SOME APPLICATIONS OF THE INEQUALITY OF ARITHMETIC AND GEOMETRIC MEANS TO POLYNOMIAL EQUATIONS

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The purpose of this note is to point out a simple generalization of the inequality

$$(z_1 z_2 \cdots z_n)^{1/n} \leqq \frac{1}{n}(z_1 + \cdots + z_n)$$

of arithmetic and geometric means, which will hold when the arguments of the complex numbers $z_1, \cdots, z_n$ are suitably restricted. We shall apply the resulting inequality to the roots of polynomial equations, obtaining first a quantitative form of the Gauss-Lucas theorem, and then some relationships between the coefficients of a polynomial and the size of a sector containing its roots.

1. The inequality.

The basic result is

THEOREM 1. *Suppose*

$$|\arg z_i| \leqq \psi < \frac{\pi}{2}, \qquad i = 1, 2, \cdots, n.$$

Then

$$|z_1 z_2 \cdots z_n|^{1/n} < (\sec \psi) \frac{1}{n} |z_1 + z_2 + \cdots + z_n| \tag{1}$$

unless n *is even and* $z_1 = \cdots = z_{n/2} = \bar{z}_{(n/2)+1} = \cdots = \bar{z}_n = re^{i\psi}$, *in which case equality holds.*

PROOF. We have

$$\begin{aligned} |z_1 + z_2 + \cdots + z_n| &\geqq |\mathrm{Re}(z_1 + \cdots + z_n)| \\ &= (|z_1| \cos \phi_1 + |z_2| \cos \phi_2 + \cdots \\ &\quad + |z_n| \cos \phi_n) \\ &\geqq (\cos \psi)(|z_1| + \cdots + |z_n|) \\ &\geqq n \cos \psi (|z_1|\,|z_2| \cdots |z_n|)^{1/n} \end{aligned} \tag{2}$$

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as claimed. All signs of equality hold only when

$$\text{(3)}\qquad \begin{aligned} &\text{(a)}\ \operatorname{Im}(z_1 + \cdots + z_n) = 0\\ &\text{(b)}\ \cos\phi_i = \cos\psi \qquad (i = 1, 2, \cdots, n)\\ &\text{(c)}\ |z_1| = |z_2| = \cdots = |z_n| \end{aligned}$$

which imply the configuration stated in the theorem. For odd n the constant $\sec\psi$ is only asymptotically best possible.

2. **Application to polynomials.** Let

$$\text{(4)}\qquad P(z) = a_0 + a_1 z + \cdots + a_n z^n = a_n(z - z_1)\cdots(z - z_n)$$

be given and let K denote the convex hull of the zeros $z_1, \cdots, z_n$ of $P(z)$. Let z be outside K, and suppose that, from z, K subtends an angle 2ψ. Then the spread in the arguments of the numbers

$$\frac{1}{z - z_1}, \cdots, \frac{1}{z - z_n}$$

is at most 2ψ, and from Theorem 1,

$$\left|\frac{1}{(z - z_1)}\cdots\frac{1}{(z - z_n)}\right|^{1/n} \leqq (\sec\psi)\frac{1}{n}\left|\sum_{\nu=1}^{n}\frac{1}{z - z_\nu}\right|.$$

But this is just the assertion that

$$\left|\frac{a_n}{P(z)}\right|^{1/n} \leqq \frac{\sec\psi}{n}\left|\frac{P'(z)}{P(z)}\right|,$$

and we have proved

THEOREM 2. *If z is a point from which the convex hull of the zeros of the polynomial $P(z)$ of degree n subtends an angle $2\psi < \pi$, then*

$$\text{(5)}\qquad |P'(z)| \geqq n|a_n|^{1/n}(\cos\psi)|P(z)|^{1-(1/n)}.$$

COROLLARY 1. *The zeros of $P'(z)$ lie in the convex hull of the zeros of $P(z)$* (Gauss-Lucas).

COROLLARY 2. *If the zeros of $P(z)$ lie in the unit circle, then we have for $|z| > 1$,*

$$\text{(6)}\qquad |P'(z)| \geqq \frac{n|a_n|^{1/n}}{\sqrt{\left(1 - \dfrac{1}{|z|^2}\right)}}|P(z)|^{1-(1/n)}.$$

THEOREM 3. *The zeros of the polynomial*

$$P(z) = a_0 + a_1 z + \cdots + a_n z^n,$$

are not contained in any sector of central angle less than

$$2\cos^{-1}\left\{\left|\frac{a_{n-1}}{na_n}\right| \min_{0\leqq k\leqq n-1}\left|\frac{a_n}{a_k}\binom{n}{k}\right|^{1/n-k}\right\}.$$

PROOF. Suppose the zeros of $P(z)$ lie in a sector of angle $2\psi<\pi$. From Theorem 1,

$$\left|\frac{a_0}{a_n}\right|^{1/n} \leqq \frac{\sec\psi}{n}\left|\frac{a_{n-1}}{a_n}\right|,$$

or

$$\sec\psi \geqq n\left|a_n\right|^{1-(1/n)}\left|a_0\right|^{1/n}\left|a_{n-1}\right|^{-1}.$$

Applying this result to

$$P^{(k)}(z) = \sum_{\nu=0}^{n-k}\frac{(\nu+k)!}{\nu!}a_{\nu+k}z^\nu,$$

which, by Corollary 1 also satisfies the hypotheses, we find

$$\sec\psi \geqq \left|\frac{na_n}{a_{n-1}}\right|\left|\frac{a_n}{a_k}\binom{n}{k}\right|^{-1/(n-k)} \qquad (k = 0, 1, \cdots, n-1),$$

and the result follows.

THEOREM 4. *Under the hypotheses of Theorem 2, let ρ denote the distance from z to the center of gravity of the zeros of $P(z)$. Then*

$$|P(z)| \leqq |a_n|(\rho\sec\psi)^n. \tag{7}$$

PROOF. Apply Theorem 1 to the numbers $z-z_1, \cdots, z-z_n$.

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