqslant \lambda_a$. Here the λ_i are non-negative integers (we shall allow any number of trailing zeros), and the size is the number of cells, $|\lambda| = \sum \lambda_i$. The term "shape" is often used in the meaning "board of shape." When a distinction is needed later, the latter is denoted $B(\lambda)$.

The conjugate shape $\lambda^* = (\lambda_1^* \geqslant \lambda_2^* \geqslant \cdots)$ is given by

$$\lambda_i^* = |\{j : \lambda_j \geqslant i\}|;$$

λ_i^* counts the number of cells in column i of a board of shape λ.

The hook $H(a, b)$ of cell (a, b) consists of all cells (u, v) such that either $u = a$ and $b \leqslant v \leqslant \lambda_a$ or $v = b$ and $a \leqslant u \leqslant \lambda_b^*$; the hook length is

$$h_{ab} = |H(a, b)| = (\lambda_a - b) + (\lambda_b^* - a) + 1. \tag{8}$$

A corner (α, β) is a cell with $h_{\alpha\beta} = 1$.

Let $p(\alpha\beta \mid ab)$ be the probability that a hook walk starting at a cell (a, b) will end up in a corner (α, β). Of course, $p(\alpha\beta \mid ab) = 0$ unless $\alpha \geqslant a$ and $\beta \geqslant b$. In [1] we showed that then

$$p(\alpha\beta \mid ab) = \left\{ \frac{1}{h_{\alpha\beta}} \prod_{a \leqslant i < \alpha} \frac{h_{i\beta}}{h_{i\beta} - 1} \right\} \left\{ \frac{1}{h_{ab}} \prod_{b \leqslant j < \beta} \frac{h_{\alpha j}}{h_{\alpha j} - 1} \right\}. \tag{9}$$

By inspection of (9) we observed that

$$p(\alpha\beta \mid ab) = p(\alpha\beta \mid a\beta)\, p(\alpha\beta \mid ab). \tag{10}$$

Noting patterns of cancellations in (9) we also observed the "constant zone effect," namely, $p(\alpha\beta \mid ab) = p(\alpha\beta \mid a'b')$ whenever $\lambda_a = \lambda_{a'}$ and $\lambda_b^* = \lambda_{b'}^*$.

The term "induction" usually refers to the integers, but may equally well be applied to locally finite partially ordered sets, i.e., p.o. sets in which for each element x the set $\{z : z \leqslant x\}$ is finite. To prove a proposition P for all members of the set if suffices to (1) prove $P(x)$ for all minimal elements x; (2) show that $P(x)$ holds for non-minimal x if $P(z)$ holds for all $z < x$.

Induction used in the p.o. set of boards ordered by inclusion will be referred to as *induction on expanding boards.*

Another twist is this: to prove a proposition P for all cells in a given board λ, it suffices (1) to prove P for the corner cells of λ; (2) if λ' is a sub-board of λ, and P holds for all cells in $\lambda - \lambda'$, to show that P holds for a corner of λ'. This method will be referred to as *induction on shrinking boards.*

The construction of a random Young tableau of a given shape described above is an example of construction by induction on shrinking boards.

4. The Hook Formula Revisited

As an illustration of a proof by induction on shrinking boards, we give a new proof of (9) and the properties immediately following it.

The definition of the hook walk implies $p(\alpha\beta \mid \alpha\beta) = 1$, while for $\alpha \geqslant a$, $\beta \geqslant b$, $(\alpha, \beta) \neq (a, b)$

$$p(\alpha\beta \mid ab) = \frac{1}{h_{ab} - 1} \left[\sum_{a < i \leqslant \alpha} p(\alpha\beta \mid ib) + \sum_{b < j \leqslant \beta} p(\alpha\beta \mid aj) \right]. \tag{11}$$

In the special case $\alpha > a$, $\beta = b$ this becomes

$$(h_{a\beta} - 1)\, p(\alpha\beta \mid a\beta) = \sum_{a < i \leqslant \alpha} p(\alpha\beta \mid ib), \tag{12}$$

while if $\alpha = a$, $\beta > b$ it becomes

$$(h_{\alpha b} - 1)\, p(\alpha\beta \mid \alpha b) = \sum_{b < j \leqslant \beta} p(\alpha\beta \mid \alpha j). \tag{12'}$$

Subtract from (12) the corresponding sum with a replaced by $a + 1$ and simplify; this yields

$$p(\alpha\beta \mid a\beta) = \frac{h_{a+1,\beta}}{h_{a\beta} - 1}\, p(\alpha\beta \mid a + 1, \beta), \tag{13}$$

and this gives rise to

$$p(\alpha\beta \mid a\beta) = \prod_{a \leqslant i < \alpha} \frac{h_{i+1,\beta}}{h_{i\beta} - 1}. \tag{14}$$

A simple re-indexing shows this to be equal to the first product on the right in (9).

