An Identity of J. S. Lomont and John Brillhart

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The following identity had been proved by Lomont and Brillhart, who asked if the methods of $A = B$ could provide a simpler proof: Let $1 \leq m \leq n$, where $n \geq 2$ and $1 \leq s \leq \min(m, n-1)$. Then

$$\sum_{j=0}^{s} \frac{(-1)^j (m+n-2j) \binom{m}{j} \binom{m-j}{m-s} \binom{n}{j} \binom{n-j}{n-s} \binom{m+n}{j} \binom{m+n-s-j-1}{s-j}}{\binom{s}{j}^2} = 0. \tag{1}$$

So we load `EKHAD` into a Maple worksheet, and we call it with

```
ct((-1)^j*(m+n-2*j)*binomial(m,j)*binomial(m-j, m-s)*binomial(n,j)*binomial(n-j,n-s)
  *binomial(m+n,j)*binomial(m+n-s-j-1,s-j)/binomial(s,j)^2,0,j,n,N);
```

thereby looking for a recurrence of order 0, i.e., using Gosper's algorithm. `EKHAD` responds with

$$-s, j*(m+n-s-j)/(m+n-2*j)$$

which means that if $F(n, j)$ is the summand in (1) above, then F satisfies

$$-sF(n, j) = G(n, j+1) - G(n, j), \tag{2}$$

where $G(n, j) = j(m+n-s-j)F(n, j)/(m+n-2j)$. If we now sum both sides of (2) over all integers j, the right side telescopes to 0, and the result (1) follows. We note that when, as in this case, Gosper's algorithm suffices, then we can do the sum in (1) not only over the range $j = 0$ to s, but over any range at all.

Note added 11/25/2003: Mr. Yong Kong has kindly called our attention to the fact that if we simply replace each binomial coefficient $\binom{a}{b}$ in (1) above by $a!/(b!(a-b)!)$, and simplify things, then (1) reduces to

$$\binom{m}{s}\binom{n}{s}\sum_{j=0}^{s}(-1)^j(m+n-2j)\binom{m+n}{j}\binom{m+n-s-j-1}{s-j} = 0,$$

which is a good bit simpler. Of course Gosper's algorithm still applies and gives the stated result. This underscores the fact that the methods of "A=B" work whether or not any human thought is exerted before applying them!