Corresponding formulas for $\alpha = a$, $\beta > b$ are

$$p(\alpha\beta \mid \alpha b) = \frac{h_{\alpha,b+1}}{h_{\alpha b} - 1}\, p(\alpha\beta \mid \alpha, b + 1) \tag{13'}$$

$$p(\alpha\beta \mid \alpha b) = \prod_{b \leqslant j < \beta} \frac{h_{\alpha,j+1}}{h_{\alpha j} - 1}. \tag{14'}$$

Formulas (13, 13′) give the first step in the proof of the "constant zone effect," e.g., if $\lambda_{a+1} = \lambda_a$ then $h_{a+1,\beta} = h_{a\beta} - 1$, hence $p(\alpha\beta \mid a\beta) = p(\alpha\beta \mid a + 1, \beta)$.

Now let Λ be the rectangular board with corner (α, β), and let λ' be any sub-board of Λ such that (10) is proven for all cells in $\Lambda - \lambda'$. Let (a, b) be a corner of λ', then we may use (10) on all terms on the right of (11). The typical summand in the first sum becomes $p(\alpha\beta \mid i\beta)\, p(\alpha\beta \mid \alpha b)$, in which the second factor is independent of i and can be factored out. The sum of the first terms equals the left side of (12). The second sum is treated similarly, using (12′). It follows that

$$\begin{aligned} p(\alpha\beta \mid ab) &= \frac{1}{h_{ab}-1}\{(h_{a\beta}-1)+(h_{\alpha b}-1)\}\, p(\alpha\beta \mid a\beta)\, p(\alpha\beta \mid ab) \\ &= p(\alpha\beta \mid a\beta)\, p(\alpha\beta \mid ab) \end{aligned}$$

upon noting that $h_{ab} = h_{a\beta} + h_{\alpha b} - 1$. This proves (10), hence by (14, 14′) also (9), and also the general "constant zone effect."

To prove the hook formula (1) [the product in the denominator extends over all cells (a, b) in the board] from (9), first calculate the probability $p(\kappa)$ that a hook walk starting from an initial cell chosen uniformly at random ends at a fixed corner $\kappa = (\alpha, \beta)$. We find

$$\begin{aligned} p(\kappa) &= \frac{1}{n} \sum_{1 \leqslant i < \alpha} \sum_{1 \leqslant j < \beta} p(\alpha\beta \mid ij) \qquad \text{(by (10))} \\ &= \frac{1}{n} \sum_{1 \leqslant i < \alpha} p(\alpha\beta \mid i\beta) \sum_{1 \leqslant j < \beta} p(\alpha\beta \mid \alpha j) \qquad \text{(by (12, 12′))} \\ &= \frac{1}{n}\{p(\alpha\beta \mid 1\beta) + (h_{1\beta}-1)\, p(\alpha\beta \mid 1\beta)\}\{p(\alpha\beta \mid \alpha 1) + (h_{\alpha 1}-1)\, p(\alpha\beta \mid \alpha 1)\} \\ &= \frac{h_{1\beta} h_{\alpha 1}}{n}\, p(\alpha\beta \mid 1\beta)\, p(\alpha\beta \mid \alpha 1) \qquad \text{(by (9))} \\ &= \frac{1}{n} \prod_{i < \alpha} \frac{h_{i\beta}}{h_{i\beta}-1} \prod_{j < \beta} \frac{h_{\alpha j}}{h_{\alpha j}-1}. \end{aligned}$$

Since we want to prove (1), temporarily denote the right side of (1) by $F(\lambda)$. Then it is easy to see, by massive cancellation, that $p(\kappa) = F(\lambda - \kappa)/F(\lambda)$. Since every play of Random Rook ends up in some corner, we have $\sum p(\kappa) = 1$; that is,

$$F(\lambda) = \sum_{\kappa} F(\lambda - \kappa).$$

This recurrence relation also holds for $f(\lambda)$ [by induction on shrinking boards, κ is the cell in which $n = |\lambda|$ is inserted]; furthermore, $F(\phi) = f(\phi) = 1$; induction on expanding boards proves $f(\lambda) = F(\lambda)$.

The fact that $p(\kappa) = f(\lambda - \kappa)/f(\lambda)$ is the key to the otherwise trivial proof by induction on expanding boards that the construction in Section 3 produces all Young tableaux of shape λ uniformly at random.

5. Complementary Boards

We recall that a shape λ is a partition $(\lambda_1 \geqslant \lambda_2 \geqslant \cdots)$ of an integer $n = |\lambda|$. The board $B(\lambda)$ we have associated with the shape consists of those cells with (row, column) numbers (x, y) such that $y \leqslant \lambda_x$; $x, y \in Z^+$. Now we associate with λ boards $\bar{B}_{pq}(\lambda)$ obtained from $B(\lambda)$ by a rotation over 180° followed by a translation which places the original (1, 1) cell at (p, q). Naturally, we require $p \geqslant \lambda_1^*$, $q \geqslant \lambda_1$. That is,

$$\bar{B}_{pq}(\lambda) = \{(x, y): 0 < q + 1 - y \leqslant \lambda_{p+1-x}, 0 < x \leqslant p\}.$$

Complementary boards arise in the following situation. Let λ be a shape, and let p, q be such that $p \geqslant \lambda_1^*$, $q \geqslant \lambda_1$, then the cells which belong to the rectangular board with corner (p, q) but not to $B(\lambda)$ constitute the board $\bar{B}_{pq}(\bar{\lambda})$, where

$$\bar{\lambda}_i = q + 1 - \lambda_{p+1-i}, \qquad 0 < i \leqslant p,$$
$$\bar{\lambda}_j^* = p + 1 - \lambda_{q+1-j}^*, \qquad 0 < j \leqslant q.$$

We now consider *special complementary hook walks* in a complementary board $\bar{B}(\bar{\lambda})$. "Special" refers to the fact that the walk starts at (p, q); "complementary" to the fact that the walk are "left" and "up" as hook walks in complementary boards should be. We assume that $\bar{B}(\bar{\lambda})$ is complementary to the board $B(\lambda)$ of shape λ. We now observe that if $p > \lambda_1^*$, $q > \lambda_1$, then every cell whose adjunction to $B(\lambda)$ would create another board lies in $\bar{B}_{pq}(\bar{\lambda})$ and these cells are all the corners of $\bar{B}_{pq}(\bar{\lambda})$. Furthermore, special complementary hook walks can terminate at any of these cells.

The *distance* between cells (u, v) and (u', v') is defined as

$$d((u, v), (u', v')) = |u - u'| + |v - v'|.$$

Hook lengths can be interpreted in terms of distances: h_{uv} is one unit larger than the distance between the extreme cells of $H(u, v)$.

Proposition. *Let λ be a shape, let p, q be such that $p > \lambda_1^*$, $q > \lambda_1$. Then the probability that a special complementary hook walk in the complementary board will terminate at the corner $\bar{\kappa}$ of $\bar{B}_{pq}(\bar{\lambda})$ equals*

$$\bar{p}(\bar{\kappa}) = \prod_{\sigma} d(\kappa, \sigma) \Big/ \prod_{\bar{\sigma}} d(\bar{\kappa}, \bar{\sigma}), \tag{15}$$

where σ (resp $\bar{\sigma}$) ranges over all corner cells of $B(\lambda)$ (resp. $\bar{B}_{pq}(\bar{\lambda})$). In particular, the probability $\bar{p}(\bar{\kappa})$ is independent of p and q.

Proof. According to (10), $p(\alpha\beta \mid 11)$ is the product of the right sides of (14) and (14′), with $a = b = 1$. An examination of a typical factor in these products shows that it is not equal to 1 iff the hook length appearing in the denominator is that of a hook one of whose extreme cells is $\kappa = (\alpha, \beta)$ and the other is another corner cell, say σ, of $B(\lambda)$. The factor in the denominator is then exactly $d(\bar{\kappa}, \sigma)$. Furthermore, in that case the factor in the numerator is the distance $d(\kappa, \bar{\sigma})$ between κ and a cell $\bar{\sigma}$ not in $B(\lambda)$ whose adjoining creates a larger board.

Transposition of these observations to the complementary board $\bar{B}_{pq}(\bar{\lambda})$ with appropriate change in notation yields (15).

PROPOSITION. *Let λ be a shape, and λ' a shape whose board $B(\lambda')$ is obtained from λ by adjoining a cell $\bar{\kappa}$ (which is a corner of $\bar{B}_{pq}(\bar{\lambda})$, if $p > \lambda_1^*$, $q > \lambda_1$). Then*

$$\bar{p}(\bar{\kappa}) = \frac{f(\lambda')}{n' f(\lambda)}, \qquad n' = |\lambda'|. \tag{16}$$

Proof. Referring to (1), the right hand side of (16) allows for cancellation of all corresponding factors except the hook lengths of those cells lying in the same row or the same column as $\kappa = (\alpha, \beta)$. Among the remaining factors there are further cancellations, namely, $h_{i\beta}$ cancels against $h'_{i+1,\beta}$ (primed quantities refer to λ') unless the terminal cell (i, λ_i) of $H(i, \beta)$ is a corner of $B(\lambda)$—similar comments apply to $h_{\alpha j}$ and $h'_{\alpha,j+1}$. The remaining factors in the numerator (resp. denominator) are now exactly those in the numerator (resp. denominator) on the right in (15).

THEOREM. *Let n, p, q be positive integers, $p > n$, $q > n$, and let T be a Young tableau resulting from the following construction. Starting with an empty tableau, a tableau of size i ($1 \leqslant i \leqslant n$) is constructed from a tableau of size $i - 1$ by inserting i into the terminal cell of a special complementary hook walk in the complementary board consisting of those cells in the rectangular board with corner (p,q) which have not been terminal cells in prior walks. Let T have shape λ. Then the probability of T being produced is $f(\lambda)/n!$*

Proof. By induction on expanding boards. The tableau T can be constructed only from the tableau $T^- = T - \{n\}$, and by induction the probability of constructing T^- as a tableau of size $n-1$ is $f(\lambda^-)/(n-1)!$, where λ^- is the shape of T^-. According to (16) the probability of transition from T^- to T is $f(\lambda^-)/(n-1)!$ Multiplication of the last two probabilities completes the proof.

